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A Harmonic Mean Inequality for the q-Gamma and q-Digamma Functions

Mohamed Bouali[a]

[a] *Institut Préparatoire aux études d'ingenieurs de Tunis.*
Faculté des Sciences de Tunis, Campus Universitaire El-Manar, 2092 El Manar Tunis, Tunisia

Abstract. We prove among others results that the harmonic mean of $\Gamma_q(x)$ and $\Gamma_q(1/x)$ is greater than or equal to 1 for arbitrary $x > 0$, and $q \in J$ where J is a subset of $[0, +\infty)$. Also, we prove that there is a unique real number $p_0 \in (1, 9/2)$, such that for $q \in (0, p_0)$, $\psi_q(1)$ is the minimum of the harmonic mean of $\psi_q(x)$ and $\psi_q(1/x)$ for $x > 0$ and for $q \in (p_0, +\infty)$, $\psi_q(1)$ is the maximum. Our results generalize some known inequalities due to Alzer and Gautschi.

1. Introduction

There exists an extensive literature on inequalities for special functions. In particular, many authors published numerous interesting inequalities for Euler's gamma and psi functions. We refer the readers to [1–3, 8–10, 13, 16] and references therein.

In the last few decades, the gamma function was generalized to the q-gamma function introduced by Jackson [17]. The q-analogue of the Γ function, denoted by $\Gamma_q(x)$, is defined for $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0 < q < 1, \tag{1}$$

$$\Gamma_q(x) = (q-1)^{1-x} q^{x(x-1)/2} \prod_{n=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q > 1. \tag{2}$$

It was proved in [21] that $\Gamma(x) = \lim_{q\to 1-} \Gamma_q(x) = \lim_{q \to 1+} \Gamma_q(x)$.

Similarly the q-digamma or q-psifunction $\psi_q(x)$ is defined by $\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$. The derivatives ψ_q', ψ_q'',...are called the q-polygamma functions. In [18], it was shown that $\lim_{q\to 1-} \psi_q(x) = \lim_{q\to 1+} \psi_q(x) = \psi(x)$. From the definitions (1) and (2) one can easily deduce that

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Email address: bouali25@laposte.net (Mohamed Bouali)

$$\psi_q(x)=\begin{cases}-\log(1-q)+(\log q)\displaystyle\sum_{n=1}^{\infty}\frac{q^{nx}}{1-q^n}, & 0<q<1,\\ -\log(q-1)+(\log q)\Big[x-\dfrac{1}{2}-\displaystyle\sum_{n=1}^{\infty}\frac{q^{-nx}}{1-q^{-n}}\Big], & q>1.\end{cases}\tag{3}$$

Differentiation of (3) gives

$$\psi_q'(x)=\begin{cases}(\log q)^2\displaystyle\sum_{n=1}^{\infty}\frac{nq^{nx}}{1-q^n}, & 0<q<1,\\ \log q+(\log q)^2\displaystyle\sum_{n=1}^{\infty}\frac{nq^{-nx}}{1-q^{-n}}, & q>1.\end{cases}\tag{4}$$

One deduces that ψ_q is strictly increasing on $(0,+\infty)$.

A short computation gives for $x>0$ and $q>0$

$$\Gamma_q(x)=q^{(x-1)(x-2)/2}\Gamma_{\frac{1}{q}}(x).\tag{5}$$

If we take logarithm of both sides of (5) and then differentiate, we find

$$\psi_q(x)=(x-\frac{3}{2})\log q+\psi_{\frac{1}{q}}(x).\tag{6}$$

We refer the redears to [5, 6, 9–11, 15, 16, 21, 22] for basic properties of the q-gamma and q-digamma functions and many monotonicity as well as some complete monotonicity properties and inequalities for the gamma, digamma, and its q-analogue, q-gamma and q-digamma functions.

Gautschi proved in [13] that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ $(x>0)$ is greater than or equal to 1, i.e.

$$\frac{2}{1/\Gamma(x)+1/\Gamma(1/x)}\geq 1,\quad x>0.$$

Our first result is a generalization of Gautschi's inequality for the q-gamma function. We show there is a subset $J\in(0,+\infty)$ such that

$$\frac{2}{1/\Gamma_q(x)+1/\Gamma_q(1/x)}\geq 1,\quad x>0,\ \ q\in J,$$

and for $q\notin J$, there is $y_q>1$ with

$$\frac{2}{1/\Gamma_q(x)+1/\Gamma_q(1/x)}\geq\frac{2}{1/\Gamma_q(y_q)+1/\Gamma_q(1/y_q)},\quad x>0.$$

Besides, in [3], Alzer proved an interesting harmonic mean inequality for the psi-function, i.e.

$$\frac{2}{1/\psi(x)+1/\psi(1/x)}\geq-\gamma,\quad x>0,$$

where $\gamma=0,57721....$ is the Euler's constant. Such inequality is generalized to higher order derivative of the polygamma function in [4]. Our second main result is an extension of Alzer's inequality for the q-psifunction. We prove, there exists $p_0\in(1,9/2)$, such that, for all $q\in(0,p_0)$ and all $x>0,\ x\neq 1$

$$\frac{2}{1/\psi_q(x)+1/\psi_q(1/x)}>\psi_q(1),$$

and for $q \in [p_0, +\infty)$, there is a unique $z_q > 1$ such that for all $x \in [1/z_q, z_q]$, $x \neq 1$

$$\frac{2}{1/\psi_q(x) + 1/\psi_q(1/x)} > \psi_q(1).$$

If $x \in (0, 1/z_q) \cup (z_q, +\infty)$, then the reversed inequality holds.

Among other monotonicity properties, we discuss some monotonicity properties of q-digamma and q-trigamma functions and others related functions with respect to the variable q.

2. Some monotonicity results

In [19], it was proved that the function $q \mapsto \psi_q(1)$ decreases on $(0, 1)$, and $q \mapsto \psi_q(2)$ increases on $(0, 1)$. Then $\psi_q(1) \leq \psi_0(1) = 0$ and $0 = \psi_0(2) \leq \psi_q(2)$. From this result and the inequality $\log(\frac{q^{x+\frac{1}{2}}-1}{q-1}) < \psi_q(x+1)$ for all $x > 0$ and $q > 0$ (see for instance [11] Corollary 2.3), one deduces the proposition below.

Proposition 2.1. *The function ψ_q $(0 < q)$ has a uniquely determined positive zero on $(1, \frac{3}{2})$, which we denote by x_q.*

In [6], Alzer proved for $x > 0$ the interesting inequality

$$\psi''(x) + (\psi'(x))^2 > 0. \tag{7}$$

The author rediscovered it in [8] and used it to prove interesting inequalities for the digamma function, see [1, 5, 6]. Alzer and Grinshpan in [1] obtained a q-analogue of (7) and proved that, for $q > 1$ and all $x > 0$,

$$\psi_q''(x) + (\psi_q'(x))^2 > 0. \tag{8}$$

The author in [12] provided another q-extension of (8) and proved that

$$\psi_q''(x) + (\psi_q'(x))^2 - \log q(\psi_q'(x)) > 0, \tag{9}$$

for all $q > 0$ and all $x > 0$. The following lemma is due to Alzer [1].

Lemma 2.2. *For every $q > 0$ and $x \geq 1$,*

$$x\psi_q'(x) + 2\psi_q(x) \geq 0.$$

We start with the following lemma which proves the convergence as $q \to 1$ of the *mth* derivatives $\psi_q^{(m)}(x)$ to $\psi^{(m)}(x)$ for all $x > 0$.

Lemma 2.3. *The function $\psi_q^{(m)}(x)$ converges uniformly to $\psi^{(m)}(x)$ as $q \to 1$ on every compact of $(0, +\infty)$ for all $m \geq 0$, where $\psi_q^{(m)}$ respectively $\psi^{(m)}$ is the mth derivatives of the q-digamma function respectively of the digamma function.*

The case $m = 0$ is proved in [18]. The proof of the general case is given below in the appendix.

Lemma 2.4. *For every $q > 0$, the functions $f(x) = x\psi_q(x)$ and $g(x) = x\Gamma_q'(x)$ increase on $[1, +\infty)$.*

The proof is a direct consequence of Lemma 2.2.

Lemma 2.5. *For all $q > 0$, and all $k \geq 1$, the function $\frac{\psi_q^{(k+1)}(x)}{\psi_q^{(k)}(x)}$ increases on $(0, +\infty)$.*

Proof. To prove the lemma, it suffices to show that the function $S_{q,k}(x) := \psi_q^{(k+2)}(x)\psi_q^{(k)}(x) - (\psi_q^{(k+1)}(x))^2$ is non negative on $(0, +\infty)$ for all $q > 0$. By the series expansion of the q-digamma function, we have for $q \neq 1$, and $k \geq 2$

$$S_{q,k}(x) = (\log q)^{2k+4}\Big(\sum_{n<m}^{\infty} n^k m^k (n-m)^2 \frac{q^{(n+m)x}}{(1-q^n)(1-q^m)}\Big) > 0.$$

For $k = 1$, and $q \in (0,1)$, the proof reminds the same as in the above case.

For $k = 1$ and $q > 1$, we have

$$S_{q,1}(x) = \psi_q'''(x)\psi_q'(x) - (\psi_q''(x))^2,$$

hence,

$$S_{q,1}(x) = \psi_{\frac{1}{q}}'''(x)(\psi_{\frac{1}{q}}'(x) + \log q) - (\psi_{\frac{1}{q}}''(x))^2 = S_{\frac{1}{q},1}(x) + \psi_{\frac{1}{q}}'''(x)\log q \geq 0$$

For $q = 1$, one has by Lemma 2.3 for all $k \in \mathbb{N}$, $\lim_{q\to 1}\psi_q^{(k)}(x) = \psi^{(k)}(x)$, and the result follows from the previous case. □

In the sequel we give some monotonic results involving the q-polygamma functions with respect to the variable q and for fixed x.

Lemma 2.6.

(1) *For $x > 0$, the function $q \mapsto \psi_q''(x)$ decreases on $(0,1)$ and increases on $[1,+\infty)$.*
(2) *For $x > 0$, the function $q \mapsto \psi_q'(x)$ increases on $(0,+\infty)$.*
(3) *The function $q \mapsto \psi_q(x)$ decreases on $(0,\infty)$ for all $x \in (0,1]$ and increases on $(0,+\infty)$ for all $x \geq 2$.*

Proof. 1) We fix $q \in (0,1)$, and for $x > 0$ we have,

$$\psi_q''(x) = \sum_{n=0}^{\infty} \frac{q^{n+x}(1+q^{x+n})(\log q)^3}{(1-q^{n+x})^3}.$$

For $q \in (0,1)$ and $a > 0$, we set

$$h_a(q) = \frac{q^a(1+q^a)(\log q)^3}{(1-q^a)^3},$$

A first differentiation with respect to q gives

$$\begin{aligned} h_a'(q) &= \frac{q^{-1+a}(\log q)^2 a(1+q^a(4+q^a))}{(-1+q^a)^4}\left(\frac{3-3q^{2a}}{a(1+q^a(4+q^a))} + \log q\right) \\ &= \frac{q^{-1+a}(\log q)^2 a(1+q^a(4+q^a))}{(-1+q^a)^4} m_a(q). \end{aligned}$$

Furthermore,

$$m_a'(q) = \frac{(-1+q^a)^4}{q(1+q^a(4+q^a))^2},$$

and $m_a(1) = 0$ for all $a > 0$. Hence, $h_a'(q) \geq 0$ for $q \geq 1$ and $h_a'(q) \leq 0$ for $q \in (0,1]$ and for all $a > 0$. Thus, the function $q \mapsto h_a(q)$ decreases on $(0,1]$ and increases on $[1,+\infty)$ for all $a > 0$. Which gives the desired result.

2) For the proof of this item see for instance [7], Theorem 4.1.

3) By the fundamental theorem of calculus, $\psi_q(x) = \psi_q(1) - \int_x^1 \psi_q'(t)dt$. Since, $q \mapsto \psi_q(1)$ decreases, one deduces that the function $x \mapsto \psi_q(x)$ decreases on $(0,+\infty)$ for all $x \in (0,1]$.

For $x \geq 2$, $\psi_q(x) = \psi_q(2) + \int_2^x \psi_q'(t)dt$, and the result follows by using item 1) and the fact that $q \mapsto \psi_q(2)$ increases on $(0,+\infty)$. □

Corollary 2.7. *The function $x\psi_q'(x)$ is strictly decreasing on $(0,+\infty)$ for all $q \in (0,1)$, and the function $x^2\psi_q'(x)$ is strictly increasing on $(0,+\infty)$ for all $q > 1$.*

Proof. 1) Let $u(x) = x\psi_q'(x)$. By the integral representation of the q-digamma function we have,

$$u'(x) = \psi_q'(x) + x\psi_q''(x) = \int_0^\infty (1-tx)e^{-xt}\frac{t}{1-e^{-t}}d\gamma_q(t).$$

The function $t \mapsto t/(1-e^{-t})$ increases on $(0,+\infty)$. By splitting the integral along the intervals $(0,1/x)$ and $(1/x,+\infty)$, it follows that

$$u'(x) \leq \frac{1}{x(1-e^{-\frac{1}{x}})}\int_0^\infty (1-tx)e^{-xt}d\gamma_q(t).$$

Since, $-\log q\dfrac{q^x}{1-q^x} = \displaystyle\int_0^\infty e^{-xt}d\gamma_q(t)$, and $-(\log q)^2\dfrac{xq^x}{(1-q^x)^2} = \displaystyle\int_0^\infty -xte^{-xt}d\gamma_q(t)$,

where $\gamma_q(t) = \begin{cases} -\log q\displaystyle\sum_{k=1}^\infty \delta(t+k\log q), & 0<q<1, \\ t, & q=1. \end{cases}$

A straightforward computation, we get

$$\int_0^\infty (1-tx)e^{-xt}d\gamma_q(t) = -\log q\frac{\frac{1}{x}}{1-e^{-\frac{1}{x}}}\frac{q^x}{1-q^x}\frac{1+x\log q-q^x}{1-q^x} < 0,$$

for all $x \in (0,+\infty)$ and $q \in (0,1)$, and then u is strictly decreasing on $(0,+\infty)$.

2) Assume $q>1$, $x>0$, and let $\varphi_q(x) = x^2\psi_q'(x)$, differentiation of φ_q yields

$$\varphi_q'(x) = x(2\psi_q'(x) + x\psi_q''(x)).$$

Applying Lemma 2.3 and Lemma 2.6, we get

$$\varphi_q'(x) \geq x(2\psi'(x) + x\psi''(x)) = (x^2\psi'(x))' > 0.$$

The last inequality follows from the relations $x^2\psi'(x) = \displaystyle\sum_{m=0}^\infty \Big(\frac{x}{m+x}\Big)^2$. □

Corollary 2.8.
For $x \geq 2$, the function $q \mapsto x\psi_q'(x) + 2\psi_q(x)$ increases on $(0,\infty)$.
In particular, for any $q \geq 1$ and $x \geq 2$ we have

$$x\psi_q'(x) + 2\psi_q(x) \geq x\psi'(x) + 2\psi(x).$$

Proof. Let $u_q(x) = x\psi_q'(x) + 2\psi_q(x)$. Differentiate $u_q(x)$ with respect to x yields $u_q'(x) = 3\psi_q'(x) + x\psi_q''(x)$. Using Lemma 2.6, we then get, $u_p'(x) \leq u_q'(x)$ whenever $1 \leq p \leq q$. Integrate on $[2,x]$ gives

$$u_p(x) - u_q(x) \leq u_p(2) - u_q(2).$$

Which is non-positive. Then, for every $x \geq 2$, $u_p(x) \leq u_q(x)$. □

Proposition 2.9. *For $q>0$ and $x>0, x \neq x_q$, define $G_q(x) = \frac{\psi_q'(x)}{\psi_q(x)}$ and $\varphi_q(x) = x\frac{\psi_q'(x)}{\psi_q(x)}$.*

(1) *For all $q>0$, the function $G_q(x)$ decreases on $(1,x_q)$ and on $(x_q,+\infty)$.*
(2) (*a*) *The function $\varphi_q(x)$ decreases on $(1,x_q)$ for all $q>0$.*
  (*b*) *The function $\varphi_q(x)$ decreases on $(x_q,+\infty)$ if and only if $q \in (0,1] \cup [q_0,+\infty)$.*

Where q_0 is the unique positive solution of $(\sqrt{q})^3 - \sqrt{q} - 1 = 0$, $q_0 = \frac{1}{3\sqrt[3]{2}}(2\sqrt[3]{2} + \sqrt[3]{25-3\sqrt{69}} + \sqrt[3]{25+3\sqrt{69}}) \simeq$ 1.75488

For the proof, see the appendix below.

Corollary 2.10. *For every $q > 0$, the function $|\psi_q(x)|$ is logarithmic concave on $(1, +\infty)$.*

Proof. For $x > 0$ and $x \neq x_q$, let $h_q(x) = \log(|\psi_q(x)|)$. Then, $h'_q(x) = G_q(x)$, and $h''_q(x) = G'_q(x) \leq 0$, and the result follows by Proposition 2.9 □

In the proposition below we provided an extension of the result of Lemma 9.

Proposition 2.11. *For $x \geq 1$, $q > 0$ and $a \in \mathbb{R}$, let $h(x, a, q) = x\psi'_q(x) + a\psi_q(x)$. Then, for a given $q > 0$, $h(x, a, q) \geq 0$ for all $x \geq 1$ if and only if $0 \leq a \leq -\frac{\psi'_q(1)}{\psi_q(1)}$.*

Observe that $-\frac{\psi'_q(1)}{\psi_q(1)} \geq 2$ for all $q > 0$, and then Proposition 2.11 gives a refinement of the result of Alzer Lemma 2.2.

Remark 2.12. *One shows that $h(x, a, q) \geq 0$ for all $x \geq 1$ and all $q > 0$ if and only if $0 \leq a \leq 2$.*

Proof. By Proposition 2.9, the function $\psi'_q(x)/\psi_q(x)$ decreases on $[1, +\infty)$ for all $q > 0$. Then, for $x \in [1, x_q)$, $\dfrac{\psi'_q(x)}{\psi_q(x)} - \dfrac{\psi'_q(1)}{\psi_q(1)} \leq 0$. Therefore, for $x \in [1, x_q)$,

$$\psi'_q(x) + a\psi_q(x) = \psi_q(x)(\frac{\psi'_q(x)}{\psi_q(x)} + a) \geq \psi_q(x)(\frac{\psi'_q(1)}{\psi_q(1)} + a) \geq 0.$$

Moreover, for $x \geq x_q$, $\psi'_q(x) \geq 0$ and $a\psi_q(x) \geq 0$ and the result follows.

The converse. If for all $x \geq 1$, $\psi'_q(x) + a\psi_q(x) \geq 0$, then for $x = 1$, we get $a \leq -\frac{\psi'_q(1)}{\psi_q(1)}$. Moreover, as $x \to +\infty$, we get for $q \in (0, 1)$, $-a\log(1 - q) \geq 0$ and then $a \geq 0$. On the other hand, for $q > 1$, we have, $\lim_{x\to\infty} \psi'_q(x)/\psi_q(x) = 0$, then $a \geq 0$.

Remark that $\psi'_q(1) + 2\psi_q(1) \geq 0$, then $-\psi'_q(1)/\psi_q(1) \geq 2$, and $\psi'_q(x) + 2\psi_q(x) \geq 0$

Moreover, $\lim_{q\to\infty} -\psi'_q(1)/\psi_q(1) = 2$, this proves the result of the remark. □

3. Harmonic mean of the q-gamma function

Our first main result is a generalization of Gautschi inequality [13]. We start by proving a useful lemma. Let's

$$J = \left\{q > 0; \psi_q(1) - (\psi_q(1))^2 + \psi'_q(1) \geq 0\right\}.$$

The set J contains the interval $[0, 4]$. Indeed, by Lemma 2.2, we have $\psi'_q(1) \geq -2\psi_q(1)$, then $\psi_q(1) - (\psi_q(1))^2 + \psi'_q(1) \geq -\psi_q(1)(1 + \psi_q(1))$. Furthermore, one shows by induction that, $4^n - 1 \geq (9/10)4^n$ for all $n \geq 2$,. Then,

$$\psi_4(1) \geq \log\frac{2}{3} - \frac{2}{3}\log 2 - \frac{20}{9}\log 2\sum_{n=2}^{\infty}\frac{1}{4^n},$$

and $1 + \psi_4(1) \geq 1 + \log\frac{2}{3} - \frac{2}{3}\log 2 - \frac{5}{27}\log 2 \simeq 0.00407 > 0$. Since, $\psi_q(1) < 0$ and the function $q \mapsto 1 + \psi_q(1)$ decreases on $(0, +\infty)$, then $[0, 4] \subset J$.

Numerical computation shows that $\psi_{10}(1) - (\psi_{10}(1))^2 + \psi'_{10}(1) \simeq -0.072$, then $J \subsetneq [0, 10)$.

Lemma 3.1. *For $q > 0$, and $x \geq 1$, let $\theta_1(x) = \frac{x\psi_q(x)}{\Gamma_q(x)}$. Then*

1) *for $q \in J$, $\theta_1(x)$ increases on $[1, x_q]$.*
2) *for $q \notin J$ there is a unique $y_q \in (1, x_q)$ such that $\theta_1(x)$ decreases on $(1, y_q)$, and increases on $[y_q, x_q]$.*

Proof. Differentiation of $\theta_1(x)$ gives,

$$\theta_1'(x) = \Big(x\psi_q(x) + x^2\psi_q'(x) - (x\psi_q(x))^2\Big)\frac{1}{x\Gamma_q(x)}.$$

1) Let $q \geq 1$, by Lemma 2.4 the function $x\psi_q(x)$ increases, moreover, it is non positive on $[1, x_q]$. Therefore, the function $(x\psi_q(x))^2$ decreases on $[1, x_q]$. By Corollary 2.7, the function $x^2\psi_q'(x)$ increases on $[1, x_q]$. Then, for all $q > 1$ and all $x \in [1, x_q]$

$$x\psi_q(x) + x^2\psi_q'(x) - (x\psi_q(x))^2 \geq \psi_q(1) - (\psi_q(1))^2 + \psi_q'(1).$$

The right hand side is positive for every $q \in J \cap [1, +\infty)$. We conclude that $\theta_1(x)$ increases on $(1, x_q)$ for all $q \in J \cap [1, +\infty)$.

If $q \in (0, 1]$, we write

$$\theta_1'(x) = \Big(\psi_q(x) + x\psi_q'(x) - x(\psi_q(x))^2\Big)\frac{1}{\Gamma_q(x)},$$

and we use the inequality $x\psi_q'(x) + 2\psi_q(x) \geq 0$ to get

$$\theta_1'(x) = \Big(1 + x\psi_q(x)\Big)\frac{-\psi_q(x)}{\Gamma_q(x)}.$$

Since, $x\psi_q(x)$ increases on $(1, +\infty)$, then $1 + x\psi_q(x) \geq 1 + \psi_q(1) \geq 1 - \gamma > 0$, and for $x \in (1, x_q]$, $\psi_q(x) \leq 0$. Which implies that $\theta_1(x)$ increases on $(1, x_q)$ for all $q \in (0, 1]$.

2) If $q \notin J$, then $q > 1$ and $\psi_q(1) - (\psi_q(1))^2 + \psi_q'(1) < 0$. Moreover, the function $x\psi_q(x) + x^2\psi_q'(x) - (x\psi_q(x))^2$ increases on $[1, x_q]$ and is positive at $x = x_q$, then there is a unique $y_q \in (1, x_q)$ such that $\theta_1(x)$ decreases on $(1, y_q)$ and increases on (y_q, x_q). □

Proposition 3.2. *For $q > 0$, $x > 0$ and $\alpha > 0$, define the functions*

$$f_q(x) = \frac{\Gamma_q(x)\Gamma_q(1/x)}{\Gamma_q(x) + \Gamma_q(1/x)}, \quad g_{q,x}(\alpha) = f_q(x^\alpha)$$

1) (*a*) *For $q \in J$, The function $f_q(x)$ decreases on $(0, 1]$, and increases on $[1, +\infty)$.*
(*b*) *For $q \notin J$, the function $f_q(x)$ decreases on $(0, 1/y_q] \cup [1, y_q]$, and increases on $[1/y_q, 1] \cup [y_q, +\infty)$.*

2) (*a*) *For every $x > 0$ and $q \in J$, the function $g_{q,x}(\alpha)$ increases on $(0, +\infty)$.*
(*b*) *For every $x > 0$ and $q \notin J$ the function $g_{q,x}(\alpha)$ decreases on $(\frac{1}{y_q}, y_q)$, and increases on $(0, \frac{1}{y_q}] \cup [y_q, +\infty)$.*

In particular, For every $q \in J$, and $x > 0$,

$$\frac{2\,\Gamma_q(x)\Gamma_q(1/x)}{\Gamma_q(x) + \Gamma_q(1/x)} > 1.$$

The sign of equalities hold if and only if $x = 1$.
For $q \notin J$, and $x > 0$,

$$\frac{\Gamma_q(x)\Gamma_q(1/x)}{\Gamma_q(x) + \Gamma_q(1/x)} \geq f_q(y_q).$$

One shows that $y_q \to 1$ as $q \to 1$.

Proof. 1) A direct calculation gives

$$f_q'(x) = (\theta_1(x) - \theta_1(\frac{1}{x}))\frac{f_q(x)}{x},$$

where $\theta_1(x) = x\frac{\psi_q(x)}{\Gamma_q(x)}$.

1) Let $q \in J$. For $x \geq 1$, we have $x \geq \frac{1}{x}$ and by performing the relation between f_q and θ_1 and using Lemma 3.1, we get $f_q'(x) \geq 0$ for $x \in [1, x_q)$. Now, for $x > x_q$, $\theta_1(x) \geq 0$, $\theta_1(1/x) \leq 0$ and $f_q(x) \geq 0$. Then f_q increases on $[1, +\infty)$. Moreover, $f_q(x) = f_q(1/x)$, hence $f_q(x)$ decreases on $(0, 1]$.

For $q \notin J$. Applying Lemma 3.1, we get for $x \in (1, y_q)$, $f_q'(x) \leq 0$ and $f_q'(x) \geq 0$ on (y_q, x_q). It follows that $f_q(x)$ decreases on $(1, y_q)$ and increases on $[y_q, x_q]$. If $x \geq x_q$, as above we have $f_q'(x) \geq 0$ and $f_q(x)$ increases. By the relation $f_q(x) = f_q(1/x)$ we get the desired result.

2) Let $\varphi(\alpha) = f_q(x^\alpha)$. Then $\varphi'(\alpha) = x^\alpha \log(x) f_q'(x^\alpha)$. Applying item 1), a) we deduce that $\varphi'(\alpha) \geq 0$ for all $\alpha \geq 0$ and $q \in J$. Also, one deduces the second result from item 1), b). This completes the proof. □

As consequence, we get the following two corollaries.

Corollary 3.3. *For every $q \in J$ and $x > 0$,*

$$\Gamma_q(x) + \Gamma_q(\frac{1}{x}) \geq 2, \text{ and } \Gamma_q(x)\Gamma_q(\frac{1}{x}) \geq 1,$$

Corollary 3.4. *The function $f(x) = \dfrac{\Gamma(x)\Gamma(1/x)}{\Gamma(x) + \Gamma(1/x)}$, decreases on $(0, 1]$ and increases on $[1, +\infty)$.*

Now we provide another generalization of the result of Proposition 3.2 when $q \in (0, 1)$.

For $m \in \mathbb{R}$ and $a, b > 0$, we set $H_m(a, b) = (\frac{a^m + b^m}{2})^{\frac{1}{m}}$.

Proposition 3.5. *Let $G_{m,q}(x) = H_m(\Gamma_q(x), \Gamma_q(\frac{1}{x}))$*

(1) *For $q \geq 1$, the function $G_{m,q}(x)$ decreases on $(0, 1)$ and increases on $(1, +\infty)$ if and only if $m \geq \frac{-\psi_q(1) - \psi_q'(1)}{(\psi_q(1))^2}$.*

(2) *For $q > 0$ and $m \geq \frac{1}{\psi_q(1)}$ the function $G_{m,q}(x)$ decreases on $(0, 1)$ and increases on $(1, +\infty)$.*

As a consequence, for $q = 1$, we get $m \geq \frac{1}{\gamma} - \frac{\pi^2}{6\gamma}$. This case is proved by Alzer [2]. For $m = -1$, we retrieve the result of Proposition 3.2.

Remark that $\psi_q'(1) + \psi_q(1) \geq \psi_q'(1) + 2\psi_q(1) \geq 0$, then $-(\psi_q'(1) + \psi_q(1))/(\psi_q(1))^2 \leq 0$ for all $q > 0$.

The proof follows the same idea used by Alzer in [2], Theorem 1.

4. Harmonic mean of the q-digamma function

In this section we give some generalization of Alzer's and Jameson's inequalities proved in [3].

Proposition 4.1. *For all $x > 0$ and $x \neq 1$, and $q \in (0, 1)$ then*

$$\psi_q(x) + \psi_q(\frac{1}{x}) < 2\psi_q(1).$$

Proof. Recall that, for $x > 0$,

$$\psi_q(x) = -\log(1-q) - \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} d\gamma_q(t),$$

Let $f(x) = \psi_q(x) + \psi_q(1/x)$. Since, $f(1/x) = f(x)$, hence it suffices to prove the proposition for $x \in (0, 1)$. Differentiate $f(x)$ yields

$$f'(x) = \psi_q'(x) - \frac{1}{x^2}\psi_q'(\frac{1}{x}) = \frac{1}{x}(u(x) - u(\frac{1}{x})),$$

where $u(x) = x\psi_q'(x)$. Let $x \in (0, 1)$, then $x < \frac{1}{x}$ and by using Corollary 2.7 we get

$$u(x) > u(\frac{1}{x}).$$

One deduces that, f is strictly increasing on $(0, 1)$. Then $f(x) < f(1)$ for all $x \in (0, 1)$. □

Lemma 4.2. *For every $x > 0$ and $q > 0$,*

$$\psi_q'''(1+x) < -\frac{q^x(1+q^x)}{(1-q^x)^3}(\log q)^3 < \psi_q'''(x),$$

$$\psi_q''(x) < -\frac{q^x}{(1-q^x)^2}(\log q)^2 < \psi_q''(1+x).$$

Proof. This follow directly by applying Lagrange mean value theorem in the interval $(x, x+1)$. □

Let

$$I = \Big\{q > 0, \psi_q'(1) + \psi_q''(1) \geq 0\Big\}.$$

Lemma 4.3. *There is a unique $p_0 \in (1, \frac{9}{2})$, such that $I = [p_0, +\infty)$*

Numerical computation shows that $p_0 \simeq 3.239945$.

Proof. We set $u(q) := \psi_q'(1) + \psi_q''(1)$, and $I = \{q > 0, u(q) \geq 0\}$. It was proved in Corollary 2.7 that $x\psi_q'(x)$ is strictly decreasing on $(0, +\infty)$ for all $q \in (0, 1)$ then $\psi_q'(1) + \psi_q''(1) < 0$. Furthermore, $\lim\limits_{q\to 1} \psi_q'(1) + \psi_q''(1) = \psi'(1) + \psi''(1) = \zeta(2) - 2\zeta(3) \simeq -0.759$. Then $I \subset (1, +\infty)$.

By Lemma 2.6, we saw that the function $q \mapsto \psi_q'(1) + \psi_q''(1)$ increases on $(1, +\infty)$. Moreover, for $q \geq 2$,

$$|\psi_q''(1)| = (\log q)^3 \sum_{n=1}^{\infty} \frac{n^2}{q^n - 1} \leq \frac{(\log q)^3}{q-1} \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Then, $\lim_{q\to+\infty} \psi_q''(1) = 0$. Also, $\psi_q'(1) \geq \log q$, and $\lim_{q\to+\infty} \psi_q'(1) = +\infty$. Then, there is a unique $p_0 > 1$ such that $\psi_q'(1) + \psi_q''(1) < 0$ for $q \in (0, p_0)$ and $\psi_q'(1) + \psi_q''(1) \geq 0$ for $p \geq p_0$.

By equation (9), we get

$$\psi_q'(1) + \psi_q''(1) \geq \psi_q'(1)(1 - \psi_{\frac{1}{q}}'(1)).$$

Since, $\psi_q'(1) \geq 0$ and the function $z(q) = 1 - \psi_{\frac{1}{q}}'(1)$ increases on $(0, +\infty)$. Furthermore, $z(9/2) = 1 - (\log 9/2)^2 \sum_{n=1}^{\infty} \frac{n}{(9/2)^n - 1}$. Since, for $n \geq 2$, $(9/2)^n - 1 \geq (6/5)4^n$. Then

$$z(9/2) \geq 1 - \frac{(\log 9/2)^2}{7/2} - \frac{35(\log 9/2)^2}{216} \simeq 0.067$$

Therefore, $p_0 < 9/2$. Which completes the proof.

Numerically $u(3) \leq \log 3 + \frac{1}{2}(\log 3)^2(1 - \log 3) + \frac{1}{4}(\log 3)^2(1 - 2\log 3) + \frac{27}{78}(\log 3)^2(1 - 3\log 3) \simeq -0.28132$. Hence, $I \subset (3, 9/2)$. □

Lemma 4.4.

(1) *For all $q > 0$ and $x > 0$, $2\psi_q''(x) + x\psi_q'''(x) \geq 0$.*
(2) *For $q \geq 1$, and $x > 0$, $\psi'(x) + x\psi''(x) \leq \psi_q'(x) + x\psi_q''(x) \leq \log q$ and for $q \in (0, 1)$ $\psi_q'(x) + x\psi_q''(x) \leq 0$*
(3) *The function $x\psi_q'(x)$ increases on $[1, +\infty)$ for every $q \in [p_0, +\infty)$ and decreases on $(0, 1)$ if $q \in (0, p_0)$.*

Proof. 1) Let $q \in (0, 1)$ and $\varphi(x) = 2\psi_q''(x) + x\psi_q'''(x)$, then

$$\varphi(1+x) - \varphi(x) = 2\psi_q''(1+x) + (1+x)\psi_q'''(1+x) - 2\psi_q''(x) - x\psi_q'''(x). \tag{10}$$

Since, $\psi_q'''(1+x) - \psi_q'''(x) = -\frac{q^x(1+q^x(4+q^x))}{(1-q^x)^4}(\log q)^4$, and $\psi_q''(1+x) - \psi_q''(x) = -\frac{q^x(1+q^x)}{(1-q^x)^3}(\log q)^3$.

Applying Lemma 4.2 and equation (12), we get

$$\varphi(1+x) - \varphi(x) \leq -\frac{q^x(\log q)^3}{(1-q^x)^4}(3(1+q^x)(1-q^x) + x\log q(1 + q^x(4+q^x)).$$

For $u \in (0,1)$, let $j(u) = 3(1+u)(1-u) + \log u(1+u(4+u))$. By successive differentiation we get $j'(u) = 4 + 1/u - 5u + 2(2+u)\log u$, $j''(u) = -3 + (-1+4u)/u^2 + 2\log u$ and $j'''(u) = 2(-1+u)^2/u^3 > 0$ on $(0,1)$. Then, $j''(u) \leq j''(1) = 0$ and $j'(u) \geq j'(1) = 0$. Thus, $j(u)$ increases on $(0,1)$ and $j(u) \leq j(1) = 0$. Hence, for all $x > 0$ and all $q \in (0,1)$

$$\varphi(1+x) - \varphi(x) \leq -\frac{q^x(\log q)^3}{(1-q^x)^4} j(q^x) \leq 0.$$

Therefore, for all $x > 0$ and all $n \in \mathbb{N}$

$$\varphi(x+n) \leq \varphi(x). \tag{11}$$

Letting $n \to +\infty$ in equation (11) yields $\varphi(x) \geq 0$ for all $x > 0$. Which gives the desired result.

For $q > 1$, we saw that $2\psi_q''(x) + x\psi_q'''(x) = 2\psi_{\frac{1}{q}}''(x) + x\psi_{\frac{1}{q}}'''(x)$ and the result follows.

2) Let $s(x) = x\psi_q'(x)$, then, $s'(x) = \psi_q'(x) + x\psi_q''(x)$ and $s''(x) = 2\psi_q''(x) + x\psi_q'''(x)$. By the previous item we deduce that $s'(x)$ increases on $(0,+\infty)$ for all $q > 0$. Since, $\lim_{x\to\infty} s'(x) = 0$ if $q \in (0,1)$ and $= \log q$ if $q > 1$. Which gives the desired result

3) We saw by item 2 that $s'(x)$ increases, then for every $x \geq 1$, $s'(x) \geq s'(1) = \psi_q'(1) + \psi_q''(1) \geq 0$ for all $q \in I$. Hence, $s(x)$ increases on $(1,+\infty)$ for $q \in I$. □

Proposition 4.5. *For all $q \in [p_0,+\infty)$ and all $x > 0$*

$$\psi_q(x) + \psi_q(1/x) \geq 2\psi_q(1).$$

For $q \in (0,p_0)$,

$$\psi_q(x) + \psi_q(1/x) \leq 2\psi_q(1).$$

Proof. Let $U(x) = \psi_q(x) + \psi_q(\frac{1}{x})$, then $U'(x) = 1/x(x\psi_q'(x) - 1/x\psi_q'(1/x))$. If $x \geq 1$, then by Lemma 4.4, and the fact that $x \geq 1/x$ we get $U'(x) \geq 0$ for all $q \geq p_0$. Hence, $U(x)$ increases on $(1,+\infty)$ and by the symmetry $U(x) = U(1/x)$, it decreases on $(0,1)$. Then $U(x) \geq U(1)$.

If $q \in (0,p_0)$, then $x\psi_q'(x)$ decreases on $(0,1)$, since $1/x \geq x$ then, $U'(x) \geq 0$ and $U(x)$ increases on $(0,1)$. By the symmetry $U(x) = U(1/x)$, it decreases on $(1,+\infty)$. Then $U(x) \leq U(1)$. Which completes the proof. □

Proposition 4.6. *For all $x > 0$ and $q > 0$,*

$$\psi_q(x)\psi_q(\frac{1}{x}) \leq (\psi_q(1))^2.$$

Proof. Firstly, remark that the function $v(x) = \psi_q(x)\psi_q(\frac{1}{x})$ is invariant by the symmetry $v(1/x) = v(x)$. So, it is enough to prove the result on $(1,+\infty)$ for all $q > 0$.

By differentiation, we have

$$v'(x) = \frac{1}{x}(x\frac{\psi_q'(x)}{\psi_q(x)} - \frac{1}{x}\frac{\psi_q'(\frac{1}{x})}{\psi_q(\frac{1}{x})})v(x) = \frac{1}{x}(w(x) - w(\frac{1}{x}))v(x),$$

where $w(x) = x\frac{\psi_q'(x)}{\psi_q(x)}$.

If $x \in (1,x_q)$ then $x > 1/x$, and By Proposition 2.9 and Proposition we have, the function $w(x)$ decreases. Then $w(x) \leq w(1/x)$. Moreover, $v(x) > 0$, hence, $v'(x) < 0$ and $v(x) < v(1)$.

For $x \geq x_q$, $\psi_q(x) \geq 0$ and $\psi_q(1/x) \leq 0$ hence, $v(x) \leq (\psi_q(1))^2$. □

Proposition 4.7. *1) For all $q \in (0,p_0)$ and all $x > 0$*

$$\frac{2\psi_q(x)\psi_q(\frac{1}{x})}{\psi_q(x) + \psi_q(\frac{1}{x})} > \psi_q(1).$$

2) For $q \in [p_0, +\infty)$, there is a unique $z_q > x_q$ such that for all $x \in [1/z_q, z_q]$

$$\frac{2\psi_q(x)\psi_q(\frac{1}{x})}{\psi_q(x) + \psi_q(\frac{1}{x})} > \psi_q(1).$$

If $x \in (0, 1/z_q) \cup (z_q, +\infty)$, then the reversed inequality holds.
The sign of equalities hold if and only if $x = 1$.

Proof. 1) Let $U(x) = \psi_q(x) + \psi_q(1/x)$. From Proposition 4.5, we conclude that for $q \in (0, p_0)$ the expression $\frac{1}{U(x)}$ is defined for all positive $x > 0$. Applying Propositions 4.5 and 4.6, we get

$$\frac{2\psi_q(x)\psi_q(\frac{1}{x})}{\psi_q(x) + \psi_q(\frac{1}{x})} \geq \frac{2(\psi_q(1))^2}{\psi_q(x) + \psi_q(\frac{1}{x})} > \psi_q(1),$$

for all $q \in (0, p_0)$ and all $x > 0$.

2) From Proposition 4.5, and the fact that for $q \geq p_0$, $U(x)$ increases on $(1, +\infty)$ and $U(1) = 2\psi_q(1) < 0$, $\lim_{x\to+\infty} U(x) = +\infty$, we deduce that there is a unique $z_q \in (1, +\infty)$ such that $U(z_q) = 0$ and $U(x)$ is defined and negative for all $x \in (1/z_q, z_q)$. The fact that $z_q > x_q$ follows from the relation $\psi_q(z_q) = -1/\psi_q(1/z_q) > 0$.

Let $H(x) = \frac{\psi_q(x)\psi_q(\frac{1}{x})}{\psi_q(x)+\psi_q(\frac{1}{x})}$. Then,

$$H'(x) = x(\frac{x\psi_q'(x)}{(\psi_q(x))^2} - \frac{\psi_q'(1/x)}{x(\psi_q(1/x))^2})(H(x))^{-2}.$$

Since, $\psi_q(x)$ increases and is negative and by Proposition 2.9 $x\psi_q'(x)/\psi_q(x)$ decreases on $(1, x_q)$ for $q > 1$ and is negative. Then, we get for $x > 1$, $H'(x) > 0$ and $H(x)$ increases on $(1, x_q)$. Thus, $H(x) \geq H(1) = 1/2\psi_q(1)$.

If $x \in (x_q, z_q)$, then $\psi_q(x)\psi_q(1/x) < 0$ and $\psi_q(x) + \psi_q(1/x) < 0$, which implies that $H(x)$ is negative and, $H(x) > \frac{1}{2}\psi_q(1)$. So, for all $q > 0$ and $x \in (1, z_q)$, $H(x) \geq \frac{1}{2}\psi_q(1)$. By the symmetry $H(x) = H(1/x)$, One deduces the result on $(1/z_q, z_q)$.

If $x \in (0, 1/z_q) \cup (z_q, +\infty)$, then $U(x) > 0$. Moreover, $\psi_q(x)\psi_q(1/x) < 0$. Then, $\psi_q(x)\psi_q(1/x) \leq (\psi_q(1/x))^2$. Or equivalently

$$\frac{2\psi_q(x)\psi_q(\frac{1}{x})}{\psi_q(x) + \psi_q(\frac{1}{x})} \leq \psi_q(\frac{1}{x}).$$

Since, the function $x \mapsto \psi_q(x)$ increases on $(0, +\infty)$. So, for $x \geq z_q > 1$, $\psi_q(\frac{1}{x}) \leq \psi_q(1)$ and on the interval $(0, 1/z_q)$ the result follows by symmetry. □

As a consequence and since, $p_0 > 1$, by letting $q \to 1$, we get the following corollary

Corollary 4.8. *For all $x > 0$,*

$$\frac{2\psi(x)\psi(\frac{1}{x})}{\psi(x) + \psi(\frac{1}{x})} > -\gamma.$$

The sign of equality hold if and only if $x = 1$.

5. Conclusion

In our present investigation, we derive in section 2 some monotonicity results for the $\Gamma_q(x)$ and $\psi_q(x)$ functions with respect to the variables x and q. The main results together with the computation of section 3 and 4 allow us to extend the results of the author's papers [13] and [3] to the case of q-gamma function and q-digamma function.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) gamma and q-hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [[23], pp. 350-351] and [[24], p. 328)]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [24], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [[24], p. 340]). This observation by Srivastava [24] will indeed apply also to any future attempt to produce the rather straightforward (p, q)-variants of the results which we have presented in this paper.

6. Appendix

Proof. (Lemma 2.3) Let $h(u) = u/(1-u)$, then for every $m \geq 1$, and $u \neq 1$,

$$h^{(m)}(u) = \frac{(-1)^{m+1} m!}{(1-u)^{m+1}}.$$

Firstly, we prove the lemma for $q \in (0,1)$. We have

$$\psi_q(x) = -\log(1-q) + (\log q) \sum_{n=0}^{\infty} h(q^{n+x}).$$

Recall the Faà di Bruno formula for the *nth* derivative of $f \circ g$, see for instance [20]. For $m \geq 0$,

$$(f \circ g)^{(m)}(t) = \sum_{k=0}^{m} \frac{m!}{k_1! k_2! ... k_m!} f^{(k)}(g(t)) \prod_{i=1}^{m} (\frac{g^{(i)}(t)}{i!})^{k_i},$$

where $k = k_1 + ... + k_m$ and summation is over all naturel integers $k_1, ..., k_m$
such that $k_1 + 2k_2 + ... + mk_m = m$. In our case, $g^{(i)}(x) = (\log q)^i q^{x+n}$ and $f^{(k)}(x) = h^{(k)}(x)$. Then,

$$\Big(h(q^{n+x})\Big)^{(m)} = (\log q)^m \sum_k \frac{m! (-1)^{k+1} k!}{\prod_{i=1}^m k_i! \prod_{i=1}^m (i!)^{k_i}} \frac{q^{k(n+x)}}{(1-q^{n+x})^{k+1}}.$$

Remark that for $k \leq m-1$, $x > 0$ and $n \geq 0$, $\lim_{q \to 1-} (\log q)^{m+1} \frac{q^{k(n+x)}}{(1-q^{n+x})^{k+1}} = 0$. That is to say, the only term which contributes in the limit of $(\log q)^{m+1} \frac{q^{k(n+x)}}{(1-q^{n+x})^{k+1}}$ as $q \to 1-$ is when $k = m$. In this case, we have $k_1 = m$, $k_2 = k_3 = ... = k_m = 0$. Then for $x > 0$, $n \geq 0$ and $m \geq 1$, we get

$$\lim_{q \to 1-} (\log q) \Big(h(q^{n+x})\Big)^{(m)} = \lim_{q \to 1-} m! \frac{(\log q)^{m+1} q^{m(n+x)}}{(1-q^{n+x})^{m+1}} = \frac{(-1)^{m+1} m!}{(n+x)^{m+1}}.$$

One shows that the functions $y \mapsto y^p(-\log(y))^{p+1}/(1-y)^{p+1}$ increase on $(0,1)$ for all $p \geq 1$, and

$$\lim_{y \to 1-} \frac{y^p(-\log(y))^{p+1}}{(1-y)^{p+1}} = 1.$$

Let $a > 0$, then for all $x \geq a$, $n \geq 0$ and all $q \in [1/2, 1]$, we have

$$\Big|(\log q)^{m+1} \frac{q^{k(n+x)}}{(1-q^{n+x})^{k+1}}\Big| = \frac{(-\log q)^{m-k}}{(n+x)^{k+1}} \frac{q^{k(n+x)}(-\log q^{n+x})^{k+1}}{(1-q^{n+x})^{k+1}} \leq \frac{(\log 2)^{m-k}}{(n+a)^{k+1}},$$

Since, $m \geq 1$ then $k \geq 1$ and

$$\Big|(\log q) \Big(h(q^{n+x})\Big)^{(m)}\Big| \leq \frac{C_m}{(n+a)^2},$$

where $C_m = \sum_k \frac{m!(-1)^{k+1}k!(\log 2)^{m-k}}{\prod_{i=1}^m k_i! \prod_{i=1}^m (i!)^{k_i}}$ is some constant independent of n. This implies that the series $\sum_{n=0}^{\infty}(\log q)\Big(h(q^{n+x})\Big)^{(m)}$ converges uniformly for $(x,q) \in [a,+\infty)\times[1/2,1]$ and $m \geq 1$, moreover

$$\lim_{q\to 1-}\psi_q^{(m)}(x) = \lim_{q\to 1-}\sum_{n=0}^{\infty}(\log q)\Big(h(q^{n+x})\Big)^{(m)} = \sum_{n=0}^{\infty}\frac{(-1)^{m+1}m!}{(n+x)^{m+1}} = \psi^{(m)}(x).$$

Assume $q > 1$, we saw that $\psi_q(x) = (x-3/2)\log q + \psi_{\frac{1}{q}}(x)$, and $\psi_q^{(m)}(x) = \log q + \psi_{\frac{1}{q}}^{(m)}(x)$ if $m = 1$, $\psi_q^{(m)}(x) = \psi_{\frac{1}{q}}^{(m)}(x)$ if $m \geq 2$. The result follows from the case $q \in (1/2, 1)$. $\square$

Proof. (Proposition 2.9) By differentiation we get

$$G_q'(x) = \frac{\psi_q''(x)\psi_q(x) - (\psi_q'(x))^2}{(\psi_q(x))^2}.$$

1) a) In order to prove $G_q'(x) \leq 0$ for $x \in (1, x_q)$, it suffices to show

$$\psi_q''(x)\psi_q(x) - (\psi_q'(x))^2 \leq 0 \quad \forall\, q > 0.$$

For $x > 0$ and $q > 0$, the inequality $\psi_q''(x) \geq -(\psi_q'(x))^2 + (\log q)\psi_q'(x)$ is proved in [10]. So,

$$\psi_q''(x)\psi_q(x) - (\psi_q'(x))^2 \leq -(\psi_q'(x))^2\psi_q(x) + (\log q)\psi_q'(x)\psi_q(x) - (\psi_q'(x))^2.$$

Which is equivalent to

$$\psi_q''(x)\psi_q(x) - (\psi_q'(x))^2 \leq -\psi_q'(x)\Big(\psi_q'(x)\psi_q(x) + \psi_q'(x) - (\log q)\psi_q(x)\Big).$$

For $x \in (1, x_q)$ and $q > 0$, let's define

$$\theta_q(x) = \psi_q'(x)\psi_q(x) + \psi_q'(x) - (\log q)\psi_q(x).$$

Differentiation of $\theta_q(x)$ gives

$$\theta_q'(x) = \psi_q''(x)\psi_q(x) + \psi_q''(x) + (\psi_q'(x))^2 - (\log q)\psi_q'(x).$$

Hence, $\theta_q'(x) \geq \psi_q''(x)\psi_q(x) \geq 0$ for all $x \in (1, x_q)$. Thus, $\theta_q(x)$ increases on $(1, x_q)$, and

$$\theta_q(x) \geq \theta_q(1) = (\psi_q'(1) - \log q)\psi_q(1) + \psi_q'(1).$$

Observe that $\theta_1(1) = \psi_1'(1)(1-\gamma) > 0$.

It remains to show that the right hand side is positive for $q > 0$, $q \neq 1$.

In [11], it is proved that the function $F_q(x) = \psi_q(x+1) - \log(\frac{1-q^{x+\frac{1}{2}}}{1-q})$ is completely monotonic on $(0,+\infty)$ for all $q > 0$. Then, $F_q'(x) \leq 0$, and $\psi_q'(x+1) \leq -(\log q)\frac{q^{x+\frac{1}{2}}}{1-q^{x+\frac{1}{2}}}$. Which implies that,

$$\theta_q(1) \geq \frac{\log q}{\sqrt{q}-1}\Big(\psi_q(1) + \frac{\sqrt{q}-1}{\log q}\psi_q'(1)\Big).$$

Since, $\frac{\log q}{\sqrt{q}-1} \geq 0$ for all $q > 0, q \neq 1$. It is enough to prove that,

$$u(q) := \psi_q(1) + \frac{\sqrt{q}-1}{\log q}\psi_q'(1) \geq 0, \quad \forall\, q > 0, q \neq 1. \tag{12}$$

Firstly, remark that

$$u(q) = \psi_{\frac{1}{q}}(1) + \sqrt{q}\frac{\sqrt{\frac{1}{q}}-1}{\log\frac{1}{q}}\psi'_{\frac{1}{q}}(1) + \sqrt{q} - 1 - \log\sqrt{q}.$$

Observe, $\sqrt{q} - 1 - \log\sqrt{q} \geq 0$ and $\frac{\sqrt{\frac{1}{q}}-1}{\log\frac{1}{q}} \geq 0$ for all $q > 0$, $q \neq 1$. Therefore, for $q > 1$, $u(q) \geq u(\frac{1}{q})$, and it suffices to show that

$$u(q) \geq 0, \qquad \forall\, q \in (0,1).$$

Using the series expansion of the functions $\psi_q(1)$ and $\psi'_q(1)$, we get

$$u(q) = \sum_{n=1}^{\infty}\frac{q^n}{n(1-q^n)}\Big(1 - q^n + n\log q + n^2(\sqrt{q}-1)\log q\Big).$$

For $x \geq 1$, and $q \in (0,1)$, define the function

$$g(x) = 1 - q^x + x\log q + x^2(\sqrt{q}-1)\log q.$$

By differentiation, we find

$$g'(x) = \Big(1 - q^x - 2x + 2x\sqrt{q}\Big)\log q, \quad g''(x) = \Big(-2 + 2\sqrt{q} - q^x\log q\Big)\log q, \quad g'''(x) = -q^x(\log q)^3.$$

For $q \in (0,1)$, we have $g'''(x) > 0$ for all $x \geq 1$, then

$$g''(x) \geq g''(1) = -\Big(2 - 2\sqrt{q} + q\log q\Big)\log q.$$

An easy computation shows that $-\Big(2 - 2\sqrt{q} + q\log q\Big)\log q \geq 0$ for all $q \in (0,1)$. Which implies that, $g''(x) \geq 0$, and then $g'(x) \geq g'(1) = -(\sqrt{q}-1)^2\log q > 0$, hence,

$$g(x) \geq g(1) = 1 - q + \sqrt{q}\log q.$$

Setting $w(q) = 1 - q + \sqrt{q}\log q$. Then, $w'(q) = -1 + \frac{1}{\sqrt{q}} + \frac{1}{2\sqrt{q}}\log q, w''(q) = -\frac{1}{4}q^{-\frac{3}{2}}\log q > 0$, and $w'(1) = 0$. Then, $w(q) \geq 0$. This implies $g(n) \geq 0$ for all $n \in \mathbb{N}$, and $u(q) \geq 0$ for all $q \in (0,1)$. Which gives the desired result.

b) If $x \in (x_q, +\infty)$ then $\psi_q(x) \geq 0$ and $\psi''_q(x) \leq 0$ and the result follows for all $q > 0$.

2) a) Assume $q > 0$ and $x \in (1, x_q)$. Since, $-\varphi_q(x) = x(-\frac{\psi'_q(x)}{\psi_q(x)})$, hence by the previous item $-\varphi_q(x)$ is a product of two positive increasing functions. Then $\varphi_q(x)$ decreases on $(1, x_q)$.

b) i) Assume that $\varphi_q(x)$ decreases on $(x_q, +\infty)$. Then,

$$(\psi_q(x))^2\varphi'_q(x) = \psi'_q(x)(\psi_q(x) - x\psi'_q(x)) + x\psi_q(x)\psi''_q(x) \leq 0.$$

For $q > 1$, and $x > 1$, we have

$$\sum_{n=1}^{\infty} n^2\frac{q^{-nx}}{1-q^{-n}} \leq q^{-x}\frac{q}{q-1}(1 + (q-1)\sum_{n=2}^{\infty} n^2\frac{1}{1-q^{-n}}).$$

Then,

$$|\psi''_q(x)| \leq a_q q^{-x}. \tag{13}$$

Following the same method, there is $b_q, c_q \geq 0$, such that,

$$|\psi_q(x)| \leq b_q + (\log q)x + c_q q^{-x}. \tag{14}$$

From inequalities (13) and (14), we get $\lim_{x\to\infty} x\psi_q(x)\psi_q''(x) = 0$, and $\lim_{x\to+\infty} \frac{x\psi_q''(x)}{\psi_q(x)} = 0$. Also we saw that $\lim_{x\to+\infty} \psi_q'(x) = \log q$, then

$$-(\log q)\log(\sqrt{q}(q-1)) \leq 0,$$

or equivalently $q^{\frac{3}{2}} - q^{\frac{1}{2}} - 1 \geq 0$. Which implies that $q \in (0,1] \cup [q_0, +\infty)$.

ii) If $q \in (0,1]$ and $x > x_q$, then by Corollary 2.7, $x\psi_q'(x)$ decreases and is positive, moreover the function $1/\psi_q(x)$ decreases and is positive. Then $\varphi_q(x)$ is a decreasing function.

iii) Assume $q \geq q_0 > 1$, we have $\varphi_q(x) = \dfrac{x\psi'_{\frac{1}{q}}(x)}{\psi_q(x)} + \log q \dfrac{x}{\psi_q(x)}$. Since, by Corollary 2.7 the function $x\psi'_{\frac{1}{q}}(x)$ decreases and is positive and $\frac{1}{\psi_q(x)}$ decreases and is positive, then $\frac{x\psi'_{\frac{1}{q}}(x)}{\psi_q(x)}$ decreases on $(x_q, +\infty)$.

Let $v_q(x) = \frac{x}{\psi_q(x)}$, then $(\psi_q(x))^2 v_q'(x) = \psi_q(x) - x\psi_q'(x)$. Furthermore, the derivative of the right hand side is $-x\psi_q''(x)$ which is positive. Then the function $\psi_q(x) - x\psi_q'(x)$ increases on $(x_q, +\infty)$, and by easy computation we have $\lim_{x\to+\infty} \psi_q(x) - x\psi_q'(x) = -\log(\sqrt{q}(q-1)) \leq 0$ for all $q \geq q_0$. One deduces that $\varphi_q(x)$ decreases on $(x_q, +\infty)$ for all $q \geq q_0$. □

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