Filomat 35:13 (2021), 4405–4416
https://doi.org/10.2298/FIL2113405A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Rough Statistical Convergence in Intuitionistic Fuzzy Normed Spaces

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**Abstract.** In this paper, we have defined rough statistical convergence in intuitionistic fuzzy normed spaces which is an useful characterization in the field of statistical convergence. We have proved some properties related to rough convergence which provides some new functional tools in the situation of uncertainty like intuitionistic fuzzy normed spaces. Further, we have established the relationship between the set of statistical limit points and set of cluster points of rough statistically convergent sequences in these spaces.

## 1. Introduction

Rough convergence deals with the approximate solution of any real life situation from numerical point of view. It helps to verify the correctness of solution obtained from computer programs and to draw conclusion from scientific experiments. The rough convergence has been initially introduced by Phu[27] as an interesting generalization of usual convergence for the sequences on finite dimensional normed linear spaces and later on introduced on infinite dimensional normed linear spaces[28]. Apart from defining the idea of rough convergence, he also contributed towards the properties like closeness and convexity of the rough limit set.

**Definition 1.1.** *[27] A sequence $x = \{x_k\}$ in a normed linear space $(\mathbb{X}, ||.||)$ is said to be rough convergent to $\xi \in \mathbb{X}$ for some non-negative number $r$ if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $||x_k - \xi|| < r + \epsilon$ for all $k \geq k_0$.*

Aytar[2] extended the rough convergence to rough statistical convergence like usual convergence is extended to statistical convergence with the help of natural density by Fast[8]. Although, natural density of set $A$, where $A \subseteq \mathbb{N}$, has given by $\delta(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{a \leq n : a \in A\} \mid$, provided limit exists, where $\mid . \mid$ designates the order of the enclosed set. Further, A sequence $x = \{x_k\}$ converges statistically to $\xi$, if $A(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| > \epsilon\}$ has natural density zero (see [9]). Moreover, Aytar[3] also examined some criteria associated with the convexity and closeness of the set of rough statistical limit points. In fact, he established the properties related to this set with the set of rough cluster points of a sequence.

**Definition 1.2.** *[2] A sequence $x = \{x_k\}$ in a normed linear space $(\mathbb{X}, ||.||)$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ for some non-negative number $r$ if for every $\epsilon > 0$ we have*

$$\delta(\{k \in \mathbb{N} : ||x_k - \xi|| \geq r + \epsilon\}) = 0,$$

2020 *Mathematics Subject Classification*. Primary 40A35; Secondary 40G99; 26E50.
*Keywords*. Statistical convergence, rough statistical convergence, intuitionistic fuzzy normed space.
Received: 09 October 2020; Revised: 04 March 2021; Accepted: 08 March 2021
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*and $\xi$ is known as r-St-limit of sequence $x = \{x_k\}$.*

This idea of rough convergence has motivated many authors to use this concept not only in usual sense but also in statistical mode in the different forms like double sequences[17, 18] and triple sequences[7], lacunary sequences[13], real valued function sequences[16], ideals[19, 26] etc. Besides these above mentioned forms it is also established for the different spaces like metric spaces[6], random normed spaces[1], cone metric spaces[4], probabilistic normed spaces[29] etc. More investigations, generalizations and applications of the rough convergence can be further revealed using statistical convergence as well as generalized statistical convergence in different directions as in [5, 10, 11, 14, 20–25].

In the literature, during last few years, considerable progress is going on this field of rough convergence which leads us to study the concept of rough statistical convergence in the intuitionistic fuzzy normed spaces. As intuitionistic fuzzy normed space itself is a well motivated area of research being a natural tool for modeling imprecision in real life situations.

Saadati and Park[30] presented the idea of intuitionistic fuzzy normed spaces as generalization of fuzzy metric spaces. Following the work of Saadati and Park[30], Lael and Nourouzi[15] have given a new variant of intuitionistic fuzzy normed spaces which is defined as follows.

**Definition 1.3.** *[15] An intuitionistic fuzzy normed space(IFNS) is the triplet $(\mathbb{X}, \varphi, \vartheta)$ with vector space $\mathbb{X}$ and fuzzy sets $\varphi, \vartheta$ on $\mathbb{X} \times \mathbb{R}$, if for each $x, y \in \mathbb{X}$ and $s, t \in \mathbb{R}$, we have*

(i) $\varphi(x, t) = 0$ *and* $\vartheta(x, t) = 1$ *for* $t \notin \mathbb{R}^+$,

(ii) $\varphi(x, t) = 1$ *and* $\vartheta(x, t) = 0$ *for* $t \in \mathbb{R}^+$ *iff* $x = 0$,

(iii) $\varphi(\alpha x, t) = \varphi(x, \frac{t}{|\alpha|})$ *and* $\vartheta(\alpha x, t) = \vartheta(x, \frac{t}{|\alpha|})$ *for* $\alpha \neq 0$,

(iv) $\min\{\varphi(x, s), \varphi(y, t)\} \leq \varphi(x + y, s + t)$ *and* $\max\{\vartheta(x, s), \vartheta(y, t)\} \geq \vartheta(x + y, s + t)$,

(v) $\lim_{t\to\infty} \varphi(x, t) = 1$, $\lim_{t\to 0} \varphi(x, t) = 0$, $\lim_{t\to\infty} \vartheta(x, t) = 0$ *and* $\lim_{t\to 0} \vartheta(x, t) = 1$.

**Example 1.4.** *[15] Let $(\mathbb{X}, \|\circ\|)$ be any normed space. For every $t > 0$ and all $x \in \mathbb{X}$, take $\varphi(x, t) = \frac{t}{t+\|x\|}$ , $\vartheta(x, t) = \frac{\|x\|}{t+\|x\|}$. Then, triplet $(\mathbb{X}, \varphi, \vartheta)$ is an IFNS which satisfies the above mentioned conditions.*

**Definition 1.5.** *[15] Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is called convergent to $\xi \in X$ with respect to the norm $(\varphi, \vartheta)$ if there exists $k_0 \in \mathbb{N}$ for every $\epsilon > 0$ and $\lambda \in (0, 1)$ such that $\varphi(x_k - \xi, \epsilon) > 1 - \lambda$ and $\vartheta(x_k - \xi, \epsilon) < \lambda$ for all $k \geq k_0$. It is denoted by $(\varphi, \vartheta) - \lim\limits_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{(\varphi,\vartheta)} \xi$.*

**Remark 1.6.** *Let $(\mathbb{X}, \|\circ\|)$ be any normed space. For every $t > 0$ and all $x \in \mathbb{X}$, take $\varphi(x, t) = \frac{t}{t+\|x\|}$ , $\vartheta(x, t) = \frac{\|x\|}{t+\|x\|}$. Then, $(\mathbb{X}, \varphi, \vartheta)$ is an IFNS.*
*Also, $x_k \xrightarrow{(\vartheta,\vartheta)} x$ if and only if $x_k \xrightarrow{\|.\|} x$.*

Karakus[12] introduced statistical convergence of sequences in intuitionistic fuzzy normed space. Now, using the technique of Lael and Nourouzi[15], we present the statistical convergence in the intuitionistic fuzzy normed space as below.

**Definition 1.7.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is called statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,*

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \xi, \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi, \epsilon) \geq \lambda\}) = 0.$$

*It is denoted by $St_{(\varphi,\vartheta)} - \lim\limits_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{St_{(\varphi,\vartheta)}} \xi$ .*

## 2. Main Results

In this section, we first define the rough convergence and rough statistical convergence in intuitionistic fuzzy normed spaces as follows:

**Definition 2.1.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ for some non-negative number $r$ if there exists $k_0 \in \mathbb{N}$ for every $\epsilon > 0$ and $\lambda \in (0, 1)$ such that*

$$\varphi(x_k - \xi; r + \epsilon) > 1 - \lambda \text{ and } \vartheta(x_k - \xi, r + \epsilon) < \lambda \text{ for all } k \geq k_0.$$

*It is denoted by $r_{(\varphi,\vartheta)} - \lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r_{(\varphi,\vartheta)}} \xi$.*

**Definition 2.2.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ for some non-negative number $r$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,*

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi, r + \epsilon) \geq \lambda\}) = 0.$$

*It is denoted by $r\text{-}St_{(\varphi,\vartheta)}\text{-}\lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} \xi$.*

**Remark 2.3.** *For the case $r = 0$, the notion rough statistical convergence with respect to the norm $(\varphi, \vartheta)$ agrees with the statistical convergence with respect to the norm $(\varphi, \vartheta)$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$.*

The $r\text{-}St_{(\varphi,\vartheta)}$-limit of a sequence may be not unique. Therefore, we consider $r\text{-}St_{(\varphi,\vartheta)}$-limit set of the sequence $x = \{x_k\}$ as $St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \{\xi : x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} \xi\}$. Moreover, sequence $x = \{x_k\}$ is $r_{(\varphi,\vartheta)}$-convergent if $LIM_x^{r_{(\varphi,\vartheta)}} \neq \phi$ where $LIM_x^{r_{(\varphi,\vartheta)}} = \{\xi^* \in \mathbb{X} : x_k \xrightarrow{r_{(\varphi,\vartheta)}} \xi^*\}$. For unbounded sequence $LIM_x^{r_{(\varphi,\vartheta)}}$ is always empty.
But in case of rough statistical convergence in $(\mathbb{X}, \varphi, \vartheta)$ which is an IFNS, we have $St_{(\varphi,\vartheta)}\text{-}LIM_x^r \neq \phi$ even though sequence may be unbounded. For this we have given the next example.

**Example 2.4.** *Consider any real normed space $(\mathbb{X}, \|.\|)$, take $\varphi(x, t) = \frac{t}{t+\|x\|}$ , $\vartheta(x, t) = \frac{\|x\|}{t+\|x\|}$ for every $t > 0$ and all $x \in \mathbb{X}$. Then, triplet $(\mathbb{X}, \varphi, \vartheta)$ is an IFNS.*
*Now, define a sequence*

$$x_k = \begin{cases} (-1)^k & k \neq n^2 \\ k & \text{otherwise} \end{cases}$$

*Then*

$$St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \begin{cases} \phi & r < 1 \\ [1 - r, r - 1] & \text{otherwise} \end{cases}$$

*and $St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \phi$ for all $r \geq 0$. Thus, this sequence is divergent in ordinary sense as it is unbounded. Also, the sequence is not rough convergent in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ for any $r$.*

Now, we are giving definition of rough statistically bounded sequence in an IFNS as follows:

**Definition 2.5.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough statistically bounded with respect to the norm $(\varphi, \vartheta)$ for some non-negative number $r$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists a real number $M > 0$ such that*

$$\delta(\{k \in \mathbb{N} : \varphi(x_k; M) \leq 1 - \lambda \text{ or } \vartheta(x_k, M) \geq \lambda\}) = 0.$$

In view of the above definitions, we obtained the following interesting results on rough statistical convergence in an IFNS.

**Theorem 2.6.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS with intuitionistic fuzzy norm $(\varphi, \vartheta)$. A sequence $x = \{x_k\}$ in $\mathbb{X}$ is statistically bounded if and only if $St_{(\varphi,\vartheta)}$-$LIM_x^r \neq \phi$ for some $r > 0$.*

*Proof. Necessary part*:
Consider the sequence $x = \{x_k\}$ which is statistically bounded in an IFNS $(\mathbb{X}, \varphi, \vartheta)$. Then, for every $\epsilon > 0$, $\lambda \in (0, 1)$ and some $r > 0$ there exists a real number $M > 0$ such that

$$\delta(\{k \in \mathbb{N} : \varphi(x_k; M) \leq 1 - \lambda \text{ or } \vartheta(x_k, M) \geq \lambda\}) = 0.$$

Let $K = \{k \in \mathbb{N} : \varphi(x_k; M) \leq 1 - \lambda \text{ or } \vartheta(x_k, M) \geq \lambda\}$.
For $k \in K^c$ we have $\varphi(x_k; M) > 1 - \lambda$ and $\vartheta(x_k, M) < \lambda$ .
Also

$$\begin{aligned}\varphi(x_k; r + M) &\geq \min\{\varphi(0; r), \varphi(x_k; M)\}\\ &= \min\{1, \varphi(x_k; M)\}\\ &> 1 - \lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_k; r + M) &\leq \max\{\vartheta(0; r), \vartheta(x_k; M)\}\\ &= \max\{0, \vartheta(x_k; M)\}\\ &< \lambda.\end{aligned}$$

Hence, $0 \in St_{(\varphi,\vartheta)}$-$LIM_x^r$. Therefore, $St_{(\varphi,\vartheta)}$-$LIM_x^r \neq \phi$.
*Sufficient Part:*
Let $St_{(\varphi,\vartheta)}$-$LIM_x^r \neq \phi$ for some $r > 0$. Then there exists $\xi \in \mathbb{X}$ such that $\xi \in St_{(\varphi,\vartheta)}$-$LIM_x^r$. For every $\epsilon > 0$ and $\lambda \in (0, 1)$ we have

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi, r + \epsilon) \geq \lambda\}) = 0.$$

Therefore, almost all $x_k$'s are contained in some ball with center $\xi$ which implies that sequence $x = \{x_k\}$ is statistically bounded in an IFNS $(\mathbb{X}, \varphi, \vartheta)$. □

Next, we discuss the algebraic characterization of rough statistically convergent sequences in an IFNS.

**Theorem 2.7.** *Let $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences in an IFNS $(\mathbb{X}, \varphi, \vartheta)$. Then for some non-negative number $r$ the following holds*

1. *If $x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} x_0$ and $\alpha \in \mathbb{N}$ then $\alpha x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} \alpha x_0$,*
2. *If $x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} x_0$ and $y_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} y_0$ then $(x_k + y_k) \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} (x_0 + y_0)$.*

*Proof.* Proof of above results are obvious so we are omitting them. □

If $x' = \{x_{k_i}\}$ be a subsequence of $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ then $LIM_{x_k}^{r(\varphi,\vartheta)} \subset LIM_{x_{k_i}}^{r(\varphi,\vartheta)}$. But this fact does not hold in case of statistical convergence. This can be justified by the next example.

**Example 2.8.** *For real normed space $(\mathbb{X}, ||.||)$, we define $\varphi(x, t) = \frac{t}{t+||x||}$ , $\vartheta(x, t) = \frac{||x||}{t+||x||}$ for every $t > 0$ and all $x \in \mathbb{X}$. Then, $(\mathbb{X}, \varphi, \vartheta)$ is an IFNS. Also the sequence*

$$x_k = \begin{cases} k & k \neq n^2 \\ 0 & \textit{otherwise} \end{cases}$$

*have $St_{(\varphi,\vartheta)}$-$LIM_x^r = [-r, r]$. And its subsequence $x' = \{1, 4, 9, .......\}$ have $St_{(\varphi,\vartheta)}$-$LIM_{x'}^r = \phi$.*

But this fact is true for nonthin subsequences of the rough statistical convergent sequence in an IFNS which is explained by the next result.

**Theorem 2.9.** *If $x' = \{x_{k_i}\}$ be a nonthin subsequence of $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ then $St_{(\varphi,\vartheta)}$-$LIM_x^r \subset St_{(\varphi,\vartheta)}$-$LIM_{x'}^r$.*

*Proof.* Proof of above result is obvious so we are omitting it. □

**Theorem 2.10.** *The set $St_{(\varphi,\vartheta)}$-$LIM_x^r$ of a sequence $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ is a closed set.*

*Proof.* We have nothing to prove as $St_{(\varphi,\vartheta)}$-$LIM_x^r = \phi$.
Let $St_{(\varphi,\vartheta)}$-$LIM_x^r \neq \phi$ for some $r > 0$ and consider $y = \{y_k\}$ be a convergent sequence in $St_{(\varphi,\vartheta)}$-$LIM_x^r$ with respect to the norm $(\varphi, \vartheta)$ to $y_0 \in \mathbb{X}$.
Then for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists a $k_1 \in \mathbb{N}$ such that

$$\varphi\left(y_k - y_0; \frac{\epsilon}{2}\right) > 1 - \lambda \text{ and } \vartheta\left(y_k - y_0; \frac{\epsilon}{2}\right) < \lambda \text{ for all } k \geq k_1.$$

Let us choose $y_m \in St_{(\varphi,\vartheta)}$-$LIM_x^r$ with $m > k_1$ such that

$$\delta\left(\{k \in \mathbb{N} : \varphi\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1 - \lambda \text{ or } \vartheta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \geq \lambda\}\right) = 0. \tag{1}$$

For $j \in \{k \in \mathbb{N} : \varphi\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1 - \lambda \text{ and } \vartheta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda\}$ we have $\varphi\left(x_j - y_m; r + \frac{\epsilon}{2}\right) > 1 - \lambda$ and $\vartheta\left(x_j - y_m; r + \frac{\epsilon}{2}\right) < \lambda$. Then, we have

$$\begin{aligned}\varphi(x_j - y_0; r + \epsilon) &\geq \min\left\{\varphi\left(x_j - y_m; r + \frac{\epsilon}{2}\right), \varphi\left(y_m - y_0; \frac{\epsilon}{2}\right)\right\}\\ &> 1 - \lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_j - y_0; r + \epsilon) &\leq \max\left\{\vartheta\left(x_j - y_m; r + \frac{\epsilon}{2}\right), \vartheta\left(y_m - y_0; \frac{\epsilon}{2}\right)\right\}\\ &< \lambda.\end{aligned}$$

Hence, $j \in \{k \in \mathbb{N} : \varphi(x_k - y_0; r + \epsilon) > 1 - \lambda \text{ and } \vartheta(x_k - y_0; r + \epsilon) < \lambda\}$. Now we have the following inclusion

$$\begin{aligned}&\{k \in \mathbb{N} : \varphi\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1 - \lambda \text{ and } \vartheta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda\}\\ &\subseteq \{k \in \mathbb{N} : \varphi(x_k - y_0; r + \epsilon) > 1 - \lambda \text{ and } \vartheta(x_k - y_0; r + \epsilon) < \lambda\}\end{aligned}$$

Therefore,

$$\begin{aligned}&\delta(\{k \in \mathbb{N} : \varphi(x_k - y_0; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - y_0; r + \epsilon) \geq \lambda\})\\ \leq\ &\delta\left(\{k \in \mathbb{N} : \varphi\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1 - \lambda \text{ or } \vartheta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \geq \lambda\}\right)\end{aligned}$$

Using (1) we get

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - y_0; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - y_0; r + \epsilon) \geq \lambda\}) = 0$$

Therefore, $y_0 \in St_{(\varphi,\vartheta)}$-$LIM_x^r$. □

In next result, we are proving the convexity of the set $St_{(\varphi,\vartheta)}$-$LIM_x^r$.

**Theorem 2.11.** *Let $x = \{x_k\}$ be a sequence in an IFNS $(\mathbb{X}, \varphi, \vartheta)$. Then, rough statistical limit set $St_{(\varphi,\vartheta)}$-$LIM_x^r$ with respect to the norm $(\varphi, \vartheta)$ is convex for some non-negative number $r$.*

*Proof.* Let $\xi_1, \xi_2 \in St_{(\varphi,\vartheta)}\text{-}LIM_x^r$. For the convexity of the set $St_{(\varphi,\vartheta)}\text{-}LIM_x^r$, we have to show that $[(1-\beta)\xi_1+\beta\xi_2] \in St_{(\varphi,\vartheta)}\text{-}LIM_x^r$ for some $\beta \in (0,1)$.
Now for every $\epsilon > 0$ and $\lambda \in (0,1)$, we define

$$M_1 = \{k \in \mathbb{N} : \varphi\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \leq 1-\lambda \text{ or } \vartheta\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \geq \lambda\},$$

$$M_2 = \{k \in \mathbb{N} : \varphi\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \leq 1-\lambda \text{ or } \vartheta\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \geq \lambda\}.$$

As $\xi_1, \xi_2 \in St_{(\varphi,\vartheta)}\text{-}LIM_x^r$, we have $\delta(M_1) = \delta(M_2) = 0$. For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}
\varphi(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) &= \varphi((1-\beta)(x_k-\xi_1) + \beta(x_k-\xi_2); r+\epsilon) \\
&\geq \min\left\{\varphi\left((1-\beta)(x_k-\xi_1); \frac{r+\epsilon}{2}\right), \varphi\left(\beta(x_k-\xi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \min\left\{\varphi\left(x_k-\xi_1; \frac{r+\epsilon}{2(1-\beta)}\right), \varphi\left(x_k-\xi_2; \frac{r+\epsilon}{2\beta}\right)\right\} \\
&> 1-\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) &= \vartheta((1-\beta)(x_k-\xi_1) + \beta(x_k-\xi_2); r+\epsilon) \\
&\leq \max\left\{\vartheta\left((1-\beta)(x_k-\xi_1); \frac{r+\epsilon}{2}\right), \vartheta\left(\beta(x_k-\xi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \max\left\{\vartheta\left(x_k-\xi_1; \frac{r+\epsilon}{2(1-\beta)}\right), \vartheta\left(x_k-\xi_2; \frac{r+\epsilon}{2\beta}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Thus,

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) \leq 1-\lambda \text{ or } \vartheta(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) \geq 1-\lambda\}) = 0.$$

Hence, $[(1-\beta)\xi_1 + \beta\xi_2] \in St_{(\varphi,\vartheta)}\text{-}LIM_x^r$ *i.e.* $St_{(\varphi,\vartheta)}\text{-}LIM_x^r$ is a convex set. □

**Theorem 2.12.** *A sequence $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ is rough statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ for some non-negative number $r$ if there exists a sequence $y = \{y_k\}$ in $\mathbb{X}$, which is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ and for every $\lambda \in (0,1)$ have $\varphi(x_k - y_k; r) > 1-\lambda$ and $\vartheta(x_k - y_k; r) < \lambda$ for all $k \in \mathbb{N}$.*

*Proof.* Let $\epsilon > 0$ and $\lambda \in (0,1)$. Consider $y_k \xrightarrow{St_{(\varphi,\vartheta)}} \xi$ and $\varphi(x_k - y_k; r) > 1-\lambda$ and $\vartheta(x_k - y_k; r) < \lambda$ for all $k \in \mathbb{N}$. For given $\lambda \in (0,1)$ define

$$A = \{k \in \mathbb{N} : \varphi(y_k - \xi; \epsilon) \leq 1-\lambda \text{ or } \vartheta(y_k - \xi; \epsilon) \geq \lambda\}$$

$$B = \{k \in \mathbb{N} : \varphi(x_k - y_k; r) \leq 1-\lambda \text{ or } \vartheta(x_k - y_k; r) \geq \lambda\}$$

Clearly, $\delta(A) = 0$ and $\delta(B) = 0$. For $k \in A^c \cap B^c$ we have

$$\begin{aligned}
\varphi(x_k - \xi; r+\epsilon) &\geq \min\{\varphi(x_k - y_k; r), \varphi(y_k - \xi; \epsilon)\} \\
&> 1-\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\vartheta(x_k - \xi; r+\epsilon) &\leq \max\{\vartheta(x_k - y_k; r), \vartheta(y_k - \xi; \epsilon)\} \\
&< \lambda.
\end{aligned}$$

Then $\varphi(x_k - \xi; r + \epsilon) > 1 - \lambda$ and $\vartheta(x_k - \xi; r + \epsilon) < \lambda$ for all $k \in A^c \cap B^c$.
This implies that $\{k \in \mathbb{N} : \varphi(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi; r + \epsilon) \geq \lambda\} \subseteq A \cup B$.
Then, $\delta(\{k \in \mathbb{N} : \varphi(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi; r + \epsilon) \geq \lambda\}) \leq \delta(A) + \delta(B)$.
Hence, we get $\delta(\{k \in \mathbb{N} : \varphi(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi; r + \epsilon) \geq \lambda\}) = 0$.
Therefore, $x_k \xrightarrow{r\text{-}St_{(\varphi,\vartheta)}} \xi$. □

**Theorem 2.13.** *Let $x = \{x_k\}$ be a sequence in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ then there does not exist elements $y, z \in St_{(\varphi,\vartheta)}\text{-}LIM^r_x$ for some $r > 0$ and every $\lambda \in (0, 1)$ such that $\varphi(y - z; mr) \leq 1 - \lambda$ or $\vartheta(y - z; mr) \geq \lambda$ for $m > 2$.*

*Proof.* We prove this result by contradiction. Assume there exists elements $y, z \in St_{(\varphi,\vartheta)}\text{-}LIM^r_x$ such that

$$\varphi(y - z; mr) \leq 1 - \lambda \text{ or } \vartheta(y - z; mr) \geq \lambda \text{ for } m > 2 \tag{2}$$

As $y, z \in St_{(\varphi,\vartheta)}\text{-}LIM^r_x$. For given $\lambda \in (0, 1)$ and every $\epsilon > 0$, we have $\delta(K_1) = \delta(K_2) = 0$ where $K_1 = \{k \in \mathbb{N} : \varphi\left(x_k - y; r + \frac{\epsilon}{2}\right) \leq 1 - \lambda \text{ or } \vartheta\left(x_k - y; r + \frac{\epsilon}{2}\right) \geq \lambda\}$ and $K_2 = \{k \in \mathbb{N} : \varphi\left(x_k - z; r + \frac{\epsilon}{2}\right) \leq 1 - \lambda \text{ or } \vartheta\left(x_k - z; r + \frac{\epsilon}{2}\right) \geq \lambda\}$. For $k \in K_1^c \cap K_2^c$ we have

$$\begin{aligned}\varphi(y - z; 2r + \epsilon) &\geq \min\left\{\varphi\left(x_k - z; r + \frac{\epsilon}{2}\right), \varphi\left(x_k - y; r + \frac{\epsilon}{2}\right)\right\} \\ &> 1 - \lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(y - z; 2r + \epsilon) &\leq \max\left\{\vartheta\left(x_k - z; r + \frac{\epsilon}{2}\right), \vartheta\left(x_k - y; r + \frac{\epsilon}{2}\right)\right\} \\ &< \lambda.\end{aligned}$$

Hence,

$$\varphi(y - z; 2r + \epsilon) > 1 - \lambda \text{ and } \vartheta(y - z; 2r + \epsilon) < \lambda. \tag{3}$$

Then, from (3) we have

$$\varphi(y - z; mr) > 1 - \lambda \text{ and } \vartheta(y - z; mr) < \lambda \text{ for } m > 2.$$

which is a contradiction to (2). Therefore, there does not exists elements $y, z \in St_{(\varphi,\vartheta)}\text{-}LIM^r_x$ such that $\varphi(y - z; mr) \leq 1 - \lambda$ or $\vartheta(y - z; mr) \geq \lambda$ for $m > 2$. □

Next, we define statistical cluster point of a sequence in IFNS and establish some results related to it.

**Definition 2.14.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS. Then $\gamma \in \mathbb{X}$ is called rough statistical cluster point of the sequence $x = \{x_k\}$ in $\mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ for some non-negative number $r$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,*

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \gamma; r + \epsilon) > 1 - \lambda \textit{ and } \vartheta(x_k - \gamma; r + \epsilon) < \lambda\}) > 0,$$

*i.e.*

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \gamma; r + \epsilon) > 1 - \lambda \textit{ and } \vartheta(x_k - \gamma; r + \epsilon) < \lambda\}) \neq 0.$$

*In this case, $\gamma$ is known as $r$-$St_{(\varphi,\vartheta)}$-cluster point of a sequence $x = \{x_k\}$.*

Let $\Gamma^r_{(\varphi,\vartheta)}(x)$ denotes the set of all $r$-$St_{(\varphi,\vartheta)}$-cluster points with respect to the norm $(\varphi, \vartheta)$ of a sequence $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$. If $r = 0$ then we get ordinary statistical cluster point with respect to the norm $(\varphi, \vartheta)$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ *i.e.* $\Gamma^r_{(\varphi,\vartheta)}(x) = \Gamma_{(\varphi,\vartheta)}(x)$.

**Theorem 2.15.** *Let $(\mathbb{X}, \varphi, \vartheta)$ be an IFNS. Then, $\Gamma^r_{(\varphi,\vartheta)}(x)$ which is the set of of all $r$-$St_{(\varphi,\vartheta)}$-cluster points with respect to the norm $(\varphi, \vartheta)$ of any sequence $x = \{x_k\}$ is closed for some non-negative real number $r$.*

*Proof.* (i) If $\Gamma^r_{(\varphi,\vartheta)}(x) = \phi$, then we have to prove nothing.

(ii) If $\Gamma^r_{(\varphi,\vartheta)}(x) \neq \phi$. Then, take a sequence $y = \{y_k\} \subseteq \Gamma^r_{(\varphi,\vartheta)}(x)$ such that $y_k \xrightarrow{(\varphi,\vartheta)} y_*$. It is sufficient to show that $y_* \in \Gamma^r_{(\varphi,\vartheta)}(x)$.

As $y_k \xrightarrow{(\varphi,\vartheta)} y_*$, then for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $k_\epsilon \in \mathbb{N}$ such that $\varphi\left(y_k - y_*; \frac{\epsilon}{2}\right) > 1-\lambda$ and $\vartheta\left(y_k - y_*; \frac{\epsilon}{2}\right) < \lambda$ for $k \geq k_\epsilon$.

Now choose $k_0 \in \mathbb{N}$ such that $k_0 \geq k_\epsilon$. Then, we have $\varphi\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) > 1-\lambda$ and $\vartheta\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) < \lambda$. Again as $y = \{y_k\} \subseteq \Gamma^r_{(\varphi,\vartheta)}(x)$, we have $y_{k_0} \in \Gamma^r_{(\varphi,\vartheta)}(x)$.

$$\Rightarrow \delta\left(\left\{k \in \mathbb{N} : \varphi\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \vartheta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}\right) > 0. \tag{4}$$

Choose $j \in \left\{k \in \mathbb{N} : \varphi\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \vartheta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}$, then we have $\varphi\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda$ and $\vartheta\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda$.

$$\begin{aligned}\varphi(x_j - y_*; r+\epsilon) &\geq \min\left\{\varphi\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right), \varphi\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right)\right\}\\ &> 1-\lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_j - y_*; r+\epsilon) &\leq \max\left\{\vartheta\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right), \vartheta\left(y_{k_0} - y_\odot; \frac{\epsilon}{2}\right)\right\}\\ &< \lambda.\end{aligned}$$

Thus, $j \in \{k \in \mathbb{N} : \varphi(x_k - y_*; r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k - y_*; r+\epsilon) < \lambda\}$.
Hence

$$\begin{aligned}&\{k \in \mathbb{N} : \varphi\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \vartheta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\}\\ &\quad\subseteq \{k \in \mathbb{N} : \varphi(x_k - y_*; r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k - y_*; r+\epsilon) < \lambda\}.\end{aligned}$$

Now,

$$\begin{aligned}&\delta(\{k \in \mathbb{N} : \varphi\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \vartheta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\})\\ &\quad\leq \delta(\{k \in \mathbb{N} : \varphi(x_k - y_*; r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k - y_*; r+\epsilon) < \lambda\}).\end{aligned} \tag{5}$$

Using equation (4), we obtained that the set on left side of (5) has natural density more than 0.

$$\Rightarrow \delta(\{k \in \mathbb{N} : \varphi(x_k - y_*; r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k - y_*; r+\epsilon) < \lambda\}) > 0.$$

Therefore, $y_* \in \Gamma^r_{(\varphi,\vartheta)}(x)$. □

**Theorem 2.16.** *Let $\Gamma_{(\varphi,\vartheta)}(x)$ be the set of all statistical cluster points with respect to the norm $(\varphi,\vartheta)$ of a sequence $x = \{x_k\}$ in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ and $r$ be some non-negative real number. Then, for an arbitrary $\gamma \in \Gamma_{(\varphi,\vartheta)}(x)$ and $\lambda \in (0,1)$ we have $\varphi(\xi - \gamma; r) > 1-\lambda$ and $\vartheta(\xi - \gamma; r) < \lambda$ for all $\xi \in \Gamma^r_{(\varphi,\vartheta)}(x)$.*

*Proof.* Let $\gamma \in \Gamma_{(\varphi,\vartheta)}(x)$. Then, for every $\epsilon > 0$ and $\lambda \in (0,1)$ we have

$$\delta(\{k \in \mathbb{N} : \varphi(x_k - \gamma; \epsilon) > 1-\lambda \text{ and } \vartheta(x_k - \gamma; \epsilon) < \lambda\}) > 0. \tag{6}$$

Now we will show that if $\xi \in \mathbb{X}$ have $\varphi(\xi-\gamma;r) > 1-\lambda$ and $\vartheta(\xi-\gamma;r) < \lambda$ then $\xi \in \Gamma^r_{(\varphi,\vartheta)}(x)$.
Let $j \in \{k \in \mathbb{N} : \varphi(x_k-\gamma;\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\gamma;\epsilon) < \lambda\}$, then $\varphi(x_j-\gamma;\epsilon) > 1-\lambda$ and $\vartheta(x_j-\gamma;\epsilon) < \lambda$. Now,

$$\begin{aligned}\varphi(x_j-\xi;r+\epsilon) &\geq \min\left\{\varphi(x_j-\gamma;\epsilon), \varphi(\xi-\gamma;r)\right\}\\ &> 1-\lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_j-\xi;r+\epsilon) &\leq \max\left\{\vartheta(x_j-\gamma;\epsilon), \vartheta(\xi-\gamma;r)\right\}\\ &< \lambda.\end{aligned}$$

We have $\varphi(x_j-\xi;r+\epsilon) > 1-\lambda$ and $\vartheta(x_j-\xi;r+\epsilon) < \lambda$. Thus $j \in \{k \in \mathbb{N} : \varphi(x_k-\xi;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\xi;\epsilon) < \lambda\}$.
Now the next inclusion holds.

$$\begin{aligned}&\{k \in \mathbb{N} : \varphi(x_k-\gamma;\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\gamma;\epsilon) < \lambda\}\\ \subseteq &\{k \in \mathbb{N} : \varphi(x_k-\xi;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\xi;r+\epsilon) < \lambda\}.\end{aligned}$$

Then

$$\begin{aligned}&\delta(\{k \in \mathbb{N} : \varphi(x_k-\gamma;\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\gamma;\epsilon) < \lambda\})\\ \leq &\delta(\{k \in \mathbb{N} : \varphi(x_k-\xi;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\xi;r+\epsilon) < \lambda\}).\end{aligned}$$

Using equation (6) we get $\delta(\{k \in \mathbb{N} : \varphi(x_k-\xi;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\xi;r+\epsilon) < \lambda\}) > 0$. Therefore, $\xi \in \Gamma^r_{(\varphi,\vartheta)}(x)$. □

**Theorem 2.17.** *If $\overline{B(c,\lambda,r)} = \{x \in \mathbb{X} : \varphi(x-c;r) \geq 1-\lambda,\ \vartheta(x-c;r) \leq \lambda\}$ represents the closure of open ball $B(c,\lambda,r) = \{x \in \mathbb{X} : \varphi(x-c;r) > 1-\lambda,\ \vartheta(x-c;r) < \lambda\}$ for some $r > 0$, $\lambda \in (0,1)$ and fixed $c \in \mathbb{X}$ then $\Gamma^r_{(\varphi,\vartheta)}(x) = \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$.*

*Proof.* Let $\gamma \in \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$ then there exists $c \in \Gamma_{(\varphi,\vartheta)}(x)$ for some $r > 0$ and given $\lambda \in (0,1)$ such that $\varphi(c-\gamma;r) > 1-\lambda$ and $\vartheta(c-\gamma;r) < \lambda$.
Fix $\epsilon > 0$. Since $c \in \Gamma_{(\varphi,\vartheta)}(x)$ then there exists a set $K = \{k \in \mathbb{X} : \varphi(x_k-c;\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-c;\epsilon) < \lambda\}$ with $\delta(K) > 0$. Now, for $k \in K$,

$$\begin{aligned}\varphi(x_k-\gamma;r+\epsilon) &\geq \min\{\varphi(x_k-c;\epsilon), \varphi(c-\gamma;r)\}\\ &> 1-\lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_k-\gamma;r+\epsilon) &\leq \max\{\vartheta(x_k-c;\epsilon), \vartheta(c-\gamma;r)\}\\ &< \lambda.\end{aligned}$$

This implies that $\delta(\{k \in \mathbb{N} : \varphi(x_k-\gamma;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\gamma;r+\epsilon) < \lambda\}) > 0$. Hence, $\gamma \in \Gamma^r_{(\varphi,\vartheta)}(x)$.
Therefore, $\bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)} \subseteq \Gamma^r_{(\varphi,\vartheta)}(x)$.
Conversely,
Let $\gamma \in \Gamma^r_{(\varphi,\vartheta)}(x)$. Then we have to show that $\gamma \in \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$.
Let if possible, $\gamma \notin \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$ *i.e.* $\gamma \notin \overline{B(c,\lambda,r)}$ for all $c \in \Gamma_{(\varphi,\vartheta)}(x)$.

Then $\varphi(\gamma - c; r) \leq 1 - \lambda$ or $\vartheta(\gamma - c; r) \geq \lambda$ for every $c \in \Gamma_{(\varphi,\vartheta)}(x)$. By Theorem 2.16 for arbitrary $c \in \Gamma_{(\varphi,\vartheta)}(x)$ we have $\varphi(\gamma - c; r) > 1 - \lambda$ and $\vartheta(\gamma - c; r) < \lambda$ for every $c \in \Gamma^r_{(\varphi,\vartheta)}(x)$ which is a contradiction to the assumption. Therefore, $\gamma \in \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$. Hence, $\Gamma^r_{(\varphi,\vartheta)}(x) \subseteq \bigcup_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$. □

**Theorem 2.18.** *Let $x = \{x_k\}$ be a sequence in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ then for any $\lambda \in (0, 1)$,*

(i) *If $c \in \Gamma_{(\varphi,\vartheta)}(x)$ then $St_{(\varphi,\vartheta)}$-$LIM^r_x \subseteq \overline{B(c,\lambda,r)}$.*

(ii) *$St_{(\varphi,\vartheta)}$-$LIM^r_x = \bigcap_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.*

*Proof.* (i) Consider $\xi \in St_{(\varphi,\vartheta)}$-$LIM^r_x$ and $c \in \Gamma_{(\varphi,\vartheta)}(x)$.
For every $\epsilon > 0$ and $\lambda \in (0, 1)$ define sets

$$A = \{k \in \mathbb{N} : \varphi(x_k - \xi : r + \epsilon) > 1 - \lambda \text{ and } \vartheta(x_k - \xi : r + \epsilon) < \lambda\} \text{ with } \delta(A^c) = 0,$$

and

$$B = \{k \in \mathbb{N} : \varphi(x_k - c; \epsilon) > 1 - \lambda \text{ and } \vartheta(x_k - c; \epsilon) < \lambda\} \text{ with } \delta(B) \neq 0.$$

Now for $k \in A \cap B$ we have

$$\begin{aligned}\varphi(\xi - c; r) &\geq \min\{\varphi(x_k - c; \epsilon), \varphi(x_k - \xi; r + \epsilon)\}\\ &> 1 - \lambda.\end{aligned}$$

and

$$\begin{aligned}\vartheta(\xi - c; r) &\leq \max\{\vartheta(x_k - c; \epsilon), \vartheta(x_k - \xi; r + \epsilon)\}\\ &< \lambda.\end{aligned}$$

Therefore, $\xi \in \overline{B(c,\lambda,r)}$. Hence, $St_{(\varphi,\vartheta)}$-$LIM^r_x \subseteq \overline{B(c,\lambda,r)}$.

(ii) By previous part we have $St_{(\varphi,\vartheta)}$-$LIM^r_x \subseteq \bigcap_{c\in\Gamma_\varphi(x)} \overline{B(c,\lambda,r)}$.

Assume $y \in \bigcap_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)}$ then $\varphi(y - c; r) \geq 1 - \lambda$ and $\vartheta(y - c; r) \leq \lambda$ for all $c \in \Gamma_{(\varphi,\vartheta)}(x)$. This implies that $\Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(y,\lambda,r)}$, *i.e.* $\bigcap_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)} \subseteq \{\xi \in \mathbb{X} : \Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.

Further, let $y \notin St_{(\varphi,\vartheta)}$-$LIM^r_x$ then for $\epsilon > 0$ we have $\delta(\{k \in \mathbb{N} : \varphi(x_k - y; r + \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - y; r + \epsilon) \geq \lambda\}) \neq 0$, which implies that a statistical cluster point $c$ exists for the sequence $x = \{x_k\}$ with $\varphi(y-c; r+\epsilon) \leq 1-\lambda$ or $\vartheta(y-c; r+\epsilon) \geq \lambda$. Thus, $\Gamma_{(\varphi,\vartheta)}(x) \nsubseteq \overline{B(y,\lambda,r)}$ and $y \notin \{\xi \in \mathbb{X} : \Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$. This implies that $\{\xi \in \mathbb{X} : \Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\} \subseteq St_{(\varphi,\vartheta)}$-$LIM^r_x$ and we get $\bigcap_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)} \subseteq St_{(\varphi,\vartheta)}$-$LIM^r_x$.

Therefore, $St_{(\varphi,\vartheta)}$-$LIM^r_x = \bigcap_{c\in\Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{(\varphi,\vartheta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.

□

**Theorem 2.19.** *Let $x = \{x_k\}$ be a sequence in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ which is statistically convergent to $\xi \in \mathbb{X}$ with respect to the norm $(\varphi, \vartheta)$ then there exists $\lambda \in (0, 1)$ such that $St_{(\varphi,\vartheta)}$-$LIM^r_x = \overline{B(\xi,\lambda,r)}$ for some $r > 0$.*

*Proof.* Let $\epsilon > 0$. Since $x_k \xrightarrow{St_{(\varphi,\vartheta)}} \xi$ then there is a set $A = \{k \in \mathbb{N} : \varphi(x_k - \xi : \epsilon) \leq 1 - \lambda \text{ or } \vartheta(x_k - \xi : \epsilon) \geq \lambda\}$ with $\delta(A) = 0$. Consider $y \in \overline{B(\xi,\lambda,r)} = \{y \in \mathbb{X} : \varphi(y - \xi; r) \geq 1 - \lambda,\ \vartheta(y - \xi; r) \leq \lambda\}$.
For $k \in A^c$

$$\begin{aligned}\varphi(x_k - y; r + \epsilon) &\geq \min\{\varphi(x_k - \xi; \epsilon), \varphi(y - \xi; r)\}\\ &> 1 - \lambda,\end{aligned}$$

and

$$\begin{aligned}\vartheta(x_k - y; r + \epsilon) &\leq \max\{\vartheta(x_k - \xi; \epsilon), \vartheta(y - \xi; r)\}\\ &< \lambda.\end{aligned}$$

This implies that $y \in St_{(\varphi,\vartheta)}\text{-}LIM_x^r$, *i.e.* $\overline{B(\xi, \lambda, r)} \subseteq St_{(\varphi,\vartheta)}\text{-}LIM_x^r$. Also $St_{(\varphi,\vartheta)}\text{-}LIM_x^r \subseteq \overline{B(\xi, \lambda, r)}$. Hence, $St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \overline{B(\xi, \lambda, r)}$. □

**Theorem 2.20.** *Let $x = \{x_k\}$ be a sequence in an IFNS $(\mathbb{X}, \varphi, \vartheta)$ which converges statistically with respect to the norm $(\varphi, \vartheta)$ then $\Gamma^r_{(\varphi,\vartheta)}(x) = St_{(\varphi,\vartheta)}\text{-}LIM_x^r$ for some $r > 0$.*

*Proof. Necessary part*:
Suppose $x_k \xrightarrow{St_{(\varphi,\vartheta)}} \xi$. Then $\Gamma_{(\varphi,\vartheta)}(x) = \{\xi\}$. By Theorem 2.17 for some $r > 0$ and $\lambda \in (0, 1)$ we have $\Gamma^r_{(\varphi,\vartheta)}(x) = \overline{B(\xi, \lambda, r)}$. Also by Theorem 2.19 we get $\overline{B(\xi, \lambda, r)} = St_{(\varphi,\vartheta)}\text{-}LIM_x^r$. Hence, $\Gamma^r_{(\varphi,\vartheta)}(x) = St_{(\varphi,\vartheta)}\text{-}LIM_x^r$.

*Sufficient part*:
Let $\Gamma^r_{(\varphi,\vartheta)}(x) = St_{(\varphi,\vartheta)}\text{-}LIM_x^r$. By Theorem 2.17 and Theorem 2.18(ii) we have

$$\bigcup_{c \in \Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c, \lambda, r)} = \bigcap_{c \in \Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c, \lambda, r)}.$$

This implies that either $\Gamma_{(\varphi,\vartheta)}(x) = \phi$ or $\Gamma_{(\varphi,\vartheta)}(x)$ is a singleton set. Then $St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \bigcap_{c \in \Gamma_{(\varphi,\vartheta)}(x)} \overline{B(c, \lambda, r)} = \overline{B(\xi, \lambda, r)}$ for some $\xi \in \Gamma_{(\varphi,\vartheta)}(x)$, further by Theorem 2.19 we get $St_{(\varphi,\vartheta)}\text{-}LIM_x^r = \{\xi\}$. □

## Acknowledgement

We express great sense of gratitude and deep respect to the referees of this paper and reviewers for their valuable suggestions.

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