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New Exact Solutions of Some Non-linear Evolution Equations via Functional Variable Method

Berfin Elma[a], Emine Misirli[a]

[a]*Ege University, Department of Mathematics, Izmir, urkey*

Abstract. Recently, many successful methods have been developed to achieve analytical solutions of non-linear partial differential equations. In this study, some new exact solutions of the non-linear coupled Klein-Gordon system and non-linear modified Benjamin-Bona-Mahony equation have been obtained by using functional variable method (FVM). Additionally, all solutions have been examined and three dimensional graphics of the obtained solutions have been drawn by using the Mathematica program. These equations have been used in various fields such as plasma physics, biophysics, and fluid dynamics. The main advantage of FVM is generate more solutions than other analytical methods and therefore, FVM is an effective and powerful method to solve evolution equations in engineering and mathematical physics.

1. Introduction

It is known that mathematical models of many physical phenomena is defined by nonlinear evolution equations (NLEEs) . NLEEs have been widely used in different areas such as optical fibers, chemical kinematics, mechanics, biology, fluid mechanics. Thus, it is very important to investigate analytical solutions of NLEEs to understand of complex phenomena. Various methods have been improved to solve of NLEEs such as $\frac{G'}{G}$ expansion method [7], sine-cosine method [12], F-expansion method [2], inverse scattering method [11], the first integral method [9], the tanh method [6], Jacobi elliptic function method [3], the modified simple equation method [15] and functional variable method [16] and so on.

A. Zerarka et al. [16] proposed the functional variable method to find to find exact solutions of NLEEs. The idea of FVM is converting nonlinear partial differential equations to nonlinear ordinary differential equations with the help of wave transformation. Therefore, FVM is a strong and reliable tool to construct exact solutions of NLEEs. In this study, FVM have been applied to obtain new traveling wave solutions of the modified Benjamin-Bona-Mahony equation [5] and the coupled Klein-Gordon System [8].

2. Fujctional Variable Method

In general, FVM is written in four step [14].

NLEE can be defined as follows:

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Email addresses: berfin-elma@hotmail.com (Berfin Elma), emine.misirli@ege.edu.tr (Emine Misirli)

$$R(u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, ..) = 0 \tag{1}$$

where R is a polynomial of $u(x, y, t)$ and its partial derivatives.

Step 1. Using wave transformation

$$u(x, y, t) = u(\xi), \xi = x + y - ct, \tag{2}$$

where $c \neq 0$, we convert the (1) to a non-linear ordinary differential equation (ODE):

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ..) = 0 \tag{3}$$

P is a polynomial of $u(\xi)$ and its total derivatives while $u_\xi = \frac{du}{d\xi}$, $u_{\xi\xi} = \frac{d^2u}{d\xi^2}$ and so on.

Step 2. Let us introduce a transformation in which the unknown function $u(\xi)$ is considered as a functional variable in the form:

$$u_\xi = F(u) \tag{4}$$

and some successive derivatives of $u(\xi)$:

$$\begin{aligned} u_\xi &= \tfrac{1}{2}(F^2)' \\ u_{\xi\xi} &= \tfrac{1}{2}(F^2)'' \sqrt{F^2} \\ u_{\xi\xi\xi} &= \tfrac{1}{2}[(F^2)''' F^2 + \tfrac{1}{2}(F^2)''(F^2)'] \end{aligned} \tag{5}$$

Step 3. After substituting (4) and (5) into (3) the ODE can be reduced as:

$$R(u, F, F', F'', ..) = 0 \tag{6}$$

After integration, (6) provides the expression of F. Also, this gives the appropriate solutions of the original equation (1).

3. Applications

3.1. Modified Benjamin-Bona-Mahony Equation

Many researchers have interested to exact solutions of the Benjamin-Bona-Mahony (BBM) equation. This equation has many application areas. Long waves which are in a nonlinear dispersive region are defined by using BBM equation. The BBM equation is used to analyze hydromagnetic waves which are in cold plasma, acoustic waves in anharmonic crystals and in compressible fluids and surface waves of long wavelength in liquids [1]. The mBBM equation is a special type of the BBM equation. BBM is converted to mBBM when $n = 2$ [10]. Modified Benjamin-Bona-Mahony (mBBM) Equation [5]

$$u_t + u_x + u^2u_x + u_{xxt} = 0 \tag{7}$$

Using the transformations

$$u(x, t) = u(\xi), \xi = x - ct \tag{8}$$

and integrating with respect to ξ equation (7) converts to the following ODE:

$$u(1 - c) + \frac{1}{3}u^3 - cu_{\xi\xi} = 0 \tag{9}$$

Substituting (5) into (9) we obtain

$$(F^2)' = \frac{2(1-c)}{c}u - \frac{2}{3c}u^3 \tag{10}$$

Integrating the eqn. (10) with respect to u , we have

$$F(u) = \sqrt{\frac{1}{6c}}u\sqrt{u^2-6(c-1)} \tag{11}$$

From (4) and (11) we deduce that

$$\int \frac{du}{u\sqrt{u^2-6(c-1)}} = \sqrt{\frac{1}{6c}}(\xi+\xi_0), \tag{12}$$

where ξ_0 is a integration constant. After integrating (12), we have the following exact solutions:
Case 1. If $6(c-1)=0$, then

$$u_1(x,t) = \pm\frac{1}{\sqrt{\frac{1}{6}}(x-ct+\xi_0)} \tag{13}$$

Case 2. If $6(c-1)>0$, then

$$u_2(x,t) = \sqrt{6(c-1)}\,sec(\sqrt{\frac{c-1}{c}}(x-ct+\xi_0)) \tag{14}$$

$$u_3(x,t) = -\sqrt{6(c-1)}\,sec(\sqrt{\frac{c-1}{c}}(x-ct+\xi_0)) \tag{15}$$

$$u_4(x,t) = \sqrt{6(c-1)}\,csc(\sqrt{\frac{c-1}{c}}(x-ct+\xi_0)) \tag{16}$$

$$u_5(x,t) = -\sqrt{6(c-1)}\,csc(\sqrt{\frac{c-1}{c}}(x-ct+\xi_0)) \tag{17}$$

Case 3. If $6(c-1)<0$, then

$$u_6(x,t) = \sqrt{6(c-1)}\,sech(\sqrt{-\frac{c-1}{c}}(x-ct+\xi_0)) \tag{18}$$

$$u_7(x,t) = -\sqrt{6(c-1)}\,sech(\sqrt{-\frac{c-1}{c}}(x-ct+\xi_0)) \tag{19}$$

$$u_8(x,t) = \sqrt{6(c-1)}\,csch(\sqrt{-\frac{c-1}{c}}(x-ct+\xi_0)) \tag{20}$$

$$u_9(x,t) = -\sqrt{6(c-1)}\,csch(\sqrt{-\frac{c-1}{c}}(x-ct+\xi_0)) \tag{21}$$

3.2. The Non-Linear Coupled Klein-Gordon Equation

The nonlinear coupled Klein-Gordon system was founded by Segal and then, the equation has been used in quantum physics and mathematics [13]. The Klein-Gordon equation that is encountered in the behavior of elementary particles and the propagation of dislocations in crystals [4] can use in models of many phenomena.

The nonlinear coupled Klein-Gordon system [8]

$$\begin{aligned} &u_{xx} + u_{tt} - u + 2u^3 + 2uv = 0 \\ &v_x + v_t - 4uu_t = 0 \end{aligned} \tag{22}$$

Using the transformations

$$u(x,t) = u(\xi), \;\; v(x,t) = v(\xi), \;\; \xi = x - ct \tag{23}$$

and after integrating with respect to ξ eqn. (22) converts to the following ODE:

$$(1 - c^2)u_{\xi\xi} - u + 2u^3 + 2uv = 0 \tag{24}$$

$$(1 + c)v_{\xi} + 4cuu_t = 0 \tag{25}$$

u and v become ODE, after integrating with respect to ξ eqn. (25)

$$v = -\frac{2c}{1+c}u^2 \tag{26}$$

Converts to following ODE after substituting (26) into (24):

$$(1 - c^2)u_{\xi\xi} - u + 2\frac{(1-c)}{(1+c)}u^3 = 0 \tag{27}$$

Substituting (5) into (27) we obtain

$$(F^2)' = \frac{2}{(1-c^2)}u - \frac{4}{(1+c)^2}u^3 \tag{28}$$

Integrating the eqn. (28) with respect to u, we have

$$F(u) = \sqrt{-\frac{1}{(1+c)^2}}\, u\sqrt{u^2 - \frac{(1+c)}{(1-c)}} \tag{29}$$

From (4) and (29) we deduce that

$$\int \frac{du}{u\sqrt{u^2 - \frac{(1+c)}{(1-c)}}} = \sqrt{-\frac{1}{(1+c)^2}}(\xi + \xi_0), \tag{30}$$

where ξ_0 is a integration constant. After integrating (30), we have the following exact solutions:

Case 1. If $\frac{(1+c)}{(1-c)} = 0$, then

$$u_1(x,t) = \pm\frac{1}{\sqrt{-\frac{1}{(c+1)^2}}(x - ct + \xi_0)} \tag{31}$$

Case 2. If $\frac{(1+c)}{(1-c)} > 0$, then

$$u_2(x,t) = \sqrt{\frac{(1+c)}{(1-c)}}\; sec\,(\sqrt{\frac{1}{(c^2-1)}}(x - ct + \xi_0)) \tag{32}$$

$$u_3(x,t) = -\sqrt{\frac{(1+c)}{(1-c)}}\, sec\,(\sqrt{\frac{1}{(c^2-1)}}(x-ct+\xi_0)) \tag{33}$$

$$u_4(x,t) = \sqrt{\frac{(1+c)}{(1-c)}}\, csc\,(\sqrt{\frac{1}{(c^2-1)}}(x-ct+\xi_0)) \tag{34}$$

$$u_5(x,t) = -\sqrt{\frac{(1+c)}{(1-c)}}\, csc\,(\sqrt{\frac{1}{(c^2-1)}}(x-ct+\xi_0)) \tag{35}$$

Case 3. If $\frac{(1+c)}{(1-c)} < 0$, then

$$u_6(x,t) = \sqrt{\frac{(1+c)}{(1-c)}}\, sech\,(\sqrt{\frac{1}{(c^2-1)}}(x-ct+\xi_0)) \tag{36}$$

$$u_7(x,t) = -\sqrt{\frac{(1+c)}{(1-c)}}\, sech\,(\sqrt{\frac{1}{(1-c^2)}}(x-ct+\xi_0)) \tag{37}$$

$$u_8(x,t) = \sqrt{\frac{(1+c)}{(1-c)}}\, csch\,(\sqrt{\frac{1}{(c^2-1)}}(x-ct+\xi_0)) \tag{38}$$

$$u_9(x,t) = -\sqrt{\frac{(1+c)}{(1-c)}}\, csch\,(\sqrt{\frac{1}{(1-c^2)}}(x-ct+\xi_0)) \tag{39}$$

4. Graphs

4.1. Graphs of Solutions for the mBBM Equation

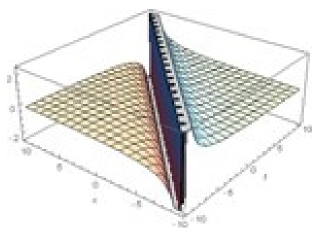

Figure 1: Eq.(13) for c = 1

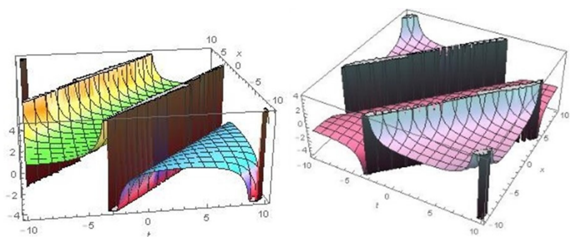

Figure 2: Eq.(14) for c = 1.06 and Eq.(15) for c = 1.06

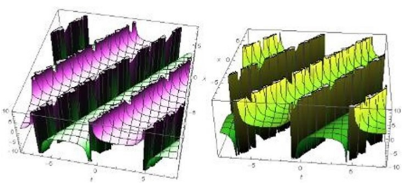

Figure 3: Eq.(16) for c = 1.25 and Eq.(17) for c = 1.26

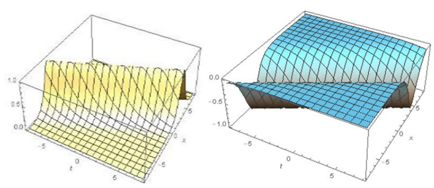

Figure 4: Eq.(18) for c = 0.41 and Eq.(19)for c = 0.55

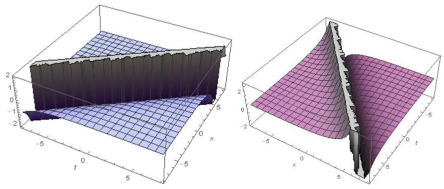

Figure 5: Eq.(20) for c = 0.75 and Eq.(21) for c = 0.83

4.2. Graphs of Solutions for the nonlinear coupled Klein-Gordon Equation

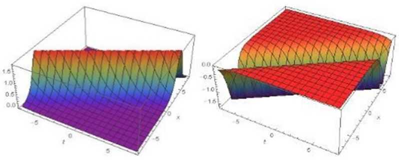

Figure 6: Eq.(32) for c = 0.54 and Eq.(33) for c = 0.7

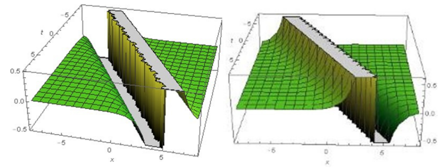

Figure 7: Eq.(14) for c = 1.06 and Eq.(15) for c = 1.06

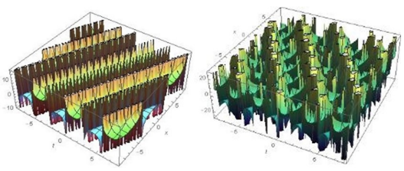

Figure 8: Eq.(16) for c = 1.25 and Eq.(17) for c = 1.26

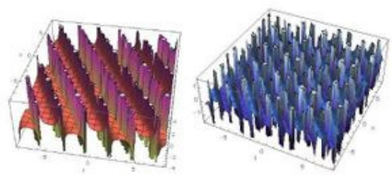

Figure 9: Eq.(18) for c = 0.41 and Eq.(19)for c = 0.55

5. Conclusion

The purpose of this study is obtaining new analytic wave solutions of some non- linear evolution equations. Some new traveling wave solutions have been successfully achieved of the non- linear coupled Klein-Gordon system and non-linear modified Benjamin-Bona-Mahony equation by using FVM. With the help of Mathematica program, the trueness of all solutions have been checked. 3D-graphs have also been drawn for suitable coefficient values. The advantage of method is give more solution functions such as periodic solutions, hyperbolic solutions and rational solutions than other popular analytical methods, therefore FVM has a wide applicability. The main effectiveness of FVM is not need to linearization and perturbation. Finally the method is flexible, reliable and straightforward to find solutions of some non-linear evolution equations arising in engineering and science.

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