Filomat 35:12 (2021), 3933–3944
https://doi.org/10.2298/FIL2112933K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Real Hypersurfaces with Reeb Invariant Structure Jacobi Operator in the Complex Quadric

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Abstract. We introduce a new notion of Reeb invariant structure Jacobi operator and two kinds of singular normal vector field N for a real hypersurface M in the complex quadric Q^m, $m \geq 3$. If the unit normal N is $\mathfrak{A}$-isotropic, we give a classification of Hopf real hypersurfaces with Reeb invariant structure Jacobi operator in the complex quadric Q^m, for $m \geq 3$.

1. Introduction

In 20th century, some classification theorems for real hypersurfaces in Hermitian symmetric spaces of rank 1 have been investigated by many differential geometers (see [1], [2], [8], [29], [30] and [31]). For instance, Okumura [17] investigated real hypersurfaces with isometric Reeb flow in complex projective space. Using this condition, Montiel and Romero [16] gave a classification for real hypersurfaces in complex hyperbolic space. Recently, many differential geometers have extended some results in Hermitian symmetric spaces of rank 1 to Hermitian symmetric spaces of rank 2 (see [3], [4], [5], [6], [11], [22], [23] and [28]).

As typical examples of Hermitian symmetric spaces of rank 2 we can consider the complex two-plane Grassmannians $SU_{m+2}/S(U_2U_m)$ and complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$. Berndt and Suh [4] gave a classification of real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians and Suh [22] did it for complex hyperbolic two-plane Grassmannians.

On the other hand, the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is another kind of rank 2 Hermitian symmetric space of compact type different from the complex two-plane Grassmannian. Indeed, the complex quadric Q^m is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$. The complex quadric can be regarded as a kind of real Grassmann manifolds of compact type with rank 2. In fact, the complex quadric admits two important geometric structures which are a complex conjugation A and a Kähler structure J. For these two structures, A and JA are self-adjoint, whereas J is skew-adjoint. Then for $m \geq 3$ the triple

2020 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C55
Keywords. complex quadric, Hopf real hypersurface, structure Jacobi operator, $\mathfrak{A}$-isotropic, $\mathfrak{A}$-principal, Reeb invariance
Received: 01 September 2020; Revised: 10 November 2020; Accepted: 14 December 2020
Communicated by Ljubica Velimirović
Gyu Jong Kim was supported by Proj. No. NRF-2020-R1G1A1A01003570, Imsoon Jeong by NRF-2021-R1F1A1064192 and Hyunjin Lee by NRF-2019-R1I1A1A01050300 from National Research Foundation of Korea.
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(Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see [9], [10], [20] and [21]).

There is another remarkable geometric structure on Q^m, namely a parallel rank 2 vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures. This is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} \,|\, \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^m$. Now, $\mathfrak{A}_{[z]}$ is a parallel rank 2-subbundle of $\mathrm{End}(T_{[z]}Q^m)$, $[z] \in Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on TQ^m, $[z] \in Q^m$, respectively (see [20]).

On the other hand, since the real structure A is an involutive automorphism on $T_{[z]}Q^m$, $[z] \in Q^m$, it can be decomposed as $T_{[z]}Q^m = V(A) \oplus JV(A)$, where $V(A) = \{X \in T_{[z]}Q^m \,|\, AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^m \,|\, AX = -X\}$ are the $(+1)$-eigenspace and the (-1)-eigenspace of A, respectively. It implies that for every unit vector $W \in T_{[z]}Q^m$ there exist $t \in [0, \frac{\pi}{4}]$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ so that

$$W = \cos(t)Z_1 + \sin(t)Z_2$$

holds (see Proposition 3 in [20]). Here, t is uniquely determined by W. In particular, the vector W is *singular* if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If W is *regular*, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and Z_1, Z_2 are uniquely determined by W.

Let M be a real hypersurface of Q^m and N be a unit normal vector field on M. From the complex structure J of (Q^m, g), we naturally obtain an almost contact metric structure (ϕ, ξ, η, g) on M. Also, by Weingarten formula which is $\bar{\nabla}_X N = -SX$ we define the shape operator S of M.

For a typical classification of real hypersurfaces in the complex quadric Q^m, we introduce a new notion of *Reeb-invariant shape operator* defined by $(\mathcal{L}_\xi S)Y = 0$ along the Reeb direction $\xi = -JN$, where $\mathcal{L}$ denotes the Lie derivative on the hypersurface. By using such notion, Suh [24] gave the following

Theorem 1.1 ([24]). *Let M be a Hopf real hypersurface with Reeb invariant shape operator in the complex quadric Q^m, $m \geq 3$. Then m is even, say $m = 2k$, and M is locally congruent to a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in Q^{2k} (this tube is called a model space of type $(\mathcal{T}_A)$).*

Here, we say that a real hypersurface M is *Hopf* if the Reeb vector field ξ is principal, that is, $S\xi = \alpha\xi$. Moreover, the smooth function $\alpha = g(S\xi, \xi)$ is called the *Reeb function* of M. If the Reeb function α identically vanishes, we say that M has *vanishing geodesic Reeb flow*. Otherwise, M has *non-vanishing geodesic Reeb flow*.

By using the expression of the curvature tensor R for a real hypersurface M in Q^m, the *structure Jacobi operator* R_ξ is an (1,1)-type tensor that can be defined as

$$R_\xi(X) = R(X, \xi)\xi$$

for any tangent vector field X on M. Pérez and Santos [19] gave a complete classification of real hypersurfaces in complex projective spaces satisfying recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for any $X, Y \in TM$. Moreover, Machado, Pérez and Suh [15] investigated real hypersurfaces in complex two-plane Grassmannians with commuting structure Jacobi operator defined by $R_\xi \circ R_Y = R_Y \circ R_\xi$, where $R_Y = R(\cdot, Y)Y$ is the Jacobi operator with respect to $Y \in TM$.

On the other hand, the structure Jacobi operator R_ξ of M is said to be *invariant* if the operator R_ξ satisfies

$$(\mathcal{L}_X R_\xi)Y = 0$$

for any $X, Y \in TM$, where the Lie derivative $(\mathcal{L}_X R_\xi)Y$ is given by

$$\begin{aligned}(\mathcal{L}_X R_\xi)Y &= [X, R_\xi Y] - R_\xi[X, Y] \\ &= \nabla_X(R_\xi Y) - \nabla_{R_\xi Y}X - R_\xi(\nabla_X Y - \nabla_Y X) \\ &= (\nabla_X R_\xi)Y - \nabla_{R_\xi Y}X + R_\xi(\nabla_Y X).\end{aligned} \tag{1.1}$$

In particular, it is said to be *Reeb invariant* if the structure Jacobi operator R_ξ of M holds

$$(\mathcal{L}_\xi R_\xi)Y = 0 \tag{*}$$

for any $Y \in TM$. Moreover, from (1.1) and using $\nabla_Y \xi = \phi SY$ for any $Y \in TM$, the condition (*) is equivalent to

$$(\nabla_\xi R_\xi)Y = \phi S R_\xi Y - R_\xi \phi S Y \tag{1.2}$$

for any $Y \in TM$.

Machado and Pérez [14] considered the notion of Reeb invariant structure Jacobi operator of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. By using this notion, they proved that *a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has Reeb invariant structure Jacobi operator if and only if M is locally congruent to a open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Related to these facts, we want to give a classification of Hopf real hypersurfaces in the complex quadric Q^m satisfying the Reeb invariance of the structure Jacobi operator R_ξ, that is, $\mathcal{L}_\xi R_\xi = 0$.

First, when the normal vector field N of M in Q^m is $\mathfrak{A}$-principal, we assert:

Theorem 1.2. *There does not exist a Hopf hypersurface with Reeb invariant structure Jacobi operator in the complex quadric Q^m, $m \geq 3$, whose normal vector field is $\mathfrak{A}$-principal.*

Next, by using the result of isometric Reeb flow in the complex quadric Q^m due to Berndt and Suh [5], we obtain:

Theorem 1.3. *Let M be a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field and non-vanishing geodesic Reeb flow in the complex quadric Q^m. Then, the structure Jacobi operator R_ξ of M is Reeb invariant if and only if M is locally congruent to a tube of radius $r \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$ around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.*

This paper is organized as follows. First, in section 2 we review some geometric structures of complex quadric Q^m and set up the notations of geometric tools in a Hopf real hypersurface M in Q^m. Also, we deal with useful materials to classify real hypersurfaces in the complex quadric Q^m in section 2. Next, in sections 3 and 4 we give a classification of real hypersurfaces in complex quadric Q^m with Reeb invariant structure Jacobi operator.

2. Complex quadrics and their Hopf real hypersurfaces

In this section, we deal with some general formulas given on a Hopf hypersurface M in the complex quadric Q^m. And we introduce some key Lemmas depending on $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic normal vector field N of M, which are used in sections 3 and 4, respectively. For more details, we can refer to [12], [13], [18], [25], [26] and [27].

As mentioned in the introduction, the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$ is a complex hypersurface in $\mathbb{C}P^{m+1}$, which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, z_1, \cdots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$ (see [5], [7], [9], [10] and [20]). It admits two geometric structures, a complex conjugation A and a Kähler structure J. Such two structures of Q^m satisfy the anti-commuting property $AJ = -JA$ for each $A \in \mathfrak{A}$. Here $\mathfrak{A}$ is a parallel rank two vector bundle which contains an S^1-bundle of real structures, that is, $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} \,|\, \lambda \in S^1 \subset \mathbb{C}\}$ for any point $[z] \in Q^m$.

The complex quadric $Q^1 = SO_3/SO_2$ is isometric to a sphere S^2 with constant curvature. And Q^2 is isometric to the Riemannian product of two complex projective spaces $\mathbb{C}P^1 \times \mathbb{C}P^1$, which is grounded in the Segre embedding $\mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow Q^2 \subset \mathbb{C}P^3$. For this reason, we will assume $m \geq 3$ hereafter.

The Gauss equation for Q^m in $\mathbb{C}P^{m+1}$ implies the Riemannian curvature tensor $\bar{R}$ of Q^m which is described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned} \tag{2.1}$$

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. By the Gauss and Weingarten formulas, the left side of (2.1) becomes

$$\bar{R}(X,Y)Z = R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY + \Big\{g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z)\Big\}N, \tag{2.2}$$

where R and S denote the Riemannian curvature tensor and the shape operator of M in Q^m, respectively.

On the other hand, at each point $[z] \in M$ we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [20]). Note that t is a function on M. From this, we have

$$AN = \cos(t)AZ_1 + \sin(t)AJZ_2 = \cos(t)Z_1 - \sin(t)JAZ_2 = \cos(t)Z_1 - \sin(t)JZ_2,$$

$$JN = \cos(t)JZ_1 + \sin(t)J^2 Z_2 = \cos(t)JZ_1 - \sin(t)Z_2$$

and

$$AJN = \cos(t)AJZ_1 - \sin(t)AZ_2 = -\cos(t)JAZ_1 - \sin(t)Z_2 = -\cos(t)JZ_1 - \sin(t)Z_2. \tag{2.3}$$

Since $JN = -\xi$, (2.3) becomes

$$A\xi = \cos(t)JZ_1 + \sin(t)Z_2. \tag{2.4}$$

Taking the inner product of (2.4) with the unit normal vector N yields

$$g(A\xi, N) = g(\cos(t)JZ_1 + \sin(t)Z_2, \cos(t)Z_1 + \sin(t)JZ_2) = 0,$$

which means the vector field $A\xi$ is tangent to M. Also, by using $JX = \phi X + \eta(X)N$ and $\eta(X) = g(X, \xi)$ for any $X \in TM$, we may put

$$AX = BX + g(AX, N)N \tag{2.5}$$

and

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N, \tag{2.6}$$

where BX and $-\phi A\xi$ are tangential parts of AX and AN, respectively. Using these notations, together with (2.2), and taking the tangential and normal components of (2.1), we obtain

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + g(AY,Z)BX - g(AX,Z)BY + g(SY,Z)SX - g(SX,Z)SY \\ &\quad + g(JAY,Z)\Big\{-B\phi X + \eta(X)\phi A\xi\Big\} - g(JAX,Z)\Big\{-B\phi Y + \eta(Y)\phi A\xi\Big\}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z) - 2\eta(Z)g(\phi X,Y) + g(X,AN)g(AY,Z) \\ &\quad - g(Y,AN)g(AX,Z) + g(X,A\xi)g(JAY,Z) - g(Y,A\xi)g(JAX,Z),\end{aligned} \tag{2.8}$$

which are called the equations of Gauss and Codazzi, respectively.

Now, we assume that M is a Hopf real hypersurface in the complex quadric. We say that M is *Hopf* if the Reeb vector field ξ of M is principal for the shape operator S, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. In particular, if the Reeb function $\alpha = g(S\xi, \xi)$ identically vanishes, we say that M has a vanishing geodesic Reeb flow. Otherwise, M has a non-vanishing geodesic Reeb flow. By virtue of the Codazzi equation (2.8), we obtain the following lemma.

Lemma 2.1 ([5]). *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. We have*

$$Y\alpha = (\xi\alpha)\eta(Y) + 2g(Y, AN)g(\xi, A\xi), \tag{2.9}$$

and

$$\begin{aligned} 0 = &\, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \end{aligned} \tag{2.10}$$

for any tangent vector fields X and Y on M.

Recall that there are two types of singular tangent vector fields for the complex quadric Q^m, one is given by $\mathfrak{A}$-principal vector fields and the other by $\mathfrak{A}$-isotropic vector fields. From such notions, we want to give two important lemmas.

First, if the unit normal vector field N of M in Q^m is $\mathfrak{A}$-principal, then we obtain that $AN = N$ and $A\xi = -\xi$. From these geometric properties we see that $AY \in T_pM$ for every tangent vector $Y \in T_{[z]}M$. Moreover, we assert:

Lemma 2.2. *Let M be a Hopf real hypersurface in Q^m with $\mathfrak{A}$-principal normal vector field N. Then following statements hold.*

(a) *α is constant on M.*
(b) *$\phi AX = -A\phi X$ for $X \in TM$.*
(c) *$q(X) = 2\alpha\eta(X)$ for $X \in TM$.*
(d) *$ASX = SAX = SX - 2\alpha\eta(X)\xi$ for $X \in TM$.*

Proof. Since the unit normal vector field N is $\mathfrak{A}$-principal, we get $A\xi = -\xi$ and $AN = N$. From this, (2.9) leads to

$$Y\alpha = (\xi\alpha)\eta(Y) \tag{2.11}$$

for any tangent vector field Y on M. This implies that $\mathrm{grad}\,\alpha = (\xi\alpha)\xi$, where $\mathrm{grad}\,\alpha$ denotes the gradient of the Reeb function α. By using the property of $g(\nabla_X \mathrm{grad}\,\alpha, Y) = g(\nabla_Y \mathrm{grad}\,\alpha, X)$, we obtain

$$\Big(X(\xi\alpha)\Big)\eta(Y) + (\xi\alpha)g(\phi SX, Y) = \Big(Y(\xi\alpha)\Big)\eta(X) + (\xi\alpha)g(\phi SY, X) \tag{2.12}$$

for any tangent vector fields X and Y on M. Putting $Y = \xi$ in (2.12) it follows $\Big(X(\xi\alpha)\Big) = \Big(\xi(\xi\alpha)\Big)\eta(X)$. From this, the equation (2.12) becomes

$$(\xi\alpha)g\Big((\phi S + S\phi)X, Y\Big) = 0.$$

On the other hand, in [12] Lee and Suh proved that there does not exist a real hypersurface with anti-commuting property, $S\phi + \phi S = 0$, in Q^m, $m \geq 3$. Thus, by virtue of this result we get $\xi\alpha = 0$ on M. From this, (2.11) implies that $Y\alpha = 0$ for any $Y \in TM$. So, this means that the Reeb function $\alpha = g(S\xi, \xi)$ is constant on M.

In order to show that $\phi AX = -A\phi X$, first we consider the equation $JAX = -AJX$ for any $X \in TM$. From $AN = N$, we obtain $AX \in TM$ for any $X \in TM$. Thus we get

$$JAX = \phi AX + \eta(AX)N = \phi AX - \eta(X)N,$$

where we have used $\eta(AX) = g(AX, \xi) = g(X, A\xi) = -g(X, \xi)$. Similarly, we see that $-AJX = -A\phi X - \eta(X)AN = -A\phi X - \eta(X)N$. Comparing these two equations yields the equality in (b).

We now prove (c). Since $g(A\xi, N) = 0$, $A\xi$ is tangent to M. Then by the Gauss formula, we get

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi)\\ &= q(X)JA\xi + A(\nabla_X\xi) + g(SX, \xi)AN - g(SX, A\xi)N,\end{aligned}$$

together with the covariant derivative of A with respect to the Riemann connection $\bar{\nabla}$ of Q^m given by $(\bar{\nabla}_X A)Y = q(X)JAY$. Taking the inner product with N, it becomes

$$q(X)g(A\xi, \xi) = -g(AN, \nabla_X\xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi). \tag{2.13}$$

Applying $g(AN, N) = 1$, $g(A\xi, \xi) = -1$ and $S\xi = \alpha\xi$, we get $q(X) = 2\alpha\eta(X)$, the end of the proof of (c).

Finally, we prove the equation (d). Since $AN = N$, we differentiate this equation with respect to a tangent vector field X. Then we see that

$$(\bar{\nabla}_X A)N + A\bar{\nabla}_X N = \bar{\nabla}_X N.$$

By using the Weingarten equation $\bar{\nabla}_X N = -SX$, we have

$$q(X)JAN - ASX = -SX.$$

From this, together with (c), it follows that

$$ASX = SX - 2\alpha\eta(X)\xi \tag{2.14}$$

for all tangent vector field X of M. Furthermore, taking the symmetric part of (2.14) implies $AS = SA$. It gives a complete proof of (d). □

Next, in the case of the normal vector field N is $\mathfrak{A}$-isotropic, we obtain $g(AN, N) = g(A\xi, \xi) = 0$. So, we can introduce the following lemma.

Lemma 2.3 ([12]). *Let M be a Hopf real hypersurface in Q^m with $\mathfrak{A}$-isotropic normal vector field. Then we have*

(a) *$\alpha = g(S\xi, \xi)$ is constant on M.*
(b) *$AN, A\xi \in TM$.*
(c) *$SAN = S\phi A\xi = SA\xi = 0$.*
(d) *If $X \in \mathcal{Q}$ is a principal curvature vector of M with $SX = \lambda X$, then $\alpha \neq 2\lambda$. Moreover, ϕX is a principal curvature vector whose principal curvature μ is given by $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ (that is, $A\phi X = \mu\phi X$).*

3. Reeb invariant structure Jacobi operator with $\mathfrak{A}$-principal normal vector field

In this section, we classify a real hypersurface M with Reeb invariant structure Jacobi operator in the complex quadric Q^m for $m \geq 3$, if N is $\mathfrak{A}$-principal.

First, we introduce the basic equation for structure Jacobi operator R_ξ of M. Putting $Y = Z = \xi$ in (2.7), the structure Jacobi operator R_ξ of M is given by

$$\begin{aligned}R_\xi Y &= R(Y, \xi)\xi\\ &= Y - \eta(Y)\xi + g(A\xi, \xi)BY - g(A\xi, Y)A\xi + g(AN, Y)\phi A\xi + \alpha SY - \alpha^2\eta(Y)\xi.\end{aligned} \tag{3.1}$$

Using this equation, we give the following lemma.

Lemma 3.1. *Let M be a Hopf hypersurface with Reeb invariant structure Jacobi operator in the complex quadric Q^m for $m \geq 3$. If M has $\mathfrak{A}$-principal unit normal vector field, then the Reeb function α identically vanishes on M. Moreover, $S\phi + \phi S = 0$ for the shape operator S of M in Q^m.*

Proof. Note that $(\mathcal{L}_\xi R_\xi)Y = 0$ is equivalent to $(\nabla_\xi R_\xi)Y = \phi S R_\xi Y - R_\xi \phi S Y$. Differentiating (3.1) along any direction X, it follows that

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) = & -g(Y, \nabla_X \xi)\eta(Z) - \eta(Y)g(Z, \nabla_X \xi) + g(A\phi SX, \xi)g(AY, Z) + g(A\xi, \nabla_X \xi)g(AY, Z) \\ & + g(A\xi, \xi)\Big\{q(X)g(JAY, Z) + g(SX, Y)g(AN, Z) + g(AN, Y)g(SX, Z)\Big\} \\ & - g(A\xi, Z)\Big\{g(A\phi SX, Y) + \alpha\eta(X)g(AN, Y)\Big\} - g(A\xi, Y)\Big\{g(A\phi SX, Z) + \alpha\eta(X)g(AN, Z)\Big\} \\ & + g(AN, Z)\Big\{g(AY, SX) + g(SX, Y)g(A\xi, \xi)\Big\} + g(AN, Y)\Big\{g(AZ, SX) + g(SX, Z)g(A\xi, \xi)\Big\} \\ & + (X\alpha)g(SY, Z) + \alpha g((\nabla_X S)Y, Z) - 2\alpha(X\alpha)\eta(Y)\eta(Z) - \alpha^2 g(Y, \nabla_X \xi)\eta(Z) - \alpha^2\eta(Y)g(Z, \nabla_X \xi), \end{aligned} \tag{3.2}$$

for any vector fields X, Y, and Z on M.

Inserting $X = \xi$ in (3.2) and using the assumption of M being Hopf, together with $\nabla_\xi \xi = \phi S \xi = 0$, we get

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = & \ g(A\xi, \xi)\Big\{q(\xi)g(JAY, Z) + g(S\xi, Y)g(AN, Z) + g(AN, Y)g(S\xi, Z)\Big\} \\ & - \alpha g(A\xi, Z)g(AN, Y) - \alpha g(A\xi, Y)g(AN, Z) \\ & + g(AN, Z)\Big\{\alpha g(AY, \xi) + \alpha\eta(Y)g(A\xi, \xi)\Big\} \\ & + g(AN, Y)\Big\{\alpha g(AZ, \xi) + \alpha\eta(Z)g(A\xi, \xi)\Big\} \\ & + (\xi\alpha)g(SY, Z) + \alpha g((\nabla_\xi S)Y, Z) - 2\alpha(\xi\alpha)\eta(Y)\eta(Z). \end{aligned} \tag{3.3}$$

On the other hand, by the equation (3.1), $\phi S R_\xi Y - R_\xi \phi S Y$ is given by

$$\begin{aligned} \phi S R_\xi Y - R_\xi \phi S Y = & \ g(A\xi, \xi)\phi SBY - g(A\xi, Y)\phi SA\xi + g(AN, Y)\phi S\phi A\xi + \alpha\phi S^2 Y \\ & - g(A\xi, \xi)B\phi SY - g(\phi A\xi, SY)A\xi - g(AN, \phi SY)\phi A\xi - \alpha S\phi SY. \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4), the Reeb invariant structure Jacobi operator R_ξ can be arranged as

$$\begin{aligned} & g(A\xi, \xi)\Big\{q(\xi)g(JAY, Z) + g(S\xi, Y)g(AN, Z) + g(AN, Y)g(S\xi, Z)\Big\} \\ & \quad + g(AN, Z)\Big\{\alpha g(AY, \xi) + \alpha\eta(Y)g(A\xi, \xi)\Big\} + g(AN, Y)\Big\{\alpha g(AZ, \xi) + \alpha\eta(Z)g(A\xi, \xi)\Big\} \\ & \quad - \alpha g(A\xi, Z)g(AN, Y) - \alpha g(A\xi, Y)g(AN, Z) + (\xi\alpha)g(SY, Z) + \alpha g((\nabla_\xi S)Y, Z) - 2\alpha(\xi\alpha)\eta(Y)\eta(Z) \\ & = g((\nabla_\xi R_\xi)Y, Z) \\ & = g(\phi S R_\xi Y - R_\xi \phi S Y, Z) \\ & = g(A\xi, \xi)g(\phi SBY, Z) - g(A\xi, Y)g(\phi SA\xi, Z) + g(AN, Y)g(\phi S\phi A\xi, Z) + \alpha g(\phi S^2 Y, Z) \\ & \quad - g(A\xi, \xi)g(B\phi SY, Z) - g(\phi A\xi, SY)g(A\xi, Z) - g(AN, \phi SY)g(\phi A\xi, Z) - \alpha g(S\phi SY, Z) \end{aligned} \tag{3.5}$$

for any tangent vector fields Y and Z on M.

Now, by our assumption of N being $\mathfrak{A}$-principal, $A\xi = -\xi$ and $AN = N$, (3.5) gives

$$-q(\xi)\phi AY + (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - 2\alpha(\xi\alpha)\eta(Y)\xi = -\phi SBY + \alpha\phi S^2 Y + B\phi SY - \alpha S\phi SY,$$

where we have used

$$AY = BY + g(AY, N)N = BY + g(Y, AN)N = BY + g(Y, N)N = BY \in TM$$

and

$$JAY = \phi AY + g(AY,\xi)N = \phi AY - g(Y,\xi)N = \phi AY - \eta(Y)N.$$

When the normal vector field N of M is $\mathfrak{A}$-principal, from Lemma 2.2, we see that the Reeb function $\alpha = g(S\xi,\xi)$ is constant on M. So, it leads to

$$-q(\xi)\phi AY + \alpha(\nabla_\xi S)Y = -\phi SBY + \alpha\phi S^2 Y + B\phi SY - \alpha S\phi SY.$$

Moreover, from Lemma 2.2 it can be rewritten as

$$-2\alpha\phi AY + \alpha(\nabla_\xi S)Y = -\phi SAY + \alpha\phi S^2 Y - \phi ASY - \alpha S\phi SY. \tag{3.6}$$

By means of the Codazzi equation, we obtain

$$(\nabla_\xi S)Y = \alpha\phi SY - S\phi SY + \phi Y - \phi AY.$$

From this, (3.6) yields

$$-3\alpha\phi AY + \alpha^2\phi SY + \alpha\phi Y = -\phi SAY + \alpha\phi S^2 Y - \phi ASY. \tag{3.7}$$

Applying the structure tensor ϕ and using $\phi^2 Y = -Y + \eta(Y)\xi$, (3.7) becomes

$$3\alpha AY - \alpha^2 SY + 2\alpha\eta(Y)\xi - \alpha Y - SAY + \alpha S^2 Y - ASY = 0 \tag{3.8}$$

for any tangent vector field Y on M.

Restricting $Y \in \mathcal{C} := \{Y \in TM \mid Y \perp \xi\}$ and using Lemma 2.2, (3.8) becomes

$$3\alpha AY - \alpha^2 SY - \alpha Y - 2SY + \alpha S^2 Y = 0. \tag{3.9}$$

Applying the complex conjugation A to (3.9) and using $A^2 = I$, it yields that

$$3\alpha Y - \alpha^2 SY - \alpha AY - 2SY + \alpha S^2 Y = 0 \tag{3.10}$$

where we have use

$$ASY = SY - 2\alpha\eta(Y)\xi = SY$$

and

$$AS^2 Y = S^2 Y - 2\alpha\eta(SY)\xi = S^2 Y - 2\alpha^2\eta(Y)\xi = S^2 Y$$

for any $Y \in \mathcal{C}$. Subtracting (3.10) from (3.9), we get that for any $Y \in \mathcal{C}$

$$4\alpha(AY - Y) = 0. \tag{3.11}$$

Assume $\alpha \neq 0$. Then (3.11) gives us $AY = Y$ for any $Y \in \mathcal{C}$. From this, the trace $\mathrm{Tr}(A)$ of a real structure A of Q^m is given by

$$\begin{aligned} \mathrm{Tr}(A) &= \sum_{i=1}^{2m} g(Ae_i, e_i) \\ &= g(AN,N) + g(A\xi,\xi) + \sum_{i=1}^{2m-2} g(Ae_i, e_i) \\ &= 2m-2, \end{aligned} \tag{3.12}$$

where $\{e_1, e_2, \cdots, e_{2m-2}, e_{2m-1} = \xi, e_{2m} = N\}$ is an orthonormal basis of TQ^m. Then it makes a contradiction. In fact, it is known that the trace $\mathrm{Tr}(A)$ of real structure A of Q^m is zero, that is, $\mathrm{Tr}A = 0$. From this and (3.12), we obtain $m = 1$. But we only consider $m \geq 3$ for Q^m in this paper.

Therefore, we see that the Reeb function α identically vanishes on M. Then the equation (3.8) becomes

$$-SAY - ASY = 0,$$

for any tangent vector field Y on M. By using Lemma 2.2, this implies $SAY = 0$, thus we conclude that

$$SY = 2\alpha\eta(Y)\xi$$

for any tangent vector field Y on M. It means that M is a totally geodesic real hypersurface with $S\phi + \phi S = 0$, which completes our proof of Lemma 3.1. □

On the other hand, in [12] Lee and Suh considered the classification problem for real hypersurfaces in the complex quadric Q^m satisfying the property $S\phi + \phi S = 0$ (which is called *anti-commuting* shape operator) and gave a non-existence theorem as follows:

Theorem 3.2. *There does not exist any real hypersurface with anti-commuting shape operator in the complex quadric Q^m for $m \geq 3$.*

Summing up Lemma 3.1 and Theorem 3.2, we give a complete proof of our Theorem 1.2 in the introduction.

4. Reeb invariant structure Jacobi operator with $\mathfrak{A}$-isotropic normal vector field

In this section, we classify Hopf real hypersurfaces with Reeb invariant structure Jacobi operator and $\mathfrak{A}$-isotropic normal vector field. First we prove the following Lemma.

Lemma 4.1. *Let M be a real hypersurface with Reeb invariant structure Jacobi operator and non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. If the normal vector field N is $\mathfrak{A}$-isotropic, then the Reeb flow of M is isometric.*

Proof. From our assumption of the unit normal vector N being $\mathfrak{A}$-isotropic, we see that $g(A\xi, \xi) = 0$. From this and using (3.3) and (3.4), we get

$$\alpha(\nabla_\xi S)Y = \alpha\phi S^2 Y - \alpha S\phi SY, \tag{4.1}$$

for any tangent vector field Y on M. On the other hand, the equation of Codazzi gives us

$$(\nabla_\xi S)Y = \alpha\phi SY - S\phi SY + \phi Y - g(AN, Y)A\xi + g(A\xi, Y)AN.$$

From this and (4.1), we obtain

$$\alpha^2\phi SY + \alpha\phi Y - \alpha g(AN, Y)A\xi + \alpha g(A\xi, Y)AN = \alpha\phi S^2 Y. \tag{4.2}$$

Taking the symmetric part of (4.2), we get

$$-\alpha^2 S\phi Y - \alpha\phi Y - \alpha g(A\xi, Y)AN + g(AN, Y)A\xi = -\alpha S^2\phi Y. \tag{4.3}$$

Comparing (4.2) and (4.3), we obtain

$$\alpha^2(\phi S - S\phi)Y = \alpha(\phi S^2 - S^2\phi)Y \tag{4.4}$$

for any tangent vector field Y on M.

Bearing in mind Lemma 2.3 in section 2, we may take a vector field U in $\mathcal{Q}$ such that $SU = \lambda U$. Then we see that $\alpha \neq 2\lambda$, and $S\phi U = \mu\phi U$, where $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then, for such vector field $U \in \mathcal{Q}$, together with $\alpha \neq 0$, (4.4) provides

$$\alpha(\lambda - \mu)\phi U = (\lambda^2 - \mu^2)\phi U. \tag{4.5}$$

Taking the inner product of (4.5) with ϕU, we get

$$(\lambda-\mu)(\lambda+\mu-\alpha)=0. \tag{4.6}$$

Suppose $(\lambda+\mu-\alpha)=0$. Since $\mu=\frac{\alpha\lambda+2}{2\lambda-\alpha}$, we obtain a quadratic equation for λ given by

$$2\lambda^2-2\alpha\lambda+\alpha^2+2=0.$$

By using the discriminant about the roots of the quadratic equation, $D/4=\alpha^2-2(\alpha^2+2)=-(\alpha^2+4)<0$, we can see that this equation has imaginary roots, which makes a contradiction.

From this and (4.6), we assert that $\lambda=\mu$. By means of this discussion, we can take an orthonormal basis $\{\xi, A\xi, \phi A\xi, e_1, e_2, \cdots, e_{m-2}, \phi e_1, \phi e_2, \cdots, \phi e_{m-2}\}$ for $T_{[z]}M$ at $[z]\in M$ so that

$$S=\operatorname{diag}(\alpha,0,0,\lambda_1,\lambda_2,\cdots,\lambda_{m-2},\lambda_1,\lambda_2,\cdots,\lambda_{m-2}),$$

that is, $Se_i=\lambda_i e_i$, $(i=1,\cdots,m-2)$ and $S\phi e_i=\mu_i\phi e_i=\lambda_i\phi e_i$, $(i=1,\cdots,m-2)$. For such orthonormal basis, any tangent vector field X of TM can be expressed by

$$X=g(X,\xi)\xi+g(X,A\xi)A\xi+g(X,\phi A\xi)\phi A\xi+\sum_{i=1}^{m-2}g(X,e_i)e_i+\sum_{i=1}^{m-2}g(X,\phi e_i)\phi e_i. \tag{4.7}$$

Taking two skew-symmetric tensors $S\phi$ and ϕS for (4.7), we obtain respectively

$$\begin{aligned}S\phi X&=\sum_{i=1}^{m-2}g(X,e_i)S\phi e_i+\sum_{i=1}^{m-2}g(X,\phi e_i)S\phi^2 e_i\\&=\sum_{i=1}^{m-2}g(X,e_i)S\phi e_i-\sum_{i=1}^{m-2}g(X,\phi e_i)Se_i\\&=\sum_{i=1}^{m-2}g(X,e_i)\lambda_i\phi e_i-\sum_{i=1}^{m-2}g(X,\phi e_i)\lambda_i e_i\end{aligned}$$

and

$$\begin{aligned}\phi SX&=\sum_{i=1}^{m-2}g(X,e_i)\phi Se_i+\sum_{i=1}^{m-2}g(X,\phi e_i)\phi S\phi e_i\\&=\sum_{i=1}^{m-2}g(X,e_i)\lambda_i\phi e_i-\sum_{i=1}^{m-2}g(X,\phi e_i)\lambda_i\phi^2 e_i\\&=\sum_{i=1}^{m-2}g(X,e_i)\lambda_i\phi e_i-\sum_{i=1}^{m-2}g(X,\phi e_i)\lambda_i e_i,\end{aligned}$$

where we have used $SA\xi=S\phi A\xi=0$ and $\phi^2 e_i=-e_i$, $i=1,2,\cdots,m-2$. From these two equations, we see that the shape operator S commutes with the structure tensor ϕ, that is, $S\phi=\phi S$. This means that the Reeb flow of M is isometric. □

On the other hand, Berndt and Suh proved:

Theorem 4.2 ([5]). *Let M be a real hypersurface of the complex quadric Q^m, $m\geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m=2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.*

By virtue of Theorem 4.2 and Lemma 4.1 we assert that *a Hopf real hypersurface M satisfying the conditions given in Lemma 4.1 is locally congruent to a tube $(\mathcal{T}_A)$ over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.*

Now, let us check the structure Jacobi operator R_ξ of the tube $(\mathcal{T}_A)$ satisfies Reeb invariance (*). That is, we check that the equation

$$(\nabla_\xi R_\xi) = \phi S R_\xi - R_\xi \phi S \tag{4.8}$$

holds on $(\mathcal{T}_A)$. In order to do this, we introduce more detailed information about the model space of type $(\mathcal{T}_A)$ as follows:

Proposition 4.3 ([5]). *Let $(\mathcal{T}_A)$ be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$ in Q^{2k}. Then the following statements hold:*

(i) *$(\mathcal{T}_A)$ is a Hopf hypersurface.*

(ii) *Every unit normal vector N of $(\mathcal{T}_A)$ is $\mathfrak{A}$-isotropic and therefore can be written in the form $N = (Z_1 + JZ_2)/\sqrt{2}$ for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $A \in \mathfrak{A}$.*

(iii) *$(\mathcal{T}_A)$ has four distinct constant principal curvatures. Their values and corresponding eigenspaces and multiplicities are given the following Table.*

principal curvature	eigenspace	multiplicity
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1
$\beta = 0$	$T_\beta = \mathbb{C}(JZ_1 + Z_2)$	2
$\lambda = -\tan(r)$	$T_\lambda = T\mathbb{C}P^k \ominus \mathbb{C}(JZ_1 + Z_2)$	$2k-2$
$\mu = \cot(r)$	$T_\mu = \nu\mathbb{C}P^k \ominus \mathbb{C}N$	$2k-2$

Here, $T\mathbb{C}P^k$ and $\nu\mathbb{C}P^k$ denote the tangent and normal bundles of $\mathbb{C}P^k$, respectively. Moreover, we have $A(T\mathbb{C}P^k \ominus \mathbb{C}(JZ_1 + Z_2)) = \nu\mathbb{C}P^k \ominus \mathbb{C}N$.

(iv) *$S\phi = \phi S$ (isometric Reeb flow).*

Bearing in mind Proposition 4.3, by using (3.3) and the Codazzi equation the left side of (4.8) is given by

$$\begin{aligned}(L.S.) &= (\nabla_\xi R_\xi)Y = \alpha(\nabla_\xi S)Y \\ &= \alpha^2 \phi SY + \alpha\phi Y - \alpha g(AN, Y)A\xi + \alpha g(A\xi, Y)AN, \\ &= \begin{cases} 0, & Y \in T_\alpha \oplus T_\beta \\ \alpha(\alpha\lambda + 1)\phi Y, & Y \in T_\lambda \\ \alpha(\alpha\mu + 1)\phi Y, & Y \in T_\mu, \end{cases}\end{aligned}$$

On the other hand, by using (3.4) the right side of (4.8) becomes

$$\begin{aligned}(R.S.) &= \phi S R_\xi Y - R_\xi \phi S Y \\ &= \alpha\phi S^2 Y \\ &= \begin{cases} 0, & Y \in T_\alpha \oplus T_\beta \\ \alpha\lambda^2 \phi Y, & Y \in T_\lambda \\ \alpha\mu^2 \phi Y, & Y \in T_\mu. \end{cases}\end{aligned}$$

By virtue of Proposition 4.3, we get $\alpha\lambda + 1 - \lambda^2 = 0$ and $\alpha\mu + 1 - \mu^2 = 0$, which implies that $\alpha(\alpha\lambda + 1) = \alpha\lambda^2$ and $\alpha(\alpha\mu + 1) = \alpha\mu^2$. So, we assert that the structure Jacobi operator R_ξ of a real hypersurface of type $(\mathcal{T}_A)$ is Reeb-invariant. Therefore, we complete the proof of our Theorem 1.3.

Acknowledgements The present authors would like to express their deep gratitude to Professors Young Jin Suh (Kyungpook Nat'l University, Korea) and Juan de Dios Pérez (University of Granada, Spain) for their valuable suggestions and comments to develop this research article.

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