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A General Fixed Point Theorem

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Abstract. In this paper we prove a theorem which ensures the existence of a unique fixed point and is applicable to contractive type mappings as well as mappings which do not satisfy any contractive type condition. Our theorem contains the well known fixed point theorems respectively due to Banach, Kannan, Chatterjea, Ćirić and Suzuki as particular cases; and is independent of Caristi's fixed point theorem. Moreover, our theorem provides new solutions to Rhoades problem on discontinuity at the fixed point as it admits contractive mappings which are discontinuous at the fixed point. It is also shown that the weaker form of continuity employed by us is a necessary and sufficient condition for the existence of the fixed point.

1. Introduction

The Banach contraction theorem [1], one of the most applied fixed point theorems, states that if a self-mapping f of a complete metric space (X, d) satisfies the condition

(i) $d(fx, fy) \le k\, d(x, y),\ 0 \le k < 1,$

for each x, y in X then f has a unique fixed point and the sequence of iterates $\{f^n x\}$ converges to the fixed point for each x. A mapping satisfying condition (i) is a uniformly continuous mapping. Kannan [18, 19] proved that if a self-mapping f of a complete metric space (X, d) satisfies the condition

(ii) $d(fx, fy) \le k\, [d(x, fx) + d(y, fy)],\ 0 \le k < \frac{1}{2},$

for each x, y in X then f has a unique fixed point and the sequence of iterates $\{f^n x\}$ converges to the fixed point for each x. A mapping satisfying condition (ii) need not be continuous in the entire domain but is continuous at the fixed point. In 1972, Chatterjea [9] proved that if a self-mapping f of a complete metric space (X, d) satisfies the condition

(iii) $d(fx, fy) \le k\, [d(x, fy) + d(y, fx)], 0 \le k < \frac{1}{2},$

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for each x, y in X then f has a unique fixed point and the sequence of iterates $\{f^n x\}$ converges to the fixed point for each x. A mapping satisfying (iii) also admits discontinuity in its domain but is continuous at the fixed point.The fixed point theorems due to Banach [1], Kannan [18, 19] and Chattertjea [9] are independent results. In 1976 Caristi [7] proved the following theorem which has turned out to be an important theorem:

Theorem 1.1 ([7]). *Let (X, d) be a complete metric space and $f : X \to X$. If there exists a lower semicontinuous function $\varphi : X \to [0, \infty)$ such that*

(iv) $d(x, fx) \leq \varphi(x) - \varphi(fx),\ x \in X,$

then f has a fixed point.

The importance of the fixed point theorems by Kannan [18, 19], Caristi [7] and Chatterjea [9] also lies in the fact that each of these theorems characterizes completeness of the metric space. The Banach contraction theorem [1], in view of an example given by Connell [12], does not characterize metric completeness. By combining some ideas in the fixed point theorems due to Kannan [18, 19], Chatterjea [9] and Caristi [7] we prove a new fixed point theorem which ensures the existence of a unique fixed point. Our theorem is independent of the Caristi's theorem [7] and contains the fixed point theorems due to Banach [1], Chatterjea [9], Ćirić [11], Kannan [18, 19] and Suzuki [43] as particular cases.

The results of Kannan [18, 19] and Chatterjea [9] motivated a large number of contractive definitions and were followed by a multitude of papers on such contractive mappings; many of these mappings admit discontinuity in their domain. An excellent example of such contractive conditions is the condition introduced by Ćirić [11]:

(v) $d(fx, fy) \leq a \max\{d(x, y), d(x, fx), d(y, fy), \frac{[d(x,fy)+d(y,fx)]}{2}\},\ 0 \leq a < 1.$

A logical extension of discontinuity in the domain of contractive mappings was the question of continuity of such mappings at their fixed point. In 1988 Rhoades [41] examined continuity of a large number of contractive mappings at their fixed points and found that all the contractive definitions studied in [41] force the mapping to be continuous at the fixed point. The question whether there exists a contractive definition which admits discontinuity at the fixed point was listed by Rhoades in [[41], p. 242] as an open problem. In continuation of the work of Rhoades [41], many more contractive mappings were studied for continuity at their fixed points by Hicks and Rhoades [17] and were found continuous at the fixed point. Pant [27–30] obtained fixed point theorems for contractive mappings which are discontinuous at the fixed point and resolved the Rhoades problem on continuity at fixed point. Recently some more solutions to the problem of continuity at fixed point and applications of such results in the study of discontinuous activation functions of neural networks have been reported (e.g. Bisht and Pant [2, 3], Bisht et al [4], Bisht and Rakočević [5, 6], Celik and Özgür [8], Özgür and Tas [25, 26], Pant and Pant [31], Pant et al [32], Pant et al [33], Pant et al [34], Pant et al [35, 36], Pant et al [37], Rashid et al [39], Tas and Özgür [44], Tas et al [45], Zheng and Wang [48]). Fixed point theorems for discontinuous mappings have found wide applications, for example application of such theorems in the study neural networks with discontinuous activation functions is presently a very active area of research (e. g. Cromme and Diener [13], Cromme [14], Ding et al [15], Forti and Nistri [16], Nie and Zheng [22–24], Wu and Shan [47]). In the present paper we give a new type of solution to the Rhoades problem on continuity of contractive mappings at the fixed point.

We now give some relevant definitions.

Definition 1.2 ([10, 11]). *If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$.*

Continuity implies orbital continuity but not conversely [10, 11].

Definition 1.3 ([31]). *A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \ldots,$ if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \to t$.*

It was shown in [31] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$ and

$$\text{continuity } \Rightarrow 2- \text{ continuity } \Rightarrow 3- \text{ continuity } \Rightarrow \ldots, \text{ but not conversely.}$$

It is also easy to see that 1-continuity is equivalent to continuity.

Definition 1.4 ([32]). *A self-mapping f of a metric space (X, d) is called weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i} y = u \Rightarrow \lim_i f f^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.*

Example 1.5. *Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by*

$$fx = \frac{(1+x)}{2} \text{ if } x < 1, \qquad fx = 0 \text{ if } 1 \le x < 2, \qquad f2 = 2.$$

Then $f^n 0 \to 1$ and $f(f^n 0) \to 1 \neq f1$. Therefore f is not orbitally continuous. However, f is weakly orbitally continuous. If we take $x = 2$ then $f^n 2 \to 2$ and $f(f^n 2) \to 2 = f2$ and, hence, f is weakly orbitally continuous. This example shows that orbital continuity implies weak orbital continuity but not conversely. If a self-mapping of X has a fixed point then it is, obviously, weakly orbitally continuous.

Using the notion of weak orbital continuity Pant et al [32] obtained the following generalisation of Caristi's theorem:

Theorem 1.6 (Theorem 2.10 of [32]). *Let f be a self-mapping of a complete metric space (X, d). Suppose $\varphi : X \to [0, \infty)$ is a function such that for each x in X we have*

(vi) $d(x, fx) \le \varphi(x) - \varphi(fx)$.

If f is weakly orbitally continuous or f^k is continuous or f is k-ontinuous for some $k \ge 1$, then f has a fixed point.

Example 1.5 satisfies the conditions of Theorem 1.6 and has a fixed point $x = 2$. In Example 1.5, the function $\varphi : X \to [0, \infty)$ can be defined in various ways. For example, as in [32], we can define $\varphi(x) = 1 - x$ if $x < 1$ and $\varphi(x) = 1 + x$ if $x \ge 1$. However, the function φ in Example 1.5 cannot be lower semi-continuous as required in Caristi's theorem. If possible, suppose that for the function f in Example 1.5 there exists a lower semicontinuous function $\varphi : X \to [0, \infty)$ such that (vi) is satisfied. Let $x = 0$. Then $f0 = \frac{1}{2}, f^2 0 = \frac{3}{2^2}, \ldots, f^n 0 = \frac{(2^n - 1)}{2^n}$. Using (vi) we get $d(f^{n-1}0, f^n 0) \le \varphi(f^{n-1}0) - \varphi(f^n 0)$, that is,$\varphi(f^n 0) \le \varphi(f^{n-1}0) - \frac{1}{2^n}$. This gives

$$\begin{aligned}\varphi(f^n 0) \le \varphi(f^{n-1}0) - \frac{1}{2^n} \le \varphi(f^{n-2}0) - \frac{1}{2^{n-1}} - \frac{1}{2^n} \le \ldots \le \varphi(0) - \frac{1}{2} - \frac{1}{2^2} - \ldots - \frac{1}{2^n} \\ = \varphi(0) - \frac{(2^n - 1)}{2^n} = \varphi(0) - f^n 0.\end{aligned}$$

From this inequality, on making $n \to \infty$ we get $\liminf_{y \to 1} \varphi(y) \le \varphi(0) - 1$. Also, using (vi) we get $d(1, f1) \le \varphi(1) - \varphi(f1)$, that is, $\varphi(1) \ge \varphi(0) + d(1, f1) = \varphi(0) + 1$. This shows that φ is not lower semicontinuous and we get a contradiction. Thus, Example 1.5 does not satisfy the conditions of Caristi's theorem and Theorem 1.6 is a proper generalization of Caristi's theorem.

2. Main Results:

Theorem 2.1. *Let f be a self-mapping of a complete metric space (X, d). Suppose $\varphi : X \to [0, \infty)$ is such that for all x, y in X*

$$d(fx, fy) \le \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy). \tag{1}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a unique fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Then

$$\begin{aligned} d(x_1, x_2) = d(fx_0, fx_1) &\leq \varphi(x_0) - \varphi(fx_0) + \varphi(x_1) - \varphi(fx_1) \\ &= \varphi(x_0) - \varphi(x_1) + \varphi(x_1) - \varphi(x_2) = \varphi(x_0) - \varphi(x_2). \end{aligned}$$

Thus

$$d(x_1, x_2) \leq \varphi(x_0) - \varphi(x_2).$$

Similarly

$$\begin{aligned} d(x_2, x_3) &\leq \varphi(x_1) - \varphi(x_3) \\ d(x_3, x_4) &\leq \varphi(x_2) - \varphi(x_4) \\ &\ldots \\ d(x_{n-1}, x_n) &\leq \varphi(x_{n-2}) - \varphi(x_n) \\ d(x_n, x_{n+1}) &\leq \varphi(x_{n-1}) - \varphi(x_{n+1}). \end{aligned}$$

Adding these inequalities we get

$$d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_n, x_{n+1}) \leq \varphi(x_0) + \varphi(x_1) - \varphi(x_n) - \varphi(x_{n+1}) \leq \varphi(x_0) + \varphi(x_1).$$

Making $n \to \infty$ we obtain

$$\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \varphi(x_0) + \varphi(x_1).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$ and $\lim_{n\to\infty} f^k x_n = t$ for each $k \geq 1$.

Suppose f is weakly orbitally continuous. Since the sequence $\{f^n x_0\}$ is convergent for each x_0 in X, weak orbital continuity of f implies that there exists y_0 in X such that $f^n y_0 \to z$ and $ff^n y_0 \to fz$ for some z in X. This implies that $z = fz$ and, hence, z is a fixed point of f. If f is orbitally continuous or if f is k-continuous for some $k \geq 1$ then f is weakly orbitally continuous and the proof follows. If u and v are fixed points of f then using (1) we get

$$\begin{aligned} d(u, v) = d(fu, fv) &\leq \varphi(u) - \varphi(fu) + \varphi(v) - \varphi(fv) \\ &= \varphi(u) - \varphi(u) + \varphi(v) - \varphi(v) = 0. \end{aligned}$$

Therefore $u = v$ and f has a unique fixed point. This proves the theorem. □

Remark 2.2. *In the setting Theorem 2.1, weak orbital continuity is a sufficient condition for the existence of the fixed point. On the other hand, suppose that a self-mapping f of a complete metric space (X, d) which satisfies condition (1) of Theorem 2.1 possesses a fixed point, say z. Then $fz = z$ and $f^n z = z$ for each $n > 1$, that is, $\lim_{n\to\infty} f^n z = z$ and $\lim_{n\to\infty} f(f^n z) = z = fz$. This means that f is weakly orbitally continuous. Therefore, weak orbital continuity is a necessary and sufficient condition for the existence of the fixed point of a mapping satisfying condition (1).*

Example 2.3. *Let $X = (-\infty, \infty)$ and d be the Euclidean metric. Define $f : X \to X$ by*

$$fx = 1 \text{ if } x \leq 1, \qquad fx = 0 \text{ if } x > 1.$$

Also let $\varphi : X \to [0, \infty)$ be defined by

$$\varphi(x) = 1 - x \text{ if } x \leq 1, \qquad \varphi(x) = 1 + x \text{ if } x > 1.$$

Then f satisfies all the conditions of the above theorem and has a unique fixed point at which f is discontinuous. It satisfies $d(fx, fy) < \max\{d(x, fx), d(y, fy)\}$ also and, hence, provides a solution to the Rhoades problem.

Example 2.4. *Let $X = [0,1]$ equipped with the Euclidean metric. Let $f : X \to X$ and $\varphi : X \to [0,\infty)$ be respectively defined by*

$$\begin{aligned} fx &= \frac{x}{2} \text{ if } x < 1, \qquad f(1) = \frac{4}{5}; \\ \varphi(x) &= x \text{ if } x < 1, \qquad \varphi(1) = 2. \end{aligned}$$

Then f satisfies all the conditions of the above theorem and has a unique fixed point $x = 0$ at which f is continuous.

Example 2.5. *Let $X = [0,2] \cup \{3\}$ and d be the usual metric. Define $f : X \to X$ by*

$$fx = 0 \text{ if } x \neq 2, \qquad f2 = 3.$$

Also, let $\varphi : X \to [0,\infty)$ be defined by

$$\varphi(x) = x \text{ if } x \neq 2, \qquad \varphi(2) = 8.$$

Then f satisfies all the conditions of Theorem 2.1 and has a unique fixed point $x = 0$ at which f is continuous. It can be verified in this example that

$$\begin{aligned} d(fx, fy) &= 0, \quad \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy) = x + y > 0 \text{ when } x \neq 2, y \neq 2; \\ d(fx, fy) &= 3, \quad \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy) = x + 5 \geq 5 \text{ when } x \neq 2, y = 2. \end{aligned}$$

It can also be verified that the mapping f does not satisfy any contractive condition. For example, if we take $1 \leq x < 2$ and $y = 2$ then $d(fx, fy) = 3$ and $\max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} = 2$. Thus f does not satisfy any contractive condition but for $x \in [1,2)$, $y = 2$ it satisfies the Lipchitz type condition:

$$d(fx, fy) \leq \frac{3}{2} \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}.$$

We now show that the well known fixed point theorems due to Banach [1], Kannan [18], Chatterjea [9], Ćirić [11], and Suzuki [43] are particular cases of Theorem 2.1.

Theorem 2.6. *If a self-mapping f of a complete metric space (X,d) satisfies the Banach contraction condition*

$$d(fx, fy) \leq a\, d(x,y), \;\; x, y \in X, \; 0 \leq a < 1, \tag{2}$$

then f also satisfies conditions of Theorem 2.1 and has a unique fixed point.

Proof. For any x in X we have $d(fx, f^2x) \leq a\, d(x, fx)$, that is,

$$\frac{1}{a} d(fx, f^2x) \leq d(x, fx). \tag{3}$$

Using (2) we get

$$\begin{aligned} d(fx, fy) &\leq a\, d(x,y) \\ &\leq a\, [d(x,fx) + d(fx,fy) + d(fy,y)]. \end{aligned}$$

This implies

$$\begin{aligned} d(fx, fy) &\leq \frac{a}{(1-a)}[d(x,fx) + d(y,fy)] \\ &= \frac{a}{(1-a)^2}[d(x,fx) + d(y,fy)] - \frac{a^2}{(1-a)^2}[d(x,fx) + d(y,fy)]. \end{aligned}$$

Using (3) the above inequality yields

$$\begin{aligned} d(fx, fy) &\leq \frac{a}{(1-a)^2}[d(x, fx) + d(y, fy)] - \frac{a^2}{(1-a)^2}\Big(\frac{1}{a}\Big)[d(fx, f^2x) + d(fy, f^2y)] \\ &= \frac{a}{(1-a)^2}[d(x, fx) + d(y, fy) - d(fx, f^2x) - d(fy, f^2y)]. \end{aligned}$$

If we define a function $\varphi : X \to [0, \infty)$ by $\varphi(x) = \frac{a}{(1-a)^2} d(x, fx)$, then the last inequality gives

$$d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy)$$

for each x, y in X. Since $d(fx, f^2x) \leq a\, d(x, fx)$, it follows that $\varphi(fx) \leq \varphi(x)$. Thus, f satisfies the conditions of Theorem 2.1 and, hence, has a unique fixed point. This proves that the Banach contraction mapping theorem is a particular case of Theorem 2.1. □

Theorem 2.7. *If a self-mapping f of a complete metric space (X, d) satisfies the Kannan contraction condition*

$$d(fx, fy) \leq \frac{a}{2}[d(x, fx) + d(y, fy)], \; x, y \in X, \; 0 \leq a < 1, \tag{4}$$

then f satisfies the conditions of Theorem 2.1 and has a unique fixed point.

Proof. For any x in X we have $d(fx, f^2x) \leq \frac{a}{2}[d(x, fx) + d(fx, f^2x)]$. This implies $(2-a)d(fx, f^2x) \leq a\, d(x, fx)$, that is,

$$\Big(\frac{2-a}{a}\Big)d(fx, f^2x) \leq d(x, fx). \tag{5}$$

Now for any x, y in X we have

$$\begin{aligned} d(fx, fy) &\leq \frac{a}{2}[d(x, fx) + d(y, fy)] \\ &= \frac{a(2-a)}{4(1-a)}[d(x, fx) + d(y, fy)] - \frac{a^2}{4(1-a)}[d(x, fx) + d(y, fy)]. \end{aligned}$$

Using (5), the above inequality yields

$$\begin{aligned} d(fx, fy) \leq \frac{a(2-a)}{4(1-a)}[d(x, fx) + d(y, fy] - \frac{a(2-a)}{4(1-a)}[d(fx, f^2x) + d(fy, f^2y)] \\ = \frac{a(2-a)}{4(1-a)}[d(x, fx) + d(y, fy) - d(fx, f^2x) - d(fy, f^2y)]. \end{aligned}$$

Let us define a function $\varphi : X \to [0, \infty)$ by $\varphi(x) = \Big(\frac{a(2-a)}{4(1-a)}\Big)d(x, fx)$, then the last inequality gives

$$d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy).$$

Since $d(fx, f^2x) \leq d(x, fx)$, it follows that $\varphi(fx) \leq \varphi(x)$. Therefore, f satisfies the conditions of Theorem 2.1 and possesses a unique fixed point. This establishes the theorem. □

Theorem 2.8. *If a self-mapping f of a complete metric space (X, d) satisfies the Chatterjea's contraction condition*

$$d(fx, fy) \leq \frac{a}{2}[d(x, fy) + d(y, fx)], \; x, y \in X, 0 \leq a < 1, \tag{6}$$

then f satisfies the conditions of Theorem 2.1 and has a unique fixed point.

Proof. For any x in X we have

$$d(fx, f^2x) \le \frac{a}{2}[d(x, f^2x) + d(fx, fx)] \le \frac{a}{2}[d(x, fx) + d(fx, f^2x)].$$

This implies

$$\frac{(2-a)}{a} d(fx, f^2x) \le d(x, fx). \tag{7}$$

Now for any x, y in X we have

$$\begin{aligned} d(fx, fy) &\le \frac{a}{2}[d(x, fy) + d(y, fx)] \\ &\le \frac{a}{2}[d(x, fx) + d(fx, fy) + d(y, fy) + d(fy, fx)]. \end{aligned}$$

This implies

$$\begin{aligned} d(fx, fy) &\le \frac{a}{2(1-a)}[d(x, fx) + d(y, fy)] \\ &= \frac{a(2-a)}{4(1-a)^2}[d(x, fx) + d(y, fy)] - \frac{a^2}{4(1-a)^2}[d(x, fx) + d(y, fy)]. \end{aligned}$$

Using (7) the above inequality yields

$$\begin{aligned} d(fx, fy) &\le \frac{a(2-a)}{4(1-a)^2}[d(x, fx) + d(y, fy)] - \frac{a^2}{4(1-a)^2}\frac{(2-a)}{a}[d(fx, f^2x) + d(fy, f^2y)] \\ &= \frac{a(2-a)}{4(1-a)^2}[d(x, fx) + d(y, fy) - d(fx, f^2x) - d(fy, f^2y)]. \end{aligned} \tag{8}$$

Let us define $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = \frac{a(2-a)}{4(1-a)^2} d(x, fx).$$

Then inequality (8) yields

$$d(fx, fy) \le \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy).$$

Since $d(fx, f^2x) \le d(x, fx)$, it follows that $\varphi(fx) \le \varphi(x)$. Therefore, f satisfies the conditions of Theorem 2.1 and, hence, possesses a unique fixed point. This shows that Theorem 2.1 contains Chatterjea's theorem as a particular case. □

We now show that the Ćirić theorem [11] is a particular case of Theorem 2.1.

Theorem 2.9. *Suppose a self-mapping f of a complete metric space (X, d) satisfies the Ćirić contraction condition*

$$d(fx, fy) \le a \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\right\},\ 0 \le a < 1, \tag{9}$$

for all x, y in X. Then f satisfies the conditions of Theorem 2.1 and has a unique fixed point.

Proof. From condition (9) it follows that

$$\frac{1}{a} d(fx, f^2x) \le d(x, fx), \tag{10}$$

and

$$d(fx, fy) \le a\,[d(x, fx) + d(y, fy) + d(fx, fy)]. \tag{11}$$

Inequality (11) gives

$$\begin{aligned} d(fx,fy) &\le \frac{a}{(1-a)}[d(x,fx)+d(y,fy)] \\ &= \frac{a}{(1-a)^2}[d(x,fx)+d(y,fy)] - \frac{a^2}{(1-a)^2}[d(x,fx)+d(y,fy)]. \end{aligned}$$

By virtue of (10), the last inequality gives

$$\begin{aligned} d(fx,fy) &\le \frac{a}{(1-a)^2}[d(x,fx)+d(y,fy)] - \frac{a^2}{(1-a)^2}\frac{1}{a}[d(fx,f^2x)+d(fy,f^2y)] \\ &= \frac{a}{(1-a)^2}[d(x,fx)+d(y,fy)-d(fx,f^2x)-d(fy,f^2y)]. \end{aligned} \tag{12}$$

Let us define $\varphi : X \to [0,\infty)$ by $\varphi(x) = \frac{a}{(1-a)^2}d(x,fx)$. Then (12) yields

$$d(fx,fy) \le \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy).$$

Since $d(fx,f^2x) \le a\,d(x,fx) \le d(x,fx)$, it follows that $\varphi(fx) \le \varphi(x)$. Therefore, f satisfies the conditions of Theorem 2.1 and, hence, possesses a unique fixed point. This completes the proof of the theorem. □

In the next theorem we show that the well known theorem due to Suzuki [43] is a particular case of Theorem 2.1. As defined in [43] let $\theta : [0,1) \to (\frac{1}{2},1]$ be such that

$$\theta(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le \frac{(\sqrt{5}-1)}{2}, \\ \frac{1-r}{r^2} & \text{if} \quad \frac{(\sqrt{5}-1)}{2} < r < \frac{1}{\sqrt{2}} \\ \frac{1}{1-r} & \text{if} \quad \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Theorem 2.10. *If a self-mapping f of a complete metric space (X,d) satisfies the Suzuki condition*

$$\theta(r)d(x,fx) \le d(x,y) \Rightarrow d(fx,fy) \le r\,d(x,y), \;\; x,y \in X,\, 0 \le r < 1, \tag{13}$$

then f also satisfies conditions of Theorem 2.1 and possesses a unique fixed point.

Proof. Since $\theta(r) \le 1$, we have $\theta(r)d(x,fx) \le d(x,fx)$ for each x in X. By virtue of (13) this implies $d(fx,f^2x) \le r\,d(x,fx)$, that is, for each x in X we have

$$\frac{1}{r}d(fx,f^2x) \le d(x,fx). \tag{14}$$

Now, using (13) we get

$$\begin{aligned} \theta(r)d(x,fx) \le d(x,y) \Rightarrow d(fx,fy) &\le r\,d(x,y) \\ &\le r\,[d(x,fx)+d(fx,fy)+d(fy,y)]. \end{aligned}$$

This implies

$$\begin{aligned} d(fx,fy) &\le \frac{r}{(1-r)}[d(x,fx)+d(y,fy)] \\ &= \frac{r}{(1-r)^2}[d(x,fx)+d(y,fy)] - \frac{r^2}{(1-r)^2})[d(x,fx)+d(y,fy)]. \end{aligned}$$

Using (14) we get

$$\begin{aligned} \theta(r)d(x,fx) &\le d(x,y) \Rightarrow d(fx,fy) \le \frac{r}{(1-r)^2})[d(x,fx)+d(y,fy)] - \frac{r^2}{(1-r)^2}\frac{1}{r}[d(fx,f^2x)+d(fy,f^2y)] \\ &= \frac{r}{(1-r)^2}[d(x,fx)+d(y,fy)-d(fx,f^2x)-d(fy,f^2y)]. \end{aligned}$$

Thus

$$\theta(r)d(x, fx) \leq d(x, y) \Rightarrow d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy),$$

where $\varphi : X \to [0, \infty)$ is defined by $\varphi(x) = \frac{r}{(1-r)^2}d(x, fx)$ for each x in X. Therefore f satisfies all the conditions of Theorem 2.1 and, hence, has a unique fixed point. Hence Theorem 2.1 contains Suzuki's theorem [43] as a particular case. □

Remark 2.11. *The proofs of Theorems 2.6 to 2.10 above depend on the fact that both sides of the respective contractive conditions contain distance terms and, therefore, we can invoke triangle inequality. Use of triangle inequality is the vital step in establishing the desired implications in Theorems 2.6 to 2.10. On the other hand, we cannot establish such implications between Theorem 2.1 and Theorem 1.6 (Theorem 2.10 of [32]) or between Theorem 2.1 and Caristi's theorem [7] because the right hand side of the inequality $d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy)$ in Theorem 2.1 or the inequality $d(x, f(x)) \leq \varphi(x) - \varphi(fx)$ in Theorem 2.10 [32] and Caristi's theorem [7] does not contain distance terms and, therefore, triangle inequality cannot be invoked. Thus, Theorem 2.1 is independent of Theorem 2.10 [32] and the Caristi's fixed point theorem [7]. Moreover, as seen in Example 2.14 below, the function $\varphi : X \to [0, \infty)$ in Theorem 2.1 need not be lower semicontinuous.*

We now prove that Theorem 2.1 characterises completeness. Several researchers have studied fixed point theorems that characterize metric completeness (e.g. Kirk [20], Liu [21], Park [38], Reich [40], Subrahmanayam [42], Suzuki [43], Weston [46]). Kirk [20] proved that Caristi's fixed theorem [7] characterizes metric completeness. Subrahmanayam [42] proved that fixed point theorems of Kannan[18, 19] and Chatterjea [9] characterise metric completeness. Suzuki [43] obtained a generalization of the Banach contraction theorem that characterises metric completeness. In view of an example given by Connell [12], the Banach contraction mapping theorem [1] does not characterise metric completeness. Park [38] gave some necessary and sufficient conditions for a metric space to be complete by combining some known characterizations of metric completeness.

Theorem 2.12. *Suppose (X, d) is a metric space and $\varphi : X \to [0, \infty)$. If every weakly orbitally continuous or k-continuous self-mapping of X satisfying the condition*

$$d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy), \ x, y \in X, \tag{15}$$

has a fixed point, then X is complete.

Proof. Suppose that every weakly orbitally continuous or k-continuous self-mapping of X satisfying condition (15) possesses a fixed point. We assert that X is complete. If possible, suppose X is not complete. Then there exists a Cauchy sequence in X, say $S = \{u_1, u_2, u_3, \ldots\}$, consisting of distinct points which does not converge. Let $x \in X$ be given. Then, since x is not a limit point of the sequence $S, d(x, S - \{x\}) > 0$ and there exists a least positive integer $N(x)$ such that $x \neq u_{N(x)}$ and for each $m \geq N(x)$ we have

$$d(u_{N(x)}, u_m) \leq \frac{1}{2}d(x, u_{N(x)}). \tag{16}$$

Thus, we can define a mapping $f : X \to X$ such that $f(x) = u_{N(x)}$. Clearly, f is a fixed point free self-mapping of X. Let us define a function $\varphi : X \to [0, \infty)$ such that

$$\varphi(x) = d(x, fx) = d(x, u_{N(x)}). \tag{17}$$

Then, for any x, y in X we get

$$d(fx, fy) = d(u_{N(x)}, u_{N(y)}) \leq \frac{1}{2}d(x, u_{N(x)}) = \frac{1}{2}d(x, fx) \text{ if } N(x) \leq N(y) \tag{18}$$

or

$$d(fx, fy) = d(u_{N(x)}, u_{N(y)}) \leq \frac{1}{2}d(y, u_{N(y)}) = \frac{1}{2}d(y, fy) \text{ if } N(x) > N(y). \tag{19}$$

$$\text{Now } \varphi(x)-\varphi(fx)+\varphi(y)-\varphi(fy) = d(x,fx)-d(fx,f^2x)+d(y,fy)-d(fy,f^2y)$$
$$= d(x,u_{N(x)})-d(u_{N(x)},u_{N(fx)})+d(y,u_{N(y)})-d(u_{N(y)},u_{N(fy)}).$$

Using (16), (18) and (19), the above equation gives

$$\varphi(x)-\varphi(fx)+\varphi(y)-\varphi(fy) \geq \frac{1}{2}d(x,u_{N(x)})+\frac{1}{2}d(y,u_{N(y)}) \geq d(fx,fy). \tag{20}$$

The mapping f is obviously weakly orbitally continuous as well as k-continuous. Thus we have a weakly orbitally continuous self-mapping f of X which satisfies conditions of Theorem 2.1 but has no fixed point. This contradicts our hypothesis. Therefore, X is complete. □

In the next theorem we generalize Theorem 2.1.

Theorem 2.13 Let f be a self-mapping of a complete metric space (X,d). Suppose $\phi : X \to [0,\infty)$ is such that

$$d(fx,fy) \leq \max\{|\phi(x)-\phi(y)|, \phi(x)-\phi(fx)+\phi(y)-\phi(fy)\}, \quad x,y \in X \tag{21}$$

$$\phi(fx) \leq \phi(x), \quad x \in X. \tag{22}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a fixed point.

Proof. We have from (22), $\varphi(fx) \leq \varphi(x)$ for each x in X. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Then using (21) we get,

$$d(x_1,x_2) = d(fx_0,fx_1) \leq \max\{|\varphi(x_0)-\varphi(x_1)|, \varphi(x_0)-\varphi(fx_0)+\varphi(x_1)-\varphi(fx_1)\}$$
$$= \varphi(x_0)-\varphi(x_1)+\varphi(x_1)-\varphi(x_2) = \varphi(x_0)-\varphi(x_2).$$

Thus $d(x_1,x_2) \leq \varphi(x_0)-\varphi(x_2)$. The remaining part of the proof follows on the lines of the proof of Theorem 2.1. □

Example 2.14 Let $X = [0,2]$ equipped with the Euclidean metric. As in Example 1.5, define $f : X \to X$ by

$$fx = \frac{(1+x)}{2} \text{ if } x < 1, \qquad fx = 0 \text{ if } 1 \leq x < 2, \qquad f2 = 2.$$

Let us define $\varphi : X \to [0,\infty)$ by

$$\varphi(x) = 1-x \text{ if } x < 1, \qquad \varphi(x) = 2+x \text{ if } x \geq 1.$$

Then f satisfies all the conditions of Theorem 2.13 and has a fixed point $x = 2$. However, f does not satisfy the condition $d(fx,fy) \leq \varphi(x)-\varphi(fx)+\varphi(y)-\varphi(fy)$ for each x,y in X. For example, if we take $x < 1$ and $y = 2$ then

$$d(fx,fy) = \frac{(3-x)}{2}, \quad \varphi(x)-\varphi(fx)+\varphi(y)-\varphi(fy) = \frac{(1-x)}{2}, \quad |\varphi(x)-\varphi(y)| = 3+x.$$

Similarly, f does not satisfy $d(fx,fy) \leq |\varphi(x)-\varphi(y)|$ for each x,y in X. If we take $1 \leq x < 2$ and $y = 2$ then

$$d(fx,fy) = 2, \quad \varphi(x)-\varphi(fx)+\varphi(y)-\varphi(fy) = 1+x \geq 2, \quad |\varphi(x)-\varphi(y)| = 2-x \leq 1.$$

It can be seen in this example that φ is not upper semicontinuous. To see this let us consider $x_0 = 1$. Then using (21) and the fact that $\varphi(x) \geq \varphi(fx)$ for each x, we get

$$d(f1,f^{n+1}1) \leq \max\{\varphi(1)-\varphi(f^n1), \varphi(1)-\varphi(f1)+\varphi(f^n1)-\varphi(f^{n+1}1)\}$$
$$\leq \max\{\varphi(1)-\varphi(f^n1), \varphi(1)-\varphi(f^{n+1}1)\}$$
$$= \varphi(1)-\varphi(f^{n+1}1) = \varphi(1)-\varphi\Big(\frac{2^n-1}{2^n}\Big).$$

This yields

$$\varphi\Big(\frac{2^n-1}{2^n}\Big) \le \varphi(1) - d(f1, f^{n+1}1) = \varphi(1) - \frac{(2^n-1)}{2^n}.$$

Making $n \to \infty$, the above inequality gives $\liminf_{x\to 1}\varphi(x) \le \varphi(1) - 1 < \varphi(1)$. Hence φ is not lower semicontinuous. This shows that Theorem 2.13 is independent of Caristi's Theorem [7]. Moreover, as noted in Example 1.5, the mapping f is weak orbitally continuous but not orbitally continuous.

Theorem 2.15 Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is such that for all $x, y \in X$

$$d(fx, fy) \le \max\{\phi(x) - \phi(fx), \phi(y) - \phi(fy)\}.$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a unique fixed point.

Theorem 2.16 Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is such that

$$d(fx, fy) \le \max\{|\phi(x) - \phi(y)|, \phi(x) - \phi(fx), \phi(y) - \phi(fy)\}, \quad x, y \in X \tag{23}$$

$$\phi(fx) \le \phi(x), \quad x \in X. \tag{24}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a fixed point.

Example 2.5 illustrates Theorem 2.15 and Example 2.14 illustrates Theorem 2.16.

As a particular case of Theorem 2.13 and Theorem 2.16 we obtain the following:

Corollary 2.17. Let f be a self-mapping of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is such that

$$d(fx, fy) \le |\phi(x) - \phi(y)|, \quad x, y \in X \tag{25}$$

$$\phi(fx) \le \phi(x), \quad x \in X. \tag{26}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a fixed point. In Corollary 2.17 if we define $\phi : X \to [0, \infty)$ by $\phi(x) = |ax|,\ a > 0$, then we get the following:

Corollary 2.18. Let f be a self-mapping of a complete subspace of real line R such that for all $x, y \in X$

$$d(fx, fy) \le |\ |ax| - |ay|\ |, \quad a > 0.$$

If $|a\phi(fx)| \le |a\phi(x)|$ for each $x \in X$ then f has a fixed point.

Example 2.19. Let $X = [0, 1]$ equipped with Euclidean metric. Define $f : X \to X$ by $fx = x^2$ for each $x \in X$. Then f satisfies the conditions of Corollary 2.18 with $\phi(x) = |2x|$ and has two fixed points at 0 and 1. In fact, f satisfies the Lipschitz condition $d(fx, fy) \le 2|x - y|$.

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