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On Distance Dominator Packing Coloring in Graphs

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Abstract. Let G be a graph and let $S = (s_1, s_2, \dots, s_k)$ be a non-decreasing sequence of positive integers. An *S-packing coloring* of G is a mapping $c : V(G) \to \{1, 2, \dots, k\}$ with the following property: if $c(u) = c(v) = i$, then $d(u, v) > s_i$ for any $i \in \{1, 2, \dots, k\}$. In particular, if $S = (1, 2, 3, \dots, k)$, then S-packing coloring of G is well known under the name *packing coloring*. Next, let r be a positive integer and $u, v \in V(G)$. A vertex u *r-distance dominates* a vertex v if $d_G(u, v) \leq r$.

In this paper, we present a new concept of a coloring, namely *distance dominator packing coloring*, defined as follows. A coloring c is a *distance dominator packing coloring* of G if it is a packing coloring of G and for each $x \in V(G)$ there exists $i \in \{1, 2, 3, \dots\}$ such that x i-distance dominates each vertex from the color class of color i. The smallest integer k such that there exists a distance dominator packing coloring of G using k colors, is the *distance dominator packing chromatic number* of G, denoted by $\chi_\rho^d(G)$. In this paper, we provide some lower and upper bounds on the distance dominator packing chromatic number, characterize graphs G with $\chi_\rho^d(G) \in \{2, 3\}$, and provide the exact values of $\chi_\rho^d(G)$ when G is a complete graph, a star, a wheel, a cycle or a path. In addition, we consider the relation between $\chi_\rho(G)$ and $\chi_\rho^d(G)$ for a graph G.

1. Introduction

The wide interest given to the concept of graph coloring is reflected in many variants of colorings derived from the classical coloring (for example: total coloring, fractional coloring, S-packing coloring, b-coloring, equitable coloring, partitioned coloring, dominator coloring, etc). While some variants of graph coloring consider only a specific property of coloring as a function, the others combine colorings with some other well known concepts. For example, such a type of coloring is a *dominator coloring* of a graph, which combines the concepts of coloring and domination. Namely, it is defined as a coloring of a graph with the property that each vertex of the graph dominates all vertices of at least one color class (see [3, 7, 8, 15, 16]). Since dominator coloring involves coloring and domination, it can be generalized from the perspective of domination and also from the perspective of coloring. For instance, *distance r-dominator coloring* generalizes dominator coloring in a way that uses r-distance dominating sets instead of standard dominating sets, and r-distance coloring instead of classical coloring (see [12]). Recall that a vertex u r-distance dominates a vertex v if $d_G(u, v) \leq r$. Next, recall that r-distance coloring is a coloring with the property that any two vertices of any color class are at distance more than r.

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In this paper, we present another generalization of dominator coloring, which includes also distance r-dominator coloring. Namely, we use the concept of S-packing coloring instead of r-distance coloring (and also instead of classical coloring, since 1-distance coloring is equivalent to classical coloring). Recall that for a given graph G and a given non-decreasing sequence of positive integers $S = (s_1, s_2, \dots, s_k)$, a *$S$-packing coloring* of G is a mapping $c : V(G) \to \{1, 2, \dots, k\}$ with the following property: if $c(u) = c(v) = i$, then $d(u, v) > s_i$ for any $i \in \{1, 2, \dots, k\}$. If $S = (1, 2, 3, \dots, k)$, then S-packing coloring of G is called *packing coloring*. Note that S-packing coloring, where $S = (1, 1, 1, \dots)$, coincides with classical coloring, and S-packing coloring, where $S = (r, r, r, \dots)$, is distance r-dominator coloring. We have to mention that the concept of S-packing coloring is used, since it presents a wide generalization of several variations of colorings (we have already mentioned a classical coloring and a distance coloring, but there are also several others such as a packing coloring). In addition, it has a very wide spectrum of possible applications, such as frequency assignments [9], applications in resource placements and biological diversity [1]. Moreover, the concept of S-packing coloring is the subject in a number of papers (e.g., [4, 6, 10, 11]), especially packing coloring (note that there is probably a non-exhaustive list of papers on packing coloring problem that were published only in the last two years, see e.g. [2, 5, 13, 14, 17]). Based on this generalization of coloring, we present a new concept of a *distance dominator S-packing coloring*. It is defined as follows. For a given non-decreasing sequence of positive integers $S = (s_1, s_2, \dots, s_k)$, a coloring c is a distance dominator S-packing coloring of G if it is an S-packing coloring of G, and in addition, each vertex from G s_i-distance dominates all vertices colored with the color i for some $i \in \{1, 2, \dots, k\}$ (note that, if there is only one vertex colored with some color i, then it s_i-distance dominates all vertices colored with i, actually only itself).

Note that the concept of dominator coloring coincides with distance dominator S-packing coloring, where $S = (1, 1, 1, \dots)$, and distance dominator S-packing coloring is equivalent to distance r-dominator coloring if $S = (r, r, r, \dots)$. Recall that S-packing coloring, when $S = (1, 2, 3, \dots)$, is called packing coloring. In this case, distance dominator S-packing coloring will be shortly called *distance dominator packing coloring*, and it will be the subject of our study. We believe that the new concept could have several possible applications, which are derived from the applications of packing coloring and distance domination. For instance, some companies have their offices located sufficiently far apart (in order to optimally cover an area), but on the other hand, the company want to have all offices of some competitive company in the vicinity such that it can check their work in order to be better as they are.

This paper is organized as follows. In the next section, we establish the notation and define the concepts used throughout the paper. We provide some lower and upper bounds on the distance dominator packing chromatic number and prove that the invariant is not hereditary (in the sense that a graph cannot have smaller distance dominator packing chromatic number than its subgraphs) in general. In Section 3, we establish characterizations of graphs with distance dominator packing chromatic number 2 or 3. Next, we consider a relation between the packing chromatic number and the distance dominator packing chromatic number of a given graph. We provide some properties of graphs with equal distance dominator packing chromatic number and packing chromatic number, but on the other hand, we prove that for any positive integer k there exists a graph G such that $\chi_\rho^d(G) - \chi_\rho(G) = k$. Finally, we provide the exact values of the distance dominator packing chromatic numbers of complete graphs, stars, wheels, paths and cycles. We end this paper with some remarks and open problems.

2. Notations and preliminaries

In this paper, we consider only finite, simple graphs. For a given graph G, we denote its vertex set by $V(G)$ and the set of its edges by $E(G)$. The *(open) neighborhood* of an arbitrary vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of all vertices adjacent to v. The number of elements in $N_G(v)$, $|N_G(v)|$, is called the *degree* of v and is denoted by $deg_G(v)$. If $deg_G(v) = 1$, then we say that v is a *leaf*. Further, the *distance* between the vertices $u, v \in V(G)$, $d_G(u, v)$, is the length of the shortest u, v-path in G. The *eccentricity* of a vertex v, $\epsilon_G(v)$, is the maximum distance between v and any other vertex of G: $\epsilon_G(v) = \max_{u \in V(G)}\{d(v, u)\}$. Next, the *diameter* of G, denoted by diam(G), is the maximum eccentricity. Note that the subscript in some of the above notations may be omitted if the graph G is clear from context.

Let $u, v \in V(G)$. A vertex u *dominates a vertex* v, if u and v are adjacent. In other words, u dominates v if $d_G(u, v) = 1$. A *dominating set* of a given graph G is a subset D of $V(G)$ such that each $x \in V(G) \setminus D$ is dominated by at least one vertex from D (is adjacent to at least one vertex from D). The cardinality of the smallest dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The described concept of domination in graphs can be generalized as follows. Let r be a positive integer and $u, v \in V(G)$. Then, we say that a vertex u *r-distance dominates* a vertex v if $d_G(u, v) \leq r$. Note that the concept of domination is equivalent to 1-distance domination.

A *proper k-coloring* of G (or shorter, k-coloring) is a mapping $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$, such that $c(u) \neq c(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest integer k such that there exists a k-coloring of G is called the *chromatic number of* G and is denoted by $\chi(G)$. The wide interest given to this concept is reflected in many graph invariants derived from the chromatic number. One of them is S-coloring, defined as follows. Let $S = (s_1, s_2, \ldots, s_k)$ be a non-decreasing sequence of positive integers. An *S-packing coloring* of a given graph G is a mapping $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ which satisfies the following property: if $c(u) = c(v) = i$, then $d(u, v) > s_i$ for any $i \in \{1, 2, \ldots, k\}$. Recall that the before defined concept of (classical) coloring coincides with S-packing coloring, if $S = (1, 1, 1, \ldots)$.

Moreover, the S-packing coloring with $S = (1, 2, 3, \ldots)$ is well known under the name *packing coloring*. The smallest integer k such that there exists an S-packing coloring of G, where $S = (1, 2, \ldots, k)$, is called the *packing chromatic number* of G and is denoted by $\chi_\rho(G)$.

A *dominator coloring* of a given graph G is a coloring of the vertices of G such that each vertex of the graph dominates all vertices of at least one color class (recall that the color class of the color i is the set all vertices colored with a color i). By the generalization of this concept from the perspectives of domination and coloring, we get a *distance dominator S-packing coloring* defined as follows. Let $S = (s_1, s_2, \ldots, s_k)$ be a non-decreasing sequence of positive integers. A coloring c is a distance dominator S-packing coloring of G if it is an S-packing coloring of G, and in addition, each vertex from G s_i-distance dominates all vertices colored with a color i for some $i \in \{1, 2, \ldots, k\}$. In particular, if $S = (1, 2, 3, \ldots)$, then such a coloring is called a *distance dominator packing coloring*. For any $i \in \{1, 2, \ldots, k\}$ and a given distance dominator packing coloring c of G, we denote by B_i the color class of color i, i.e. the set of all vertices which receive a color i by c ($B_i = c^{-1}(i)$). So, in other words, c is a distance dominator packing coloring of G if it is a packing coloring and for each $x \in V(G)$ there exists $i \in \{1, 2, 3, \ldots\}$ such that x i-distance dominates each vertex from B_i. The smallest integer d for which there exists a distance dominator packing coloring of a given graph G using d colors (d-distance dominator packing coloring), is called the *distance dominator packing chromatic number* of G and is denoted by $\chi_\rho^d(G)$.

Clearly, every distance dominator packing coloring of a given graph is also a packing coloring, which implies the following proposition.

Proposition 2.1. *Let G be a graph. Then,*

$$\chi_\rho^d(G) \geq \chi_\rho(G).$$

While the written proposition gives us the lower bound on the distance dominator packing chromatic number of a given graph, with the following two propositions, we provide also some upper bounds on this number.

Proposition 2.2. *For any graph G,* $\chi_\rho^d(G) \leq \chi_\rho(G) + \gamma(G)$.

Proof. In order to prove the statement, we construct a $(\chi_\rho(G) + \gamma(G))$-distance dominator packing coloring of G. Let $D = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a smallest dominating set of G and let c be any optimal packing coloring of G. Then, by setting $c'(u) = c(u)$ for any $u \in V(G) \setminus D$, and $c'(u_i) = \chi_\rho(G) + i$ for any $i \in \{1, 2, \ldots, \gamma(G)\}$, we form a distance dominator packing coloring c' of G. Namely, each vertex u_i distance dominates the color class of color $\chi_\rho(G) + i$. Furthermore, since D is a dominating set of G, each vertex $u \in V(G) \setminus D$ is adjacent to some vertex $u_i \in D$ and hence u distance dominates the color class of color $\chi_\rho(G) + i$. Clearly, since c' is a packing coloring of G, the statement holds. □

Proposition 2.3. *For any connected graph G of order at least* 2, $\chi_\rho^d(G) \le |V(G)| - \alpha(G) + 1$. *In addition, the bound is sharp.*

Proof. Let G be a connected graph of order at least 2. In order to prove that $\chi_\rho^d(G) \le |V(G)| - \alpha(G) + 1$, we construct a $(|V(G)| - \alpha(G) + 1)$-distance dominator packing coloring c of G. Let A be an independent set of G such that $|A| = \alpha(G)$. Then, set $c(a) = 1$ for any vertex $a \in A$ and let the vertices from $V(G) \setminus A$ receive pairwise different colors from $\{2, 3, \ldots, |V(G)| - \alpha(G) + 1\}$ by c. Clearly, this is a packing coloring of G. Moreover, each vertex $u \in V(G) \setminus A$ distance dominates the color class of color $c(u)$. Next, since G is a connected graph, each vertex a from A is adjacent to at least one vertex $u \in V(G) \setminus A$ colored with unique color $c(u)$, which implies that a distance dominates the color class of color $c(u)$.

Finally, in order to prove that the bound is sharp, consider an arbitrary graph G with diameter 2. Namely, by [9, Proposition 2.1], $\chi_\rho(G) = |V(G)| - \alpha(G) + 1$, and by Corollary 4.2, $\chi_\rho^d(G) = \chi_\rho(G)$. This completes the proof. □

Next, recall that the invariant of packing chromatic number is hereditary in the sense that a graph cannot have smaller packing chromatic number than its subgraphs. Since every distance dominator packing coloring of a given graph is also its packing coloring, there arises a question of whether also the distance dominator packing chromatic number is hereditary. As we will see, this is not the case. Indeed, consider the cycle C_8 and the path P_8. As we will see in the sequel of this paper (see Theorems 5.2 and 5.3), $\chi_\rho^d(C_8) = 3$ and $\chi_\rho^d(P_8) = 4$. Since P_8 is a subgraph of C_8, our claim holds. But then, there arises a question of whether $\chi_\rho^d(H) \le \chi_\rho^d(G)$ for some family of subgraphs H of a given graph G. Recall that a graph H is an *isometric subgraph* of a given graph G, if for any two vertices $u, v \in V(H)$, $d_H(u, v) = d_G(u, v)$. We will show that a graph cannot have smaller distance dominator packing chromatic number than its isometric subgraphs.

Proposition 2.4. *Let G be a graph. If H is an isometric subgraph of G, then* $\chi_\rho^d(H) \le \chi_\rho^d(G)$.

Proof. Let H be an isometric subgraph of a given graph G, c an arbitrary optimal distance dominator packing coloring of G and x an arbitrary vertex of H. Then, in G x distance dominates all vertices of one color class, say B. Using the fact that $d_H(x, y) = d_G(x, y)$ for any $y \in V(H)$, we infer that x distance dominates all vertices from color class B also in subgraph H. Thus, c is a distance dominator packing coloring of H and the claim holds. □

3. Characterizations of graphs with small distance dominator packing chromatic numbers

In this section, we characterize graphs with small distance dominator packing chromatic numbers, namely graphs G with $\chi_\rho^d(G) \in \{2, 3\}$. Several times we use the fact that every distance dominator packing coloring of a given graph is also its packing coloring (namely that $\chi_\rho(G) \le \chi_\rho^d(G)$ holds for any graph G) and the results of Goddard et al. [9], which characterize graphs G with $\chi_\rho(G) \in \{2, 3\}$.

With the following theorem we prove that connected graphs with distance dominator packing chromatic number 2 are exactly stars.

Theorem 3.1. *For any connected graph G,* $\chi_\rho^d(G) = 2$ *if and only if G is a star.*

Proof. First, let G be a star $K_{1,n}$, $n \ge 1$. If $n \ge 2$, then color all its leaves with color 1 and the other vertex with color 2. Otherwise, color one vertex with color 1 and the other with color 2. In this way, 2-distance dominator packing coloring of G is formed, hence $\chi_\rho^d(G) \le 2$. Further, using the fact that $\chi_\rho(G) = 2$ [9, Proposition 3.1] and Proposition 2.1 we infer the result.

Next, suppose that G is a connected graph such that $\chi_\rho^d(G) = 2$. By Proposition 2.1, $\chi_\rho(G) \le 2$. Using characterizations of graphs with packing chromatic numbers 1 or 2 [9, Proposition 3.1], we derive that G is a complete graph K_1 or a star. Since $\chi_\rho^d(K_1) = 1$ and $\chi_\rho^d(K_{1,n}) = 2$ for any $n \ge 1$, the result follows. □

Before characterizing graphs G with $\chi_\rho^d(G) = 3$, recall the characterization of graphs G with $\chi_\rho(G) = 3$, which was proven by Goddard and co-authors [9].

Proposition 3.2. *[9] Let G be a graph. Then $\chi_\rho(G) = 3$ if and only if G can be formed by taking some bipartite multigraph H with bipartition (V_2, V_3), subdividing every edge exactly once, adding leaves to some vertices in $V_2 \cup V_3$, and then performing a single T-add to some vertices in V_3.*

Note that *T-add to a vertex v* is defined as follows. First, we take a vertex w_v and a set X_v of independent vertices, then add an edge between vertices v and w_v, and finally add some of the edges between $\{v, w_v\}$ and X_v.

Theorem 3.3. *For any connected graph G, $\chi^d_\rho(G) = 3$ if and only if one of the following four possibilities holds for G.*

1. *G can be formed by taking bipartite multigraph $K_{1,2}$ with bipartition (V_2, V_3), subdividing every edge exactly once, adding leaves to some vertices in $V_2 \cup V_3$, and then performing a single T-add to a vertex in V_3.*
2. *G can be formed by taking bipartite multigraph $K_{i,j}$, $i, j \geq 2$, with bipartition (V_2, V_3), subdividing every edge exactly once and adding leaves to some vertices in V_2.*
3. *G is a subgraph of a graph from 1 and contains P_4, K_3 or C_4 as an induced subgraph.*
4. *G is an isometric subgraph of a graph from 2 which contains P_4, K_3 or C_4 as an induced subgraph.*

Proof. First, suppose that G is a connected graph such that $\chi^d_\rho(G) = 3$. By Proposition 2.1, $\chi_\rho(G) \leq 3$, which implies that G is a complete graph of order 1, a star or a graph with $\chi_\rho(G) = 3$. Clearly, since $\chi^d_\rho(K_1) = 1$ and $\chi^d_\rho(K_{1,n}) = 2$ for any $n \geq 1$, we have $\chi_\rho(G) = 3$, which means that G satisfies condition from Proposition 3.2. Since every distance dominator packing coloring of a graph G is also packing coloring of G (but not necessarily optimal), the graphs that satisfy $\chi^d_\rho(G) = 3$ are exactly the graphs from the characterization of graphs that satisfy $\chi_\rho(G) = 3$, for which there exists a distance dominator packing coloring with three colors. Thus, we are looking for graphs with $\chi_\rho(G) = 3$, described in Proposition 3.2, for which there exists a distance dominator packing coloring with three colors.

First, consider the case when G is a graph formed as is described in Proposition 3.2 for which $H = K_{1,2}$ and with single T-add. It is easy to check that the packing coloring shown in the Figure 1 is unique for G, and that it is also the distance dominator packing coloring for G. Therefore, $\chi^d_\rho(G) = 3$.

Further, let G' be formed as is described in Proposition 3.2 for which $H = K_{i,j}$, $i, j \geq 2$. We observe that there exists only one packing coloring c of G' using 3 colors: all vertices in V_2 and each vertex w_v that belongs to T-add of some vertex v, receive color 2, all vertices in V_3 color 3, and all the remaining vertices color 1. Clearly, c is not a distance dominator packing coloring of G' since vertices in T-adds and leaves added to vertices in V_3 do not distance dominate any color class. Hence, in order to get a graph G from G' with $\chi^d_\rho(G) = 3$, we remove some vertices from G'. Clearly, the deleted vertices are those which are problematic, namely vertices in T-adds and leaves added to vertices in V_3. In this way, we obtain a graph G described in 2. Coloring $c|_G$ is a distance dominator packing coloring for G using three colors. Thus, $\chi^d_\rho(G) \leq 3$. Clearly, G contains P_4 as an induced subgraph, so we conclude that $\chi^d_\rho(G) = 3$.

Next, consider the family of proper subgraphs of graphs which are described in Proposition 3.2. First, supppose that G is a proper subgraph of a graph described in 1. Using Theorem 3.1, we infer that G contains P_4, C_4 or K_3 as an induced subgraph which provides that $\chi^d_\rho(G) \geq 3$. On the other hand, G is a subgraph of the graph shown in Fig. 1 and the shown coloring restricted to G is a distance dominator packing coloring for G using 3 colors. This provides that $\chi^d_\rho(G) = 3$. Therefore, if $\chi^d_\rho(G) = 3$ and G is a proper subgraph of a graph described in 1, then it contains P_4, C_4 or K_3 as an induced subgraph. Next, let G be a proper subgraph of a graph H described in 2. Denote by $a_1, a_2, \ldots$ the vertices obtained by subdivision in H. Note that, if G is an isometric subgraph of H, then from Proposition 2.4 follows that $\chi^d_\rho(G) \leq \chi^d_\rho(H) = 3$. Using Theorem 3.1 we infer that G contains an induced subgraph isomorphic to P_4, C_4 or K_3. This proves the description 4. Next, suppose that G is not an isometric subgraph of H. This means that: *a*) there exists $a_i \in V(H)$ such that a_i is a leaf in G or *b*) $a_i \notin V(G)$ (in this case, since G is not an isometric subgraph of H, a_i do not belong to C_4 in H). If a_i is a leaf adjacent to some vertex from V_3, then it is easy to check that there does not exist distance dominator packing coloring for G using 3 colors. If a_i is a leaf adjacent to some vertex from V_2 and a_i belong to some C_4 in H, then G is not a proper subgraph of H, a contradiction. Hence, we can assume that a_i is a leaf adjacent to some vertex from V_2 but a_i does not belong to some C_4 in H. It is easy to check that there does not exist distance dominator packing coloring for G using 3 colors. Finally, consider the case when $a_i \notin V(G)$. Again, there does not exist distance dominator packing coloring for G using 3 colors.

The converse implication is trivial. Let G be a graph from 1 or 2 or any of theirs subgraphs described in 3 or 4. We have already constructed the 3-distance dominator packing coloring for each of these graphs and hence $\chi_\rho^d(G) \leq 3$. Next, since in each case these graphs contain P_4, K_3 or C_4 as an induced subgraph, we have $\chi_\rho^d(G) \geq \chi_\rho(G) \geq 3$. □

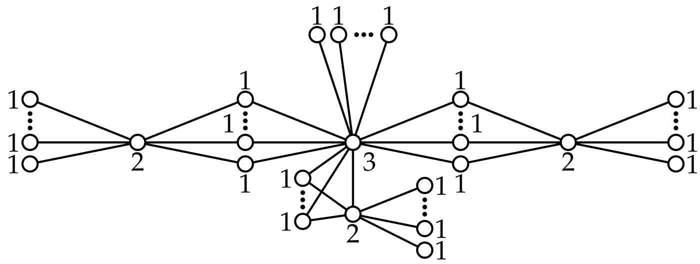

Figure 1: Distance dominator packing coloring of graphs, described in 1, with three colors.

Interestingly, we have proven that the characterizations of graphs G with $\chi_\rho^d(G) = 2$ (resp., $\chi_\rho^d(G) = 1$) and graphs H with $\chi_\rho(H) = 2$ (resp., $\chi_\rho(H) = 1$) coincide, while this is not true for graphs G and H with $\chi_\rho^d(G) = 3$ and $\chi_\rho(H) = 3$. Therefore, the difference of the concepts of packing coloring and distance dominator packing coloring can be noticeable only for graphs that are not 1- or 2-packing colorable, which is a motivation for the following studies.

We conclude the section by proving a lemma that gives the property of graphs with $\chi_\rho^d(G) = 3$. The lemma will be used later in the paper.

Lemma 3.4. *Let G be a connected graph. If $\chi_\rho^d(G) = 3$, then $diam(G) \leq 6$. In addition, this bound is sharp.*

Proof. Let G be a connected graph with $\chi_\rho^d(G) = 3$ and suppose that $\text{diam}(G) \geq 7$. Then, there exist vertices $u, v \in V(G)$ such that $d_G(u, v) = 7$. Denote by $P : u = x_0, x_1, x_2, \ldots x_6, v = x_7$ one of the shortest u,v-paths in G. Since $\chi_\rho(P) = 3$ and P is an isometric subgraph of G, by Proposition 2.1 and 2.4, $\chi_\rho^d(P) = 3$. Hence, there exists an optimal packing coloring of P which is also a distance dominator packing coloring. It is easy to check that any optimal packing coloring c of P assigns a color 3 to exactly two vertices, say x_a and x_b, where $0 \leq a < b \leq 7$ and $b - a \geq 4$. Since $\chi_\rho(P_n) > 2$ for any $n \geq 4$, it is clear that at most three consecutive vertices of the path P receive a color different from 3 by c. Using these facts we infer that $b - a = 4$. In other words, there are exactly 3 vertices between x_a and x_b that belong to P and moreover, there is only one possibility to color them using only colors 1 and 2, namely $c(x_{a+1}) = 1 = c(x_{a+3})$ and $c(x_{a+2}) = 2$. Recall that this holds also for any optimal distance dominator packing coloring of P. We infer that x_a does not 1-distance dominate all vertices of a color class of color 1, neither 3-distance dominate all vertices of a color class of color 3, which implies that it 2-distance dominates all vertices from a color class of color 2. By analogous consideration, the same holds also for x_b. Hence, $c(x_i) \neq 2$ for any existing x_i, where $i \in \{a-2, a-1, b+1, b+2\}$ (and also $c(x_i) \neq 3$), which implies that x_a is a leaf or is adjacent to leaf and the same holds for x_b. Then, P has at most 7 vertices, a contradiction to our assumption.

In order to prove that the bound is sharp, consider the path P_7. Namely, $\text{diam}(P_7) = 6$ and later we will see that $\chi_\rho^d(P_7) = 3$. □

4. Packing chromatic number and distance dominator packing chromatic number

Recall that each distance dominator packing coloring of a given graph is also its packing coloring. Hence, for any graph G, $\chi_\rho^d(G) \geq \chi_\rho(G)$. In this section, we provide some sufficient conditions for the equality of the packing chromatic number and the distance dominator packing chromatic number of a given graph. On the other hand, we prove that the difference between both of the mentioned numbers can be arbitrarily large.

Theorem 4.1. *Let G be a graph. If $\chi_\rho(G) \geq diam(G)$, then $\chi_\rho(G) = \chi_\rho^d(G)$.*

Proof. Let G be a graph such that $\chi_\rho(G) \geq$diam$(G) = k$. Then, any packing coloring of G assigns a color $\chi_\rho(G)$ to exactly one vertex $x \in V(G)$. Consequently, x $\chi_\rho(G)$-distance dominates its own color class. Any other vertex of G is at distance at most k from x and thus $\chi_\rho(G)$-distance dominates the color class of color $\chi_\rho(G)$. Therefore, any packing coloring of G is also a distance dominator packing coloring of G, thus $\chi_\rho^d(G) \leq \chi_\rho(G)$. Using Proposition 2.1 we derive the result. □

Note that, if G has diameter 2, it contains at least one edge (actually two edges), which implies that $\chi_\rho(G) \geq 2$. Therefore, every graph with diameter 2 satisfies the condition from Theorem 4.1 and hence the following claim holds.

Corollary 4.2. *Let G be a graph with diameter 2. Then, $\chi_\rho(G) = \chi_\rho^d(G)$.*

Next, we show that the difference between the packing chromatic number and the distance dominator packing chromatic number of a graph can be arbitrary large.

Theorem 4.3. *For any integer $k \geq 1$ there exists a graph G such that $\chi_\rho^d(G) - \chi_\rho(G) = k$.*

Proof. Let k be an arbitrary positive integer. In order to prove the statement, consider the path P_n, where $n = k^2 + 8k$. Since $n \geq 9$, $\chi_\rho(P_n) = 3$ [9, Proposition 2.3]. If $k = 1$, then $n = 9$ and Theorem 5.2 will show us that $\chi_\rho^d(P_9) = 4$, hence the $\chi_\rho^d(P_9) - \chi_\rho(P_9) = 1$. Analogously consider the case when $k = 2$. Namely, by Theorem 5.2, $\chi_\rho^d(P_{20}) = 5$ and thus the result follows. Next, if $k \geq 3$, then $n \geq 33$. In this case, Theorem 5.5 will show us that $\chi_\rho^d(P_n) = \left\lceil \sqrt{n+16} \right\rceil - 1 = k + 3$. Therefore, $\chi_\rho^d(P_n) - \chi_\rho(P_n) = (k+3) - 3 = k$ and hence the proof is done. □

5. Distance dominator packing chromatic numbers of some known families of graphs

In this section, we determine the distance dominator packing chromatic numbers of complete graphs, stars, wheels, paths and cycles.

Using Theorem 4.1 (or Corollary 4.2) and some results by Goddard and co-authors [9], we infer the following results.

Proposition 5.1. *Let n be a positive integer. Then,*
$\chi_\rho^d(K_n) = n$,
$\chi_\rho^d(K_{1,n}) = 2$,
$\chi_\rho^d(W_n) = \begin{cases} 4; & n = 3 \text{ or } n = 4k, k \in \mathbb{N}\,, \\ 5; & \text{otherwise}\,. \end{cases}$

Next, we continue with determining the distance dominator packing chromatic numbers of paths and cycles.

Theorem 5.2. *Let P_n be a path. Then,*

$$\chi_\rho^d(P_n) = \begin{cases} 2; & n \in \{2,3\}\,, \\ 3; & 4 \leq n \leq 7\,, \\ 4; & 8 \leq n \leq 11\,, \\ 5; & 12 \leq n \leq 20\,. \end{cases}$$

Proof. Since $\chi_\rho(P_2) = 2$, $\chi_\rho(P_3) = 2$ [9], $diam(P_2) = 1$ and $diam(P_3) = 2$, Theorem 4.1 implies that $\chi_\rho^d(P_2) = \chi_\rho^d(P_3) = 2$.

Next, let $n \in \{4,5,6,7\}$. Since in this case $\chi_\rho(P_n) = 3$ [9], Proposition 2.1 implies that $\chi_\rho^d(P_n) \geq 3$. In order to prove that $\chi_\rho^d(P_n) \leq 3$ for any $n \in \{4,5,6,7\}$, color the vertices of P_7 one after another with colors $1,2,1,3,1,2,1$. Clearly, such coloring is a distance dominator packing coloring of P_7, thus $\chi_\rho^d(P_7) = 3$. Next, since for any $n \in \{4,5,6\}$, P_n is an isometric subgraph of P_7, from Proposition 2.4 it follows that $\chi_\rho^d(P_n) = 3$, $n \in \{4,5,6\}$.

Now, consider the case when $n \in \{8,9,10,11\}$. Color all vertices of P_{11} one after another with colors $1,2,1,3,1,2,1,4,1,2,1$. Since such coloring is a distance dominator packing coloring, we infer that $\chi_\rho^d(P_{11}) \leq 4$. Moreover, using Proposition 2.4 and the fact that $\chi_\rho^d(P_7) = 3$, we infer that $3 \leq \chi_\rho^d(P_n) \leq 4$ for any $n \in \{8,9,10,11\}$. Next, since $diam(P_n) \geq 7$ for any $n \geq 8$, Lemma 3.4 yields the result.

Finally, let $n \in \{12,13,\ldots,20\}$ and prove that $\chi_\rho^d(P_n) = 5$. Suppose to the contrary that $\chi_\rho^d(P_{12}) = 4$ and denote by c an arbitrary 4-distance dominator packing coloring of P_{12}. Let $u_1,u_2,\ldots,u_{12}$ be the consecutive vertices of P_{12}.

First, consider the case when $c(u_i) = c(u_j) = 4$ for some i,j, where $1 \leq i < j \leq 12$. Note that $j - i \geq 5$ and without loss of generality assume that $i - 1 \leq 12 - j$. The latter means that the number of vertices on the left hand side of u_i is less than or equal to the number of vertices on the right hand side of u_j, and implies that $i \leq 4$. Since $\chi_\rho(P_4) = 3$, the vertices $u_{i+1},u_{i+2},u_{i+3},u_{i+4}$ receive colors from $\{1,2,3\}$ in a way that each of the listed colors is used. This fact implies that u_{i+8} (note that $i+8 \leq 12$ since $i \leq 4$) does not dominate all vertices of any color class and hence there does not exist a 4-distance dominator packing coloring of P_{12} which assigns to (at least) two distinct vertices a color 4. Therefore, $c(u_i) = 4$ for exactly one $i \in \{1,2,\ldots,12\}$. Without loss of generality we may assume that $i \leq 6$. Again, since $\chi_\rho(P_4) = 3$, the vertices $u_{i+1},u_{i+2},u_{i+3},u_{i+4}$ receive colors from $\{1,2,3\}$ by c in a way that each of the listed colors is used. If $i \leq 4$, then u_{12} does not dominate any color class, a contradiction to c being a 4-distance dominator packing coloring. Otherwise, if $i \in \{5,6\}$, then also the vertices u_1,u_2,u_3 and u_4 receive colors from $\{1,2,3\}$ such that each of the listed colors is used. Again, u_{12} does not distance dominate all vertices of any color class, a contradiction. Therefore, there does not exist any 4-distance dominator packing coloring of P_{12} and thus, $\chi_\rho(P_{12}) \geq 5$. Using Proposition 2.4 we also infer that $\chi_\rho(P_n) \geq 5$ for any $n \geq 12$.

In order to prove that $\chi_\rho(P_n) \leq 5$ for any $n \in \{12,13,\ldots,20\}$ form a 5-distance dominator packing coloring c of P_n as follows. If $n \geq 15$, then let $c(u_5) = 4$ and $c(u_{15}) = 5$. Otherwise, let $c(u_5) = 4$ and $c(u_n) = 5$. In both cases, all of the other vertices of P_n are colored one after another with the following pattern of colors: $1,2,1,3$. Clearly, c is a 5-distance dominator packing coloring of P_n and thus $\chi_\rho(P_n) = 5$ for any $n \in \{12,13,\ldots,20\}$. This completes the proof. □

Theorem 5.3. *Let C_n be a cycle. Then,*

$$\chi_\rho^d(C_n) = \begin{cases} 3; & n \in \{3,4,8\}, \\ 4; & n \in \{5,6,7,9,10,11,12\}, \\ 5; & n \in \{13,14,\ldots,20\}. \end{cases}$$

Proof. First, recall that $\chi_\rho(C_3) = \chi_\rho(C_4) = 3$, $\mathrm{diam}(C_3) = 1$ and $\mathrm{diam}(C_4) = 2$. Then, using Theorem 4.1, we infer that $\chi_\rho^d(C_3) = \chi_\rho^d(C_4) = 3$. Next, consider a cycle C_8. By Proposition 2.1 we derive that $\chi_\rho^d(C_8) \geq 3$. In order to see that $\chi_\rho^d(C_8) \leq 3$, form a 3-distance dominator packing coloring of C_8 as follows. Color all vertices of C_8 one after another using the following pattern of colors: $1,2,1,3,1,2,1,3$. It is easy to check that such coloring is 3-distance dominator packing coloring of C_8, hence $\chi_\rho^d(C_8) = 3$.

Next, recall that $\chi_\rho(C_5) = \chi_\rho(C_6) = \chi_\rho(C_7) = \chi_\rho(C_9) = 4$, $\mathrm{diam}(C_5) = 2$, $\mathrm{diam}(C_6) = 3$, $\mathrm{diam}(C_7) = 3$ and $\mathrm{diam}(C_9) = 4$. Again, applying Theorem 4.1 we derive that $\chi_\rho^d(C_5) = \chi_\rho^d(C_6) = \chi_\rho^d(C_7) = \chi_\rho^d(C_9) = 4$.

Further, consider a cycle C_{10}. If we color its vertices one after another using colors $1,3,1,4,1,2,3,1,4,2$, we get 4-distance dominator packing coloring of C_{10}, which implies that $\chi_\rho^d(C_{10}) \leq 4$. Next, suppose that

$\chi_\rho^d(C_{10}) \leq 3$. Since $\chi_\rho(P_4) > 2$, at most three consecutive vertices of C_{10} receive colors 1 or 2. This implies that three vertices of C_{10} receive a color 3 and hence none of the vertices of C_{10} distance dominates all vertices of a color class of color 3. In addition, none of the vertices distance dominates a color class of color 1 or 2 and thus, there does not exist 3-distance dominator packing coloring of C_{10}, what completes the proof in this case. Analogously we prove that $\chi_\rho^d(C_i) \geq 4$ for any $i \geq 11$. In order to prove that $\chi_\rho^d(C_{11}) = \chi_\rho^d(C_{12}) = 4$, we form a 4-distance dominator packing colorings of C_{11} and C_{12} as follows. Color all vertices of C_{11} one after another with the following patterns of colors: $1, 2, 1, 3, 1, 2, 4, 1, 2, 1, 3$, and all vertices of C_{12} one after another with colors: $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3$. Clearly, these are 4-distance dominator packing colorings, hence $\chi_\rho^d(C_{11}) = \chi_\rho^d(C_{12}) = 4$.

Next, let $n \in \{13, 14, \ldots, 20\}$ and prove that $\chi_\rho^d(C_n) = 5$. First, color one (arbitrary) vertex of C_n with color 4 and one of its diametral vertices with color 5. Further, color the remaining vertices using colors $1, 2, 3$ in a way that we get a packing coloring of C_n (it is easy to check that such packing coloring exists). Clearly, such packing coloring of C_n is 5-distance dominator packing coloring, since each vertex of C_n distance dominates color class of color 4 or 5 (note that the vertices, colored by 4 and 5 are pairwise at distance at most 10). Thus, $\chi_\rho^d(C_n) \leq 5$. Next, prove that $\chi_\rho^d(C_n) \geq 5$ for any $n \in \{13, 14, \ldots, 20\}$. Suppose to the contrary that $\chi_\rho^d(C_n) \leq 4$ and denote by c any 4-distance dominator packing coloring of C_n. Then, it is easy to check that none of the vertices from $V(C_n)$ distance dominates all vertices of the color class of color 1 or 2. If $|c^{-1}(4)| \geq 3$, then $|c^{-1}(3)| \geq 3$ since $\chi_\rho(P_4) > 2$. Next, we derive that none of the vertices distance dominates a color class of a color 3 or 4. Hence, c cannot be a 4-distance dominator packing coloring. Therefore, $|c^{-1}(4)| \in \{1, 2\}$. If $|c^{-1}(4)| = 1$, then at least 12 consecutive vertices of C_n receive colors different from 4, which implies that $|c^{-1}(3)| \geq 3$. Thus, none of the vertices of C_n distance dominates a color class of a color 3 and there are only 9 vertices of C_n which distance dominate color class of color 4, a contradiction since $n \geq 13$. Next, suppose that $|c^{-1}(4)| = 2$. Then, $|c^{-1}(3)| \geq 2$. Further, we observe that at most 5 vertices distance dominate the color class of color 4 and at most 3 vertices distance dominate the color class of color 3. Again, a contradiction, since at least 5 vertices of C_n do not distance dominate all vertices from any color class. Therefore, $\chi_\rho^d(C_n) \geq 5$ and the proof is completed. □

The following lemma will help us to determine $\chi_\rho^d(P_n)$ and $\chi_\rho^d(C_n)$ for any $n \geq 21$.

Lemma 5.4. *Let G be the path P_n or the cycle C_n, $n \geq 21$. Further, let $x \geq 5$ be a positive integer. Then, there does not exist an x-distance dominator packing coloring of G if $n = x^2 + 2x - 14$.*

Proof. Let G be the path P_n or the cycle C_n, $n \geq 21$. Suppose to the contrary that there exists x-distance dominator packing coloring c of G, where $n = x^2 + 2x - 14$ and $x \geq 5$. Let $B_1, B_2, \ldots, B_x$ be the color classes determined by c. Clearly, there are no vertices of G, which distance dominate the color class B_1 or B_2. Denote the number of vertices that distance dominate color class B_3 by t. Note that if $|B_3| = 1$, then $t \leq 7$, if $|B_3| = 2$, then $t \leq 3$, and otherwise, $t = 0$.

First, consider the case when $|B_j| \geq 3$ for some $j \in \{4, 5, \ldots, x\}$. Then, none of the vertices of G distance dominates all vertices from color class B_j. Further, note that, if u of G is colored with $i \in \{4, 5 \ldots, x\}$, then only the vertices which are at distance at most i from u can i-distance dominate color class B_i. This means that the number of the vertices which can distance dominate B_i is at most $2i + 1$. Therefore, the number of the vertices that distance dominate all vertices from some color class is at most $t + \sum_{i=4, i\neq j}^{x}(2i + 1)$, which is less than $n = x^2 + 2x - 14$, a contradiction to c being a x-distance dominator packing coloring. Therefore, each of the colors from $\{4, 5, \ldots, x\}$ is assigned to at most two vertices of G by c. If a color $j \in \{4, 5, \ldots, x\}$ is assigned to exactly one vertex of G by c, then at most $2j + 1$ vertices distance dominate color class B_j. Further, consider the case when a color $j \in \{4, 5, \ldots, x\}$ is assigned to exactly two vertices. If G is the path P_n, then it is clear that there are at most j vertices distance dominating color class B_j (vertices between two vertices colored by j). Next, suppose that G is the cycle C_n. Let $u, v \in V(C_n)$ be colored by j. Then, there are exactly two disjoint paths, P and P', between u and v in C_n. If only the vertices from P (resp., P') distance dominate both vertices from B_j, then clearly the number of such vertices is at most j. Otherwise, there are some vertices belonging to P and some to P', which distance dominate the color class B_j, and we will prove that also in this case there cannot be more than j such vertices. Suppose to the contrary that there are

$j+1$ vertices that distance dominate all vertices from color class B_j, and denote by a the number of them belonging to P and by b the number of them belonging to P' (clearly, $a+b=j+1$). Then, the number of the vertices of C_n is $3j+1 \leq 3x+1$. Since $3x+1<n$ for any $x \geq 5$, we have a contradiction and hence the claim holds.

In the sequel of this proof, we distinguish three cases.

Case 1. $|B_i|=1$ for every $i \in \{4,5,\dots,x\}$.

Subcase 1.1. $|B_3|=1$. Since $\chi_\rho(P_4) \geq 3$, there exist at most three consecutive vertices of G which receive a color 1 or 2. This implies that c assigns a color 1 or 2 to at most $3(x-2)+3$ vertices if G is the path P_n, and to at most $3(x-2)$ vertices if G is the cycle C_n. Further, c assigns to $x-2$ vertices colors from $\{3,4,5,\dots,x\}$. Therefore, c colors at most $4x-5$ vertices if G is the path P_n, and at most $4x-8$ vertices if G is the cycle C_n. In both cases, c colors less than n vertices, a contradiction.

Subcase 1.2. $|B_3|=2$.

Again, the fact that $\chi_\rho(P_4) \geq 3$, implies that there exist at most three consecutive vertices of G which receive a color 1 or 2. Thus, c assigns to two vertices a color 3, to $x-3$ vertices colors from $\{4,5,\dots,x\}$ and color 1 or 2 to at most $3(2+x-3)+3$ vertices (more precisely, to at most $3(2+x-3)+3$ vertices if G is the path P_n, and to at most at most $3(2+x-3)$ vertices if G is the cycle C_n). Therefore, c colors at most $4x-1$ vertices, what is less than n, a contradiction.

Subcase 1.3. $|B_3| \geq 3$. In this case there are no vertices of G which distance dominate color class B_3, thus the number of the vertices that distance dominate all vertices of any color class is at most $\sum_{i=4}^{x}(2i+1) = x^2+2x-15$, what is less than n, a contradiction to c being a distance dominator packing coloring.

Case 2. $|B_j|=2$ for some $j \in \{4,5,\dots,x\}$ and $|B_i|=1$ for every $i \in \{4,5,\dots,x\} \setminus \{j\}$.

In this case there are at most $t+\sum_{i=4,i\neq j}^{x}(2i+1)+j$ vertices of G which distance dominate all vertices of some color class. If $t \leq 5$, then $t+\sum_{i=4,i\neq j}^{x}(2i+1)+j<n$, a contradiction to c being x-distance dominator packing coloring of G. Therefore, we only need to consider the case when $t=6$ or $t=7$, which both mean that $|B_3|=1$.

Denote by a,b the vertices colored by j and recall that the color class B_j is distance dominated by at most j vertices. The fact $j \geq 4$ implies that there are at least 4 vertices on each path between a and b. Since $\chi_\rho(P_4)>2$, we infer that there are at most three consecutive vertices of G which receive a color 1 or 2. Hence, there is at least one vertex on each path between a and b, which receives a color $k>2$ by c (and this color is used only once by c). Therefore, if j vertices distance dominate color class B_j, then at least 4 of them also distance dominate a color class B_k, which implies that there are at most $t+\sum_{i=4,i\neq j}^{x}(2i+1)+j-4$ vertices distance dominating some color class, what is less than n for any $x \geq 5$, a contradiction. Otherwise, there are at most $j-2$ vertices which distance dominate color class B_j (note that the case when $j-1$ vertices distance dominate color class B_j is not possible). Hence, at most $t+\sum_{i=4,i\neq j}^{x}(2i+1)+(j-2)$ vertices of G distance dominate all vertices from some color class. Since this is less than n, again we have a contradiction.

Case 3. $|B_j|=2$ for at least two colors $j \in \{4,5,\dots,x\}$.

Let $j_1,j_2,\dots,j_k \in \{4,5,\dots,x\}$ be the colors for which $|B_{j_1}|=|B_{j_2}|=\dots=|B_{j_k}|=2$. Then there are at most j_l vertices that distance dominate color class B_{j_l} for any $l \in \{1,\dots k\}$, and at most $2l+1$ vertices that distance dominate color class B_l for any $l \in \{4,5,\dots,x\} \backslash \{j_1,j_2,\dots,j_k\}$. Therefore, the number of vertices of G that distance dominate some color class is at most $t+\sum_{i=4,i\notin\{j_1,\dots,j_k\}}^{x}(2i+1)+\sum_{i\in\{j_1,\dots,j_k\}} i$ which is at most $t+\sum_{i=4,i\notin\{j_1,j_2\}}^{x}(2i+1)+\sum_{i\in\{j_1,j_2\}} i$. But $t+\sum_{i=4,i\notin\{j_1,j_2\}}^{x}(2i+1)+\sum_{i\in\{j_1,j_2\}} i<n$, a contradiction to c being a x-distance dominator packing coloring. So, in all cases we got a contradiction to c being a distance dominator packing coloring, hence we can conclude that there does not exist an x-distance dominator packing coloring of G if $n=x^2+2x-14$. $\square$

Theorem 5.5. *Let G be the path P_n or the cycle C_n, $n \geq 21$. Then,*

$$\chi_\rho^d(G) = \left\lceil \sqrt{n+16} \right\rceil - 1.$$

Proof. Let G be the path P_n or the cycle C_n, $n \geq 21$. Further, let $u_1,u_2,\dots,u_n$ be its consecutive vertices. Let

$\left\lceil \sqrt{n+16} \right\rceil - 1 = r$.

First, in order to prove that $\chi_\rho^d(G) \le r$, we form a r-distance dominator packing coloring c of G in the following way. First, for all $l \in \{5, \ldots, r\}$ and $i = 5 + \sum_{j=5}^{l} 2j$, set $c(u_i) = l$. Further, let $c(u_5) = 4$. Finally, color the other vertices of G one after another using the following pattern of colors: $1, 2, 1, 3$. Clearly, the described coloring is a packing coloring of G with the color classes $B_1, B_2, \ldots, B_r$. Next, it is easy to observe that any two vertices $u, u' \in V(G)$ such that $c(u) = k-1$ and $c(u') = k$, where $k \in \{5, 6, \ldots, r\}$, are at distance $2k$ (in other words, there are $2k-1$ consecutive vertices of G between u and u'). In particular, vertices u_i, where $i \le 4$, are at distance at most 4 from u_5, which is colored by 4. Next, recall that $c(u_{r^2+r-15}) = r$ and then it is easy to check that all vertices u_j, where $r^2 + r - 15 < j \le n$, are at distance at most r from u_{r^2+r-15}. These facts imply that each vertex u_i colored with a color from $\{1, 2, 3\}$ l-distance dominates all vertices from a color class B_l for some $l \in \{4, 5, \ldots, r\}$. Clearly, each u_i with the property that $c(u_i) = l \ge 4$, l-distance dominates the color class B_l. Thus, c is an r-distance dominator packing coloring of G and $\chi_\rho^d(G) \le r$.

Next, prove that $\chi_\rho^d(G) \ge r$. Suppose to the contrary that $\chi_\rho^d(G) \le r - 1$ (recall that $r - 1 \ge 5$).

Then, $\chi_\rho^d(G) + 2 \le \left\lceil \sqrt{n+16} \right\rceil$. Therefore, $\chi_\rho^d(G) + 2 < \sqrt{n+16} + 1$, and we get $\chi_\rho^d(G)^2 + 2\chi_\rho^d(G) - 15 < n$, a contradiction to Lemma 5.4. □

6. Remarks and open problems

Note that in the case of distance r-dominator coloring (and hence, also in the case of dominator coloring), all colors are somehow equivalent (namely, vertices in each color class are pairwise at distance more than r). Since each vertex of a given graph must r-distance dominate all vertices of one color class, it is not clear, if r refers to its color or to the color of the vertices from the mentioned color class. Clearly, this is not true for distance dominator S-packing coloring as is defined in this paper. Therefore, the concept of dominator coloring (and distance r-dominator coloring) can be generalized via S-packing coloring in two different ways. One of them is presented in our paper, but we could also demand that each vertex u of a given graph x-distance dominates all vertices from some color class, where x is the color of u (not of the mentioned color class). Hence, there arises a natural aspiration to study the second type of generalization of dominator coloring via S-packing coloring.

Further, in Section 3, we have characterized graphs with distance dominator packing chromatic number 2 respectively 3, but there is still the problem of characterization of graphs with distance dominator packing chromatic number 4. Using the fact that $\chi_\rho(G) \le \chi_\rho^d(G)$ for any graph G, we know that graphs H with $\chi_\rho^d(H) = 4$ are those with $\chi_\rho(H) = 3$ or $\chi_\rho(H) = 4$. Since a characterization of graphs with packing chromatic number 4 is not known yet, we think that the task is harder as in the case of graphs with χ_ρ^d equal to 3.

Next, in Section 5, we have determined the distance dominator packing chromatic numbers of complete graphs, stars, wheels, paths and cycles. There is still the open question about the exact values or bounds for the distance dominator packing chromatic numbers for some others well known graphs. For instance, for the family of trees it is already known that the distance dominator packing chromatic number is unbounded. Namely, each tree contains a path as an isometric subgraph and the distance dominator packing chromatic number in the family of paths is unbounded.

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