On the Difference of Coefficients of Univalent Functions

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Abstract. For \( f \in S \), the class of normalized functions, analytic and univalent in the unit disk \( D \) and given by
\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,
\]
we give an upper bound for the coefficient difference \( |a_4| - |a_3| \) when \( f \in S \). This provides an improved bound in the case \( n = 3 \) of Grinspan’s 1976 general bound \( ||a_n+1| - |a_n|| \leq 3.61 \ldots \). Other coefficients bounds, and bounds for the second and third Hankel determinants when \( f \in S \) are found when either \( a_2 = 0 \), or \( a_3 = 0 \).

1. Introduction. preliminaries and definitions

Let \( A \) be the class of functions \( f \) which are analytic in the open unit disc \( D = \{ z : |z| < 1 \} \) of the form
\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,
\]
and let \( S \) be the subclass of \( A \) consisting of functions that are univalent in \( D \).

Although the famous Bieberbach conjecture \( |a_n| \leq n \) for \( n \geq 2 \), was proved by de Branges in 1985 [1], a great many other problems concerning the coefficients \( a_n \) remain open. The main aim of this paper (Section 3), is by use of the Grunsky inequalities, to find an upper for the difference of coefficients \( |a_4| - |a_3| \) for \( f \in S \), which improves the well-known general bound of Grispan \( ||a_{n+1}| - |a_n|| \leq 3.61 \ldots \) [4], when \( n = 3 \). We also obtain information concerning the initial coefficients of \( f(z) \), and of the second and third Hankel determinants when either \( a_2 = 0 \), or \( a_3 = 0 \).

For \( f \in S \), the Grunsky coefficients \( \omega_{p,q} \) as defined in N. A. Lebedev [6] are given by
\[
\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,
\]
where \( \omega_{p,q} = \omega_{q,p} \), and satisfy the so-called Grunsky inequalities [2, 6].

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In this paper we will use these expressions to obtain information concerning the coefficients of the form \( \omega_{2p-1,2q-1} \), and inequalities (2) take the form

\[
\sum_{q=1}^{\infty} \left( 2q - 1 \right) \left( \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right)^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1},
\]

(4)  

(Note that in this paper, we omit the upper index (2) in \( \omega_{2p-1,2q-1} \) in Lebedev’s notation).  

The following similar inequality follows from the relation (15) on page 57 in [6].

\[
\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} x_{2q-1} \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}.
\]

(5)  

Thus for example, from (4) and (5) when \( x_{2p-1} = 0 \) and \( p = 3, 4, \ldots \), we obtain

\[
|a_{11}x_1 + a_{31}x_3|^2 + 3|a_{13}x_1 + a_{33}x_3|^2 + 5|a_{15}x_1 + a_{35}x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3},
\]

(6)  

and

\[
|a_{11}^2x_1^2 + 2a_{13}x_1x_3 + a_{33}x_3^2| \leq |x_1|^2 + \frac{|x_3|^2}{3},
\]

(7)  

respectively.

It was also shown in [6, p.57], that if \( f \in S \) is given by (1), then the coefficients \( a_2, a_3, a_4 \) and \( a_5 \) can be expressed in terms of the Grunsky coefficients \( \omega_{2p-1,2q-1} \) of the function \( f_2 \) given by (3) as follows.

\[
\begin{align*}
a_2 &= 2a_{11}, \\
a_3 &= 2a_{13} + 3a_{11}^2, \\
a_4 &= 2a_{33} + 8a_{11}a_{13} + \frac{10}{3}a_{11}^3, \\
a_5 &= 2a_{35} + 8a_{11}a_{33} + 5a_{13}^2 + 18a_{11}^2a_{13} + \frac{7}{3}a_{11}^4, \\
o &= 3a_{15} - 3a_{11}a_{13} + a_{11}^3 - 3a_{33}.
\end{align*}
\]

(8)

In this paper we will use these expressions to obtain information concerning the coefficients \( a_2, a_3, a_4, \) and \( a_5 \) when \( f \in S \).

In recent years a great deal of attention has been given to finding upper bounds for the modulus of the second and third Hankel determinants \( H_2(2) \) and \( H_3(1) \), defined as follows who’s elements are the coefficients of \( f \in S \) (see e.g. [8]).
For $f \in S$

$$H_2(2) = a_2a_4 - a_3^2$$

and

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

(9)

Almost all results have concentrated on finding bounds for $|H_2(2)|$ and $|H_3(1)|$ for subclasses of $S$, and only recently has a significant bound been found for the whole class $S$ [7] for $|H_2(2)|$ and $|H_3(1)|$. However finding exact sharp bounds remains an open problem.

We begin by using the Grunsky inequalities in (5) to obtain bounds for the modulus of some initial coefficients and $|H_2(2)|$ and $|H_3(1)|$ when $f \in S$ provided either $a_2$, or $a_3 = 0$.

2. Coefficient bounds and Hankel determinants

Obtaining sharp bounds for the modulus of the coefficients for odd functions in $S$ has long been been an open problem. If $f_2$, given by (3) is an odd function in $S$, then the only known sharp bounds for $|c_{2n-1}|$ for $n \geq 2$ are $|c_3| \leq 1$, and $|c_5| \leq 1/2 + e^{-2/3} = 1.013 \ldots$. In general the best bound to date is $|c_{2n-1}| \leq 1.14$ for $n \geq 2$, (see e.g. [2]).

In our first theorem, we give bounds for $|a_3|$, $|a_4|$ and $|a_5|$ when $f \in S$ assuming only that only $a_2 = 0$, thus providing bounds for a wider class of functions than the odd functions in $S$. We also give bounds for $|H_2(2)|$ and $|H_3(1)|$ in this case.

**Theorem 2.1.** Let $f \in S$ and be given by (1) with $a_2 = 0$. Then

(i) $|a_3| \leq 1$,

(ii) $|a_4| \leq \frac{2}{3} = 0.666 \ldots$,

(iii) $|a_5| \leq \sqrt{\frac{19}{15}} = 1.67666 \ldots$,

(iv) $|H_2(2)| \leq 1$,

(v) $|H_3(1)| \leq \frac{21}{20} = 1.05$.

Proof.

(i) The classical inequality $|a_3 - a_2^2| \leq 1$ for $f$ in $S$ when $a_2 = 0$, gives $|a_3| \leq 1$, which from (8) gives

$$|\omega_{13}| \leq \frac{1}{2}.$$  

(10)
(ii) Next choose $x_1 = 0$ and $x_3 = 1$ in (7), which gives

$$|\omega_{33}| \leq \frac{1}{3}. \quad (11)$$

Also, since $\omega_{11} = 0 \Leftrightarrow a_2 = 0$, then from (8) and (11) we obtain

$$|a_4| = 2|\omega_{33}| \leq \frac{2}{3} = 0.666 \ldots$$

(iii) Again since $\omega_{11} = 0$, from (8) we obtain

$$|a_5| = |2\omega_{35} + 5a_{13}^2|. \quad (12)$$

From (6) with $x_1 = 0$ and $x_3 = 1$ we have ($\omega_{11} = 0$)

$$|\omega_{13}|^2 + 3|\omega_{33}|^2 + 5|\omega_{35}|^2 \leq \frac{1}{3}$$

and from here

$$|\omega_{35}| \leq \frac{1}{\sqrt{15}} \sqrt{1 - 3|\omega_{13}|^2}. \quad (13)$$

From (12) and (13) we have

$$|a_5| \leq 2|\omega_{35}| + 5|\omega_{13}|^2 \leq \frac{1}{\sqrt{15}} \sqrt{1 - 3|\omega_{13}|^2} + 5|\omega_{13}|^2 \leq \frac{503}{300} = 1.67666 \ldots$$

(iv) Since we are assuming $a_2 = 0$, (i) shows that $|H_2(2)| \leq 1$ is trivial.

(v) When $\omega_{11} = 0$, from the last relation in (8) we have $\omega_{33} = \omega_{15}$, and from (9),

$$|H_3(1)| = |2a_{13}^3 + 4a_{13}\omega_{35} - 4\omega_{33}^2| \leq 2|a_{13}|^3 + 4 + |a_{13}\omega_{35} - \omega_{15}|^2. \quad (14)$$

Now choose $x_1 = -\omega_{15}$, and $x_3 = \omega_{13}$, and since $\omega_{33} = \omega_{15}$, from (6) we obtain

$$|\omega_{13}|^4 + 5E_1^2 \leq |\omega_{15}|^2 + \frac{|\omega_{13}|^2}{3} \leq \frac{1}{5} - \frac{3}{5}|\omega_{13}|^2 + \frac{1}{3}|\omega_{13}|^2,$$

(since by (6) $3|\omega_{13}|^2 + 5|\omega_{15}|^2 \leq 1$ for $x_1 = 1, x_3 = 0$ and $\omega_{11} = 0$), which implies $5E_1^2 \leq \frac{1}{5} - \frac{4}{15}|\omega_{13}|^2 - |\omega_{13}|^4$, i.e., $E_1 \leq \frac{1}{5}$.

Finally from (10) and (14), it follows that

$$|H_3(1)| \leq 2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{5} = \frac{21}{20} = 1.05.$$  

This completes the proof of Theorem 2.1. \hfill \Box

We next prove a similar result, this time assuming that $a_3 = 0$. 
Theorem 2.2. Let \( f \in S \) and be given by (1), with \( a_3 = 0 \). Then

(i) \(|a_2| \leq 1\),

(ii) \(|a_4| \leq \frac{\sqrt{37} + 13}{12} = 1.59023\ldots\),

(iii) \(|a_5| \leq \frac{1}{4} \frac{\sqrt{757} + 85}{64} = 3.10412\ldots\),

(iv) \(|H_2(2)| \leq \frac{13 + \sqrt{37}}{12} = 1.59023\ldots\),

(v) \(|H_3(1)| \leq \frac{24 + \sqrt{645}}{30} = 1.64656\ldots\).

Proof.

(i) Since \(|a_3 - a_2^2| \leq 1\) and \(a_3 = 0\), then \(|a_2^2| \leq 1\), i.e., \(|a_2| \leq 1\). Also, since by (8), \(a_3 = 2\omega_{13} + 3\omega_{11}^2 = 0\), it follows that

\[
\omega_{13} = -\frac{3}{2}\omega_{11} \quad \left( \Leftrightarrow \omega_{11}^2 = \frac{2}{3}\omega_{13} \right). \tag{15}
\]

Because \(|a_2| = |2\omega_{11}| \leq 1\), we have

\[
|\omega_{11}| \leq \frac{1}{2} \quad \text{and} \quad |\omega_{13}| \leq \frac{3}{8} \quad (\text{by (15)}. \tag{16}
\]

(ii) By using (8) and (15), we obtain

\[
|a_4| = \left| 2\omega_{33} + 8\omega_{11} \left( -\frac{3}{2}\omega_{11}^2 \right) + \frac{10}{3} \omega_{11}^3 \right|
= \left| 2\omega_{33} - \frac{26}{3} \omega_{11}^3 \right|
\leq 2|\omega_{33}| + \frac{26}{3} |\omega_{11}|^3. \tag{17}
\]

From (6), using \(x_1 = 0\) and \(x_3 = 1\), we have

\[
|\omega_{13}|^2 + 3|\omega_{33}|^2 \leq \frac{1}{3},
\]

which implies (with \(\omega_{13} = -\frac{3}{2} \omega_{11}^2\), see (15))

\[
|\omega_{33}| \leq \sqrt{\frac{1}{9} - \frac{3}{4} |\omega_{11}|^4}. \tag{18}
\]

Combining (17) and (18) we obtain

\[
|a_4| \leq 2 \sqrt{\frac{1}{9} - \frac{3}{4} |\omega_{11}|^4 + \frac{26}{3} |\omega_{11}|^3} =: \varphi(|\omega_{11}|), \tag{19}
\]

where \(\varphi(t) = 2 \sqrt{\frac{1}{9} - \frac{3}{4} t^4 + \frac{26}{3} t^3}, 0 \leq t = |\omega_{11}| \leq \frac{1}{2} \) (by (16)). Since \(\varphi\) is increasing function on \([0, 1/2]\),

\[
\varphi(t) \leq \varphi(1/2) = \frac{\sqrt{37} + 13}{12},
\]

which, together with (19), gives the desired result.
(iii) From the last relation in (8), using (15) we have $\omega_{33} = \omega_{15} + \frac{11}{6} \omega_{11}^3$, which with the expression for $a_5$ in (8), gives

$$|a_5| = |2\omega_{35} + 8\omega_{11}\omega_{15} + 5\omega_{13}^2 - 10\omega_{11}^4| \leq 2 |\omega_{35} + 4\omega_{11}\omega_{15}| + 5|\omega_{13}^2| + 10|\omega_{11}^4|. \quad (20)$$

Once again, using (6) choosing $x_1 = 4\omega_{11}$, $x_3 = 1$ and $\omega_{13} = -\frac{3}{2} \omega_{11}^2$, we have

$$(C_1)^2 = |4\omega_{11}\omega_{15} + \omega_{33}|^2 \leq -\frac{5}{4}|\omega_{11}|^4 + \frac{16}{5}|\omega_{11}|^2 + \frac{1}{15} \leq \frac{757}{64} \cdot 15,$$

since $|\omega_{11}| \leq \frac{1}{2}$. Thus

$$C_1 \leq \frac{1}{8} \sqrt{\frac{757}{15}}.$$

Next, since $\omega_{13} = -\frac{3}{2} \omega_{11}^2$ and $|\omega_{11}| \leq \frac{1}{2}$, we have

$$C_2 = 5 \cdot \frac{9}{4} |\omega_{11}|^4 + 10|\omega_{11}|^2 = \frac{85}{4} |\omega_{11}|^4 \leq \frac{85}{4} \cdot \frac{1}{16} = \frac{85}{64},$$

since $|\omega_{11}| \leq \frac{1}{2}$.

Finally from (20) we have

$$|a_5| \leq \frac{1}{4} \sqrt{\frac{757}{15}} + \frac{85}{64} = 3.10412 \ldots.$$

(iv) By using (9), (8) and (15), we have

$$H_2(2) = 4\omega_{11}\omega_{33} + 4\omega_{11}^2\omega_{13} - 4\omega_{13}^2 - \frac{7}{3} \omega_{11}^4$$

$$= 4\omega_{11}\omega_{33} - \frac{52}{3} \omega_{11}^4 \quad (21)$$

and from here

$$|H_2(2)| \leq 4|\omega_{11}| |\omega_{33}| + \frac{52}{3} |\omega_{11}|^4. \quad (22)$$

From (18) and (22) we have

$$|H_2(2)| \leq 4|\omega_{11}| \sqrt{\frac{1}{9} - \frac{3}{4} |\omega_{11}|^4 + \frac{52}{3} |\omega_{11}|^4} =: \varphi_1(|\omega_{11}|),$$

where

$$\varphi_1(t) = 4t \sqrt{\frac{1}{9} - \frac{3}{4} t^4 + \frac{52}{3} t^4},$$

with $0 \leq t = |\omega_{11}| \leq \frac{1}{2}$. Finally, it can be checked that $\varphi_1$ is an increasing function on the interval $(0, 1/2)$, and so

$$|H_2(2)| \leq \varphi_1(1/2) = \frac{13 + \sqrt{37}}{12} = 1.59023 \ldots.$$
(v) By using the last relation from (8) with \( \omega_{13} = -\frac{3}{2} \omega_{11}^2 \), it follows that \( \omega_{33} = \omega_{15} + \frac{11}{4} \omega_{11}^2 \), and so using (9), after some calculations we obtain

\[
H_3(1) = -12\omega_{11}^2 \left( \omega_{11} \omega_{15} + \frac{2}{3} \omega_{35} \right) - 4\omega_{15}^2 - 30\omega_{11}^6
\]

which gives

\[
|H_3(1)| \leq 12|\omega_{11}|^2 \left| \omega_{11} \omega_{15} + \frac{2}{3} \omega_{35} \right| + 4|\omega_{15}|^2 + 30|\omega_{11}|^6.
\]  

(23)

Now choose \( x_1 = \omega_{11} \) and \( x_3 = \frac{2}{3} \) in (6), then (since \( \omega_{13} = -\frac{3}{2} \omega_{11}^2 \)),

\[
\left| \omega_{11} \omega_{15} + \frac{2}{3} \omega_{35} \right| \leq \sqrt{\frac{1}{5} \left( |\omega_{11}|^2 + \frac{4}{27} \right)}.
\]

and so

\[
D_1 \leq 12|\omega_{11}|^2 \sqrt{\frac{1}{5} \left( |\omega_{11}|^2 + \frac{4}{27} \right)}
\]

\[
\leq 12 \cdot \frac{1}{4} \sqrt{\frac{1}{5} \left( \frac{1}{4} + \frac{4}{27} \right)} = \sqrt{\frac{43}{60}} = \frac{\sqrt{645}}{30} = 0.84656 \ldots,
\]

since \( |\omega_{11}| \leq \frac{1}{2} \).

Also, as in the proof of (iii), we have

\[
5|\omega_{15}|^2 \leq 1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 = 1 - |\omega_{11}|^2 - \frac{27}{4} |\omega_{11}|^4,
\]

where we have once again used \( \omega_{13} = -\frac{3}{2} \omega_{11}^2 \). Now

\[
D_2 \leq \frac{4}{5} - \frac{4}{5} |\omega_{11}|^2 + \frac{27}{5} |\omega_{11}|^4 + 30|\omega_{11}|^6 =: \varphi_2(|\omega_{11}|^2),
\]

where

\[
\varphi_2(t) = \frac{1}{5} \left( 4 - 4t - 27t^2 + 150t^3 \right),
\]

and \( 0 \leq t = |\omega_{11}|^2 \leq \frac{1}{4} \). Since \( \varphi_2 \) attains its maximum at \( t_0 = 0 \),

\[
D_2 \leq \varphi_2(0) = \frac{4}{5}.
\]  

(25)

Finally, by using (23), (24) and (25) we obtain

\[
|H_3(1)| \leq D_1 + D_2 \leq \frac{24 + \sqrt{645}}{30} = 1.64656 \ldots.
\]

\( \square \)
3. Coefficient differences for \( f \in S \)

A long standing problem in the theory of univalent functions is to find sharp upper and lower bounds for \( |a_{n+1}| - |a_n| \), when \( f \in S \). Since the Koebe function has coefficients \( a_n = n \), it is natural to conjecture that \( |a_{n+1}| - |a_n| \leq 1 \). As early as 1933, this was shown to be false even when \( n = 2 \), when Fekete and Szegő [3] obtained the sharp bounds

\[
-1 \leq |a_3| - |a_2| \leq \frac{3}{4} e^{-\lambda_0(2e^{-\lambda_0} - 1)} = 1.029 \ldots,
\]

where \( \lambda_0 \) is the unique value of \( \lambda \) in \( 0 < \lambda < 1 \), satisfying the equation \( 4\lambda = e^\lambda \).

Hayman [5] showed that if \( f \in S \), then \( |a_{n+1}| - |a_n| \leq C \), where \( C \) is an absolute constant. The exact value of \( C \) is unknown, the best estimate to date being \( C = 3.61 \ldots [4] \), which because of the sharp estimate above when \( n = 2 \), cannot be reduced to 1.

We now use the methods of this paper to obtain a better upper bound in the case \( n = 3 \).

**Theorem 3.1.** Let \( f \in S \) and be given by (1). Then

\[
|a_4| - |a_3| \leq 2.1033299 \ldots.
\]

**Proof.** By using (8) we have

\[
|a_4| - |a_3| \leq |a_4| - |\omega_{11}| |a_3| \leq |a_4 - \omega_{11}a_3| = 2\left|\frac{\omega_{33} + 3\omega_{11}\omega_{33} + 1}{6\omega_{11}}\right|.
\]

From (7) with \( x_1 = \frac{1}{\sqrt{6}}\omega_{11} \) and \( x_3 = 1 \), we obtain

\[
\left|\frac{\omega_{33} + 2\sqrt{6}\omega_{11}\omega_{13} + 1}{6\omega_{11}}\right| \leq \frac{1}{3} |\omega_{11}|^2 + \frac{1}{3}
\]

\[
\Rightarrow \left|B + \left(\frac{2\sqrt{6}}{3} - 3\right)\omega_{11}\omega_{13}\right| \leq \frac{1}{6} |\omega_{11}|^2 + \frac{1}{3}
\]

\[
\Rightarrow |B| \leq \left|3 - \frac{\sqrt{6}}{3}\right| |\omega_{11}| |\omega_{13}| + \frac{1}{6} |\omega_{11}|^2 + \frac{1}{3}
\]

\[
\Rightarrow |B| \leq \left|3 - \frac{\sqrt{6}}{3}\right| |\omega_{11}| \cdot \frac{1}{\sqrt{3}} \sqrt{1 - |\omega_{11}|^2} + \frac{1}{6} |\omega_{11}|^2 + \frac{1}{3}
\]

\[
\Rightarrow |B| \leq \frac{1}{3}\left[\left(3\sqrt{3} - \sqrt{2}\right)|\omega_{11}| \sqrt{1 - |\omega_{11}|^2} + \frac{1}{2} |\omega_{11}|^2 + 1\right] =: \varphi(|\omega_{11}|),
\]

where \( \varphi(t) = \frac{1}{3}\left[\left(3\sqrt{3} - \sqrt{2}\right)t \sqrt{1 - t^2} + \frac{1}{2} t^2 + 1\right] \) for \( 0 \leq t \leq 1 \), and where we have used that \( |\omega_{13}| \leq \frac{1}{\sqrt{3}} \sqrt{1 - |\omega_{11}|^2} \). Since the function \( \varphi \) attains its maximum at

\[
t_0 = \sqrt{\frac{1}{2} + \frac{1}{6} \sqrt{\frac{1}{379}(39 + 8\sqrt{6})}} = 0.75202 \ldots,
\]

and since \( \varphi(t_0) = \frac{1}{12}\left[5 + \sqrt{117 - 24\sqrt{6}}\right] \), it follows that

\[
|a_4| - |a_3| \leq 2\varphi(t_0) = 2.10495 \ldots.
\]
References