Filomat 35:13 (2021), 4483–4494
https://doi.org/10.2298/FIL2113483T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Statistical Weighted Mean Summability of Double Sequences of Fuzzy Numbers

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Abstract. In this paper, we present statistical weighted mean (briefly, $(\overline{N}, p, q, 1, 1)$) summability method for double sequences of fuzzy numbers and give necessary and sufficient Tauberian conditions under which statistical convergence of a double sequence of fuzzy numbers follows from its statistical $(\overline{N}, p, q, 1, 1)$ summability. Furthermore, we apply our new method of summability to prove a fuzzy Korovkin type approximation theorem for a double sequence of fuzzy positive linear operators.

1. Introduction

The concept of statistical convergence of a double sequence has been presented by Mursaleen and Edely [20], Tripathy [31] and Móricz [18] independently. From these papers. it is know that Ordinary convergence implies statistical convergence, while the converse is not true, in general. Firstly, Edely and Mursaleen [15] gave Tauberian conditions under which the ordinary convergence of double sequences follows from the statistical convergence. Then many authors defined and studied statistical summability methods for double sequences and proved Tauberian Theorems for these methods. Móricz [19] defined statistical $(C, 1, 1)$ summability method and gave Tauberian theorems for double sequences that are statistically $(C, 1, 1)$ summable. Chen ve Chang [5] and Fekete [17] presented statistical $(\overline{N}, p, q, 1, 1)$ summability of a double sequence and generalized the results obtained by Móricz [19]. Chen and Chan [6] improved earlier results and gave Tauberian conditions under which the original convergence of double sequences follows from its statistical $(\overline{N}, p, q, 1, 1)$ summability. By means of a four-dimensional non-negative RH-regular summability matrix, a more general summability method for double sequences was given by Belen et al. [3]. This method was called statistical A summability. Belen et al. [3] applied their new method of summability to prove a Korovkin type approximation theorem for a function of two variables.

After fuzzy set theory was founded by Zadeh [37] in 1965, summability theory has developed a growing interest in fuzzy set theory. Authors have introduced various types of summability methods for sequences of fuzzy numbers and obtained corresponding Tauberian conditions [7–11, 14, 22, 25–29, 32, 35, 36].

2020 *Mathematics Subject Classification*. Primary 03E72; Secondary 40A05, 40G15, 40E05
Keywords. Double sequence of fuzzy numbers, Statistical convergence, Weighted mean method, Tauberian theorem, Fuzzy Korovkin theory
Received: 15 October 2020; Revised: 23 December 2020; Accepted: 24 December 2020
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Since the concept of statistical convergence for double sequences of fuzzy numbers defined by Savaş and Mursaleen [24], many results obtained for double sequences of real numbers have been extended to double sequences of fuzzy numbers. First Tauberian theorems for statistically convergent sequences of fuzzy numbers have been given by Talo and Beyazit [30]. After this Yapalı ve Talo [34] and Önder at al. [21] presented statistical $(C, 1, 1)$ summability method for double sequences of fuzzy numbers.

Demirci and Karakuş [12] proved a fuzzy Korovkin-type approximation theorem for double sequences of fuzzy positive linear operators by using statistical convergence. Yapalı ve Talo [34] applied statistical (C,1,1) summability method to prove a fuzzy Korovkin-type approximation theorem.

In this paper, we extend the results obtained by Chen [5] and by Chen and Chan [6] to double sequences of fuzzy numbers. First, we present statistical $(\overline{N}, p, q, 1, 1)$ summability of a double sequence of fuzzy numbers and later we give Tauberian theorems for this method. By using this summability method we prove a fuzzy Korovkin-type approximation theorem. Also we have generalized the results of Yapalı ve Talo [34] from statistical (C,1,1) summability to statistical $(\overline{N}, p, q, 1, 1)$ summability.

2. Preliminaries

In this section, the basic notions and results related to fuzzy numbers without proof from [2, 13, 16] are summarized.

Throughout this paper, $\mathbb{R}$ is the real line and $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all natural numbers. The term fuzzy set of $\mathbb{R}$ stands for a function from $\mathbb{R}$ to $[0, 1]$. Let u be a fuzzy set of $\mathbb{R}$. For all $\alpha \in (0, 1]$, α-level set $[u]_\alpha$ is defined by $\{x \in \mathbb{R} : u(x) \geq \alpha\}$ and the support of u, denoted by $[u]_0$, is the set $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$. A fuzzy number is a fuzzy set of $\mathbb{R}$ which is normal, has bounded support, and is upper semi-continuous and quasi-concave as a function on its support. We denote the set of all fuzzy numbers by E^1 and called it the space of fuzzy numbers. $u \in E^1$ if and only if $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0, 1]$. We denote $[u]_\alpha = [u^-(\alpha), u^-(\alpha)]$ for $\alpha \in [0, 1]$.

It is evident that a real number r can be considered as a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(x) := \begin{cases} 1 & , \quad \text{if } x = r, \\ 0 & , \quad \text{if } x \neq r. \end{cases}$$

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then addition and scalar multiplication of fuzzy numbers are defined by

$$[u + v]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{and} \quad [ku]_\alpha = k[u]_\alpha,\ \alpha \in [0, 1].$$

These operations have the following properties.

Lemma 2.1. *[2] The following equalities hold true:*

(i) $\tilde{0} + u = u + \tilde{0} = u$ *for any* $u \in E^1$.

(ii) $(k + l)u = ku + lu$ *for any* $k, l \in \mathbb{R}$ *with* $kl \geq 0$ *and* $u \in E^1$.

(iii) $k(u + v) = ku + kv$ *for any* $k \in \mathbb{R}$ *and* $u, v \in E^1$.

(iv) $k(lu) = (kl)u$ *for any* $k, l \in \mathbb{R}$ *and* $u \in E^1$.

For $u, v \in E^1$ set

$$D(u, v) := \sup_{\alpha \in [0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\},$$

then D is a metric on E^1 and has the following properties.

Proposition 2.2. *[2] Let* $u, v, w, z \in E^1$ *and* $k \in \mathbb{R}$. *Then,*

(i) (E^1, D) *is a complete metric space.*

(ii) $D(ku, kv) = |k|D(u, v)$.

(iii) $D(u + v, w + v) = D(u, w)$.

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.

The partial ordering relation on E^1 is defined as follows:

$$\begin{aligned} u \leq v \quad &\Longleftrightarrow \quad [u]_\alpha \leq [v]_\alpha \\ &\Longleftrightarrow \quad u^-(\alpha) \leq v^-(\alpha) \text{ and } u^+(\alpha) \leq v^+(\alpha) \quad \text{for all } \alpha \in [0, 1]. \end{aligned}$$

The concept of ordinary convergence of a double sequence of fuzzy numbers was presented by Savaş [23] as follows:

A double sequence $u = (u_{mn})_{m,n=0}^{\infty}$ of fuzzy numbers is called convergent to μ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $D(u_{mn}, \mu) < \varepsilon$; whenever $\min(m, n) \geq n_0$ and this is written as $u_{mn} \to \mu$ or $\lim_{m,n\to\infty} u_{mn} = \mu$.

In the remainder of this paper, we assume (u_{mn}) to be a double sequence of fuzzy numbers and μ to be a fuzzy number.

We call (u_{mn}) bounded if $\sup_{m,n\in\mathbb{N}} D(u_{mn}, \tilde{0}) < \infty$.

Savaş and Mursaleen [24] have presented the concept of statistical convergence of a double sequence of fuzzy numbers. $u = (u_{mn})$ is called statistically convergent to μ if for every $\varepsilon > 0$

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left|\{m \leq M \text{ and } n \leq N : D(u_{mn}, \mu) \geq \varepsilon\}\right| = 0$$

holds, where $|\cdot|$ denotes the cardinality of a given set, and this is denoted as

$$u_{mn} \xrightarrow{st} \mu \quad \text{or} \quad st - \lim_{m,n\to\infty} u_{mn} = \mu.$$

Let $p = (p_j)$ and $q = (q_k)$ be two sequences of nonnegative real numbers with $p_0 > 0$, $q_0 > 0$ and

$$P_m = \sum_{j=0}^{m} p_j \ (m = 0, 1, 2, \ldots) \quad \text{ve} \quad Q_n = \sum_{k=0}^{n} q_k \ (n = 0, 1, 2, \ldots).$$

The weighted means t_{mn}^{11} of (u_{jk}) are defined by

$$t_{mn}^{11} := \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k u_{jk}, \quad m, n = 0, 1, 2, \ldots$$

We write $u_{mn} \longrightarrow \mu \ (\overline{N}, p, q, 1, 1)$ if $t_{mn}^{11} \to \mu$. We also write $u_{mn} \xrightarrow{st} \mu \ (\overline{N}, p, q, 1, 1)$ if $t_{mn}^{11} \xrightarrow{st} \mu$.

We know that $u_{mn} \to \mu \Rightarrow u_{mn} \xrightarrow{st} \mu$ and $u_{mn} \to \mu \Rightarrow u_{mn} \longrightarrow \mu \ (\overline{N}, p, q, 1, 1)$. However, the converses of these implications is false, in general. For the implication that $u_{mn} \xrightarrow{st} \mu \Rightarrow u_{mn} \to \mu$ several Tauberian results were found (see [21, 30]). Also Totur and Çanak [33] have given Tauberian conditions under which

$$u_{mn} \longrightarrow \mu \ (\overline{N}, p, q, 1, 1) \Rightarrow u_{mn} \longrightarrow \mu. \tag{1}$$

It is known that there is no implication from one of $u_{mn} \xrightarrow{st} \mu$ and $u_{mn} \xrightarrow{st} \mu \ (\overline{N}, p, q, 1, 1)$ to another in E^1. First, we prove that if a double sequence (u_{mn}) of fuzzy numbers is bounded, then the implication

$$u_{mn} \xrightarrow{st} \mu \Rightarrow u_{mn} \xrightarrow{st} \mu \ (\overline{N}, p, q, 1, 1) \tag{2}$$

holds. Then we introduce Tauberian conditions for the converse of (2). Furthermore, by combining our results with results of [30], we deduce that under Hardy conditions, $u_{mn} \xrightarrow{st} \mu \ (\overline{N}, p, q, 1, 1)$ implies validity of $u_{mn} \longrightarrow \mu$. Our results include the corresponding results in [21, 34] for the case $p_j = 1$ for all j and $q_k = 1$ for all k.

3. Main Results

Chen and Hsu [4] defined the class SVA_+. SVA_+ is the set of all nonnegative real sequences p with the property that $P_m \neq 0$ for all $m \geq 0$ and

$$\liminf_{m\to\infty}\left|\frac{P_{\lambda_m}}{P_m}-1\right|>0 \quad \text{for all } \lambda>0 \text{ with } \lambda\neq 1.$$

Here and subsequently, $\lambda_m := [\lambda m]$ and $[\cdot]$ denotes the integral part.

Lemma 3.1. *[4]. Let $p=(p_m)$ be a nonnegative sequence with $p_0>0$. Then $p \in SVA_+$ is equivalent to any of the following assertions:*

$$\begin{aligned}
&\liminf_{m\to\infty}\frac{P_{\lambda_m}}{P_m}>1 \qquad (\lambda>1),\\
&\limsup_{m\to\infty}\frac{P_{\lambda_m}}{P_m}<1 \qquad (0<\lambda<1),\\
&\liminf_{m\to\infty}\frac{P_m}{P_{\lambda_m}}>1 \qquad (0<\lambda<1),\\
&\limsup_{m\to\infty}\frac{P_m}{P_{\lambda_m}}<1 \qquad (\lambda>1).
\end{aligned}$$

First, we show that statistical $(\overline{N},p,q,1,1)$ summability method is a regular method for bounded double sequences of fuzzy numbers.

Theorem 3.2. *Let $p,q \in SVA_+$ and (u_{mn}) be a bounded double sequence of fuzzy numbers with $u_{mn} \xrightarrow{st} \mu$. Then $u_{mn} \xrightarrow{st} \mu\ (\overline{N},p,q,1,1)$.*

Proof. By Lemma 2.2, we have

$$\begin{aligned}
D(t^{11}_{mn},\mu) &= D\left(\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_ku_{jk},\mu\right)\\
&\leq \frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_kD(u_{jk},\mu).
\end{aligned}$$

We know that double real sequence $\{D(u_{jk},\mu)\}$ is bounded and $D(u_{jk},\mu) \xrightarrow{st} 0$. So we have

$$\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_kD(u_{jk},\mu) \xrightarrow{st} 0.$$

This implies that $t^{11}_{mn} \xrightarrow{st} \mu$. □

Note that for the case $p_j=1$ for all j and $q_k=1$ for all k, statistical $(\overline{N},p,q,1,1)$ summability reduces to statistical $(C,1,1)$ summability. From Example 2 of [34] we know that there exists a double sequence of fuzzy numbers which is statistical $(C,1,1)$ summable to a fuzzy number, but not statistical convergent to any fuzzy number.

Lemma 3.3. *Let $u_{mn} \xrightarrow{st} \mu\ (\overline{N},p,q,1,1)$. Then for every $\lambda>0$, we have $t^{11}_{\lambda_m,\lambda_n} \xrightarrow{st} \mu$, $t^{11}_{m,\lambda_n} \xrightarrow{st} \mu$ and $t^{11}_{\lambda_m,n} \xrightarrow{st} \mu$ where $\lambda_n := [\lambda n]$.*

Proof. Replacing absolute value with metric D in Lemma 2.3. of [5], proof can be obtained easily. □

The following lemma gives a fuzzy version of Theorem 3.1 in [5]. Also, It generalizes Lemma 4 of [34].

Lemma 3.4. *Let $p, q \in SVA_+$ and $u_{mn} \xrightarrow{st} \mu\ (\overline{N}, p, q; 1, 1)$. Then for $\lambda > 1$*

$$st - \lim_{m,n\to\infty} \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k u_{jk} = \mu \tag{3}$$

and for $0 < \lambda < 1$

$$st - \lim_{m,n\to\infty} \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k u_{jk} = \mu. \tag{4}$$

Proof. Let $\lambda > 1$. By Lemma 4 in [33],

$$\begin{aligned} D\left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k u_{jk}, \mu\right) &\le D\left(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k u_{jk}, t_{mn}^{11}\right) + D(t_{mn}^{11}, \mu) \\ &\le \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} D\left(t_{\lambda_m,\lambda_n}^{11}, t_{\lambda_m,n}^{11}\right) \\ &+ \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} D\left(t_{mn}^{11}, t_{m,\lambda_n}^{11}\right) \\ &+ \frac{P_{\lambda_m}}{(P_{\lambda_m} - P_m)} D\left(t_{\lambda_m,n}^{11}, t_{mn}^{11}\right) \\ &+ \frac{Q_{\lambda_n}}{(Q_{\lambda_n} - Q_n)} D\left(t_{m,\lambda_n}^{11}, t_{mn}^{11}\right) + D(t_{mn}^{11}, \mu). \end{aligned} \tag{5}$$

By Lemma 3.1 we obtain

$$\begin{aligned} \limsup_{m\to\infty} \frac{P_{\lambda_m}}{(P_{\lambda_m} - P_m)} &= \left(1 - \limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}}\right)^{-1} < \infty \\ \limsup_{n\to\infty} \frac{Q_{\lambda_n}}{(Q_{\lambda_n} - Q_n)} &= \left(1 - \limsup_{n\to\infty} \frac{Q_n}{Q_{\lambda_n}}\right)^{-1} < \infty \end{aligned}$$

Therefore (3) follows from inequality (5), Lemma 3.3 and the statistical convergence of (t_{mn}^{11}).

Let $0 < \lambda < 1$. Again by Lemma 4 in [33]

$$\begin{aligned} D\left(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} u_{jk}, \mu\right) &\le D\left(\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} u_{jk}, t_{mn}^{11}\right) + D(t_{mn}^{11}, \mu) \\ &\le \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} D\left(t_{mn}^{11}, t_{m,\lambda_n}^{11}\right) \\ &+ \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} D\left(t_{\lambda_m,\lambda_n}^{11}, t_{\lambda_m,n}^{11}\right) \\ &+ \frac{P_{\lambda_m}}{(P_m - P_{\lambda_m})} D\left(t_{mn}^{11}, t_{\lambda_m,n}^{11}\right) \\ &+ \frac{Q_{\lambda_n}}{(Q_n - Q_{\lambda_n})} D\left(t_{mn}^{11}, t_{m,\lambda_n}^{11}\right) + D(t_{mn}^{11}, \mu). \end{aligned} \tag{6}$$

From Lemma 3.1 we have

$$\limsup_{m\to\infty}\frac{P_{\lambda_m}}{(P_m-P_{\lambda_m})}=\left(\liminf_{m\to\infty}\frac{P_m}{P_{\lambda_m}}-1\right)^{-1}<\infty,$$
$$\limsup_{n\to\infty}\frac{Q_{\lambda_n}}{(Q_n-Q_{\lambda_n})}=\left(\liminf_{n\to\infty}\frac{Q_n}{Q_{\lambda_n}}-1\right)^{-1}<\infty.$$

Now (4) follows from inequality (6), Lemma 3.3 and the statistical convergence of (t^{11}_{mn}). □

Now as a consequence of Lemma 3.4, we give the following main theorem.

Theorem 3.5. *Let $p,q\in SVA_+$ and $u_{mn}\xrightarrow{st}\mu$ $(\overline{N},p,q;1,1)$. Then $u_{mn}\xrightarrow{st}\mu$ if and only if one of the following two conditions hold: for $\varepsilon>0$,*

$$\inf_{\lambda>1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:\\ D\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},u_{mn}\right)\ge\varepsilon\Bigg\}\Bigg|=0 \tag{7}$$

or

$$\inf_{0<\lambda<1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:\\ D\left(\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_ku_{jk},u_{mn}\right)\ge\varepsilon\Bigg\}\Bigg|=0. \tag{8}$$

Proof. Necessity. Suppose that $u_{mn}\xrightarrow{st}\mu$ $(\overline{N},p,q;1,1)$ and $u_{mn}\xrightarrow{st}\mu$ are satisfied. Applying Lemma 3.4 yield (7) for all $\lambda>1$, and (8) for all $0<\lambda<1$.

Sufficiency. Assume that $u_{mn}\xrightarrow{st}L$ $(\overline{N},p,q;1,1)$ and (7) are satisfied. In order to prove $u_{mn}\xrightarrow{st}L$, it is enough to show

$$D(u_{mn},t^{11}_{mn})\xrightarrow{st}0. \tag{9}$$

For $\lambda>1$, we have

$$\begin{aligned}D(t^{11}_{mn},u_{mn})\ &\le\ D\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},u_{mn}\right)\\ &\quad+D\left(\frac{1}{((P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},t^{11}_{mn}\right).\end{aligned}$$

From inequality (5) we obtain

$$\begin{aligned}D(t^{11}_{mn},u_{mn})\le{}&\frac{Q_{\lambda_n}P_{\lambda_m}}{(Q_{\lambda_n}-Q_n)(P_{\lambda_m}-P_m)}D\left(t^{11}_{\lambda_m,\lambda_n},t^{11}_{\lambda_m,n}\right)+\frac{Q_{\lambda_n}P_{\lambda_m}}{(Q_{\lambda_n}-Q_n)(P_{\lambda_m}-P_m)}D\left(t^{11}_{mn},t^{11}_{m,\lambda_n}\right)\\ &+\frac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}D\left(t^{11}_{\lambda_m,n},t^{11}_{mn}\right)+\frac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}D\left(t^{11}_{m,\lambda_n},t^{11}_{mn}\right)\\ &+D\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},u_{mn}\right).\end{aligned} \tag{10}$$

We define

$$A_{MN}(\varepsilon)=\left\{m\le M \text{ and } n\le N: D\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},u_{mn}\right)\ge\frac{\varepsilon}{2}\right\}$$

and

$$B_{MN}(\varepsilon)=\Big\{m\le M \text{ and } n\le N: \frac{Q_{\lambda_n}P_{\lambda_m}}{(Q_{\lambda_n}-Q_n)(P_{\lambda_m}-P_m)}D\left(t^{11}_{\lambda_m,\lambda_n},t^{11}_{\lambda_m,n}\right)+\frac{Q_{\lambda_n}P_{\lambda_m}}{(Q_{\lambda_n}-Q_n)(P_{\lambda_m}-P_m)}D\left(t^{11}_{mn},t^{11}_{m,\lambda_n}\right)$$
$$+\frac{P_{\lambda_m}}{(P_{\lambda_m}-P_m)}D\left(t^{11}_{\lambda_m,n},t^{11}_{mn}\right)+\frac{Q_{\lambda_n}}{(Q_{\lambda_n}-Q_n)}D\left(t^{11}_{m,\lambda_n},t^{11}_{mn}\right)\ge\frac{\varepsilon}{2}\Big\}.$$

Then we have

$$\left\{m\le M \text{ and } n\le N: D(t^{11}_{mn},u_{mn})\ge\varepsilon\right\}\subseteq A_{MN}(\varepsilon)\cup B_{MN}(\varepsilon).$$

For $\delta>0$, from (7) there exist $\lambda>1$ such that

$$\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|A_{MN}(\varepsilon)|\le\delta.$$

From Lemma 3.4,

$$\lim_{M,N\to\infty}\frac{1}{(M+1)(N+1)}|B_{MN}(\varepsilon)|=0.$$

So, we obtain

$$\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\left|\left\{m\le M \text{ and } n\le N: D(t^{11}_{mn},u_{mn})\ge\varepsilon\right\}\right|\le\delta.$$

Since $\delta>0$ is arbitrary, for $\varepsilon>0$ we arrive at

$$\lim_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\left|\left\{m\le M \text{ and } n\le N: D(t^{11}_{mn},u_{mn})\ge\varepsilon\right\}\right|=0.$$

This implies that $D(u_{mn},t^{11}_{mn})\xrightarrow{st}0$.

In case of $0<\lambda<1$, using the same way as in the preceding $\lambda>1$ and by (8) we have (9). □

Now we give statistically slow oscillation condition for a double sequence of fuzzy numbers. We say that (u_{mn}) is statistically slowly oscillating with respect to the first index if, for every $\varepsilon>0$,

$$\inf_{\lambda>1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\left\{m\le M \text{ and } n\le N: \max_{m<j\le\lambda_m}D(u_{jn},u_{mn})\ge\varepsilon\right\}=0 \tag{11}$$

and that (u_{mn}) is statistically slowly oscillating in the strong sense with respect to the first index if (11) is satisfied with

$$\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}}D(u_{jk},u_{mk}) \quad \text{in place of} \quad \max_{m<j\le\lambda_m}D(u_{jn},u_{mn}).$$

The statistically slow oscillation property with respect to the second index is defined analogously.

Taking Lemma 2.1 and Proposition 2.2 into consideration we have the following inequality:

$$D\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_ku_{jk},u_{mn}\right)\le\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}}D(u_{jk},u_{mk})+\max_{n<k\le\lambda_n}D(u_{mk},u_{mn}).$$

Hence, if (u_{mn}) is statistically slowly oscillating with respect to the second index and statistically slowly oscillating in the strong sense with respect to the first index, then (7) is satisfied for all $\varepsilon>0$. By Theorem 3.5, we arrive at the following result.

Corollary 3.6. *Let $p, q \in SVA_+$ and $u_{mn} \xrightarrow{st} \mu$ $(\overline{N}, p, q; 1, 1)$. If (u_{mn}) is statistically slowly oscillating with respect to both indices, in addition, in the strong sense with respect to one of the indices, then $u_{mn} \xrightarrow{st} \mu$.*

Now we give two-sided Landau's conditions for double sequences of fuzzy numbers. We consider that there exist constants $n_0 \geq 1$ and $H > 0$ such that

$$jD(u_{jn}, u_{j-1,n}) \leq H \quad (j, n > n_0), \tag{12}$$

$$kD(u_{mk}, u_{m,k-1}) \leq H \quad (m, k > n_0). \tag{13}$$

Then for$\lambda > 1$ and $m, k > n_0$, we obtain

$$\max_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} D(u_{jk}, u_{mk}) \leq \max_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} \left\{ \left(\sum_{l=m+1}^{j} \frac{1}{l} \right) \left(\sup_{m<l\leq j} lD(u_{lk}, u_{l-1,k}) \right) \right\} \leq \sum_{l=m+1}^{\lambda_m} \frac{H}{l} \leq H \log \lambda.$$

So if (12) is satisfied, then (u_{mn}) is statistically slowly oscillating in the strong sense with respect to the first index. Similarly (13) implies the statistically slow oscillation property in the strong sense with respect to the second index. As a result of (3.6) we give the next corollary.

Corollary 3.7. *Let $p, q \in SVA_+$ and $u_{mn} \xrightarrow{st} \mu$ $(\overline{N}, p, q; 1, 1)$. If conditions (12) and (13) are satisfied for some $n_0 \geq 1$ and some $H > 0$, then $u_{mn} \xrightarrow{st} \mu$.*

The following corollary is given by Talo and Bayazit [30].

Corollary 3.8. *[30] Let $u_{mn} \xrightarrow{st} \mu$. If conditions (12) and (13) are satisfied, then $u_{mn} \to \mu$.*

Combining corollaries 3.7 and 3.8 yields the next result.

Corollary 3.9. *Let $p, q \in SVA_+$ and $u_{mn} \xrightarrow{st} \mu$ $(\overline{N}, p, q; 1, 1)$. If conditions (12) and (13) are satisfied for some $n_0 \geq 1$ and some $H > 0$, then $u_{mn} \to \mu$.*

4. Aplications to Fuzzy Korovkin Type Approximation Theorem

Demirci and Karakuş [12] firstly proved a fuzzy version of Korovkin type approximation theorem for functions of two variables. In this chapter, we proved this theorem by using statically $(\overline{N}, p, q; 1, 1)$ summability method.

Let K be a compact subset of $\mathbb{R}^2$ and we denote by $C(K)$ the space of all continuous real functions on K. $C(K)$ is a Banach spaces with the norm:

$$\|h\| = \sup_{(x,y)\in K} |h(x, y)|.$$

A fuzzy-number-valued function of two variable $f : K \to E^1$ has the parametric representation

$$[f(x, y)]_\alpha = [f_\alpha^-(x, y), f_\alpha^+(x, y)],$$

for each $(x, y) \in K$ and $\alpha \in [0, 1]$. The set of all continuous fuzzy-number-valued functions on K is denoted by $C_{\mathcal{F}}(K)$ and $C_{\mathcal{F}}(K)$ is a metric space with the metric

$$\begin{aligned} D^*(f, g) &= \sup_{(x,y)\in K} D(f(x, y), g(x, y)) \\ &= \sup_{(x,y)\in K} \sup_{\alpha\in[0,1]} \max\{|f_\alpha^-(x, y) - g_\alpha^-(x, y)|, |f_\alpha^+(x, y) - f_\alpha^+(x, y)|\}. \end{aligned}$$

Now let $L : C_{\mathcal{F}}(K) \to C_{\mathcal{F}}(K)$ be an operator. Then L is said to be fuzzy linear if for every $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}(K)$ and $(x, y) \in K$,

$$L(\lambda_1 f_1 + \lambda_2 f_2; x, y) = \lambda_1 L(f_1; x, y) + \lambda_2 L(f_2; x, y)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and the condition $L(f; x, y) \preceq L(g; x, y)$ is satisfied for any $f, g \in C_{\mathcal{F}}(K)$ and all $(x, y) \in K$ with $f(x, y) \preceq g(x, y)$.

Theorem 4.1. *Let $L_{mn} : C_{\mathcal{F}}(K) \to C_{\mathcal{F}}(K)$ be a fuzzy positive linear operator for each $(m, n) \in \mathbb{N}^2$. Suppose that there exist corresponding positive linear operators $\widetilde{L}_{mn} : C(K) \to C(K)$ fulfiling*

$$\{L_{mn}(f; x, y)\}_\alpha^\pm = \widetilde{L}_{mn}(f_\alpha^\pm; x, y) \tag{14}$$

for all $(x, y) \in K, \alpha \in [0, 1], (m, n) \in \mathbb{N}^2$ and $f \in C_{\mathcal{F}}(K)$. Suppose further that

$$st - \lim_{m,n\to\infty} \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_i) - g_i \right\| = 0, \quad i = 0, 1, 2, 3 \tag{15}$$

where $g_0(x, y) = 1$, $g_1(x, y) = x$, $g_2(x, y) = y$, $g_3(x, y) = x^2 + y^2$. Then, for all $f \in C_{\mathcal{F}}(K)$, we have

$$st - \lim_{m,n\to\infty} D^* \left(\frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k L_{jk}(f), f \right) = 0. \tag{16}$$

Proof. By $f \in C_{\mathcal{F}}(K)$, there is a $M > 0$ such that $D(f(x, y), \tilde{0}) \leq M$ for all $(x, y) \in K$ and for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $D(f(s, t), f(x, y)) < \varepsilon$ for all $(s, t) \in K$ fulfilling $\sqrt{(s - x)^2 + (t - y)^2} < \delta$. On the other hand, if $\sqrt{(s - x)^2 + (t - y)^2} \geq \delta$, then

$$D(f(s, t), f(x, y)) \leq D(f(s, t), \tilde{0}) + D(f(x, y), \tilde{0}) \leq \frac{2M}{\delta^2} \left((s - x)^2 + (t - y)^2 \right)$$

Hence, for fixed (x, y) we attain

$$D(f(s, t), f(x, y)) < \varepsilon + \frac{2M}{\delta^2} \left((s - x)^2 + (t - y)^2 \right).$$

From definition of the metric D, for $\alpha \in [0, 1]$ the following inequality

$$|f_\alpha^\pm(s, t) - f_\alpha^\pm(x, y)| < \varepsilon + \frac{2M}{\delta^2} \left((s - x)^2 + (t - y)^2 \right) \tag{17}$$

holds. We know that $\widetilde{L}_{mn}$ is linear and positive operator on $C(K)$ and $f_\alpha^\pm \in C(K)$ for $\alpha \in [0, 1]$. By (17) we obtain

$$\begin{aligned}
&\left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^\pm; x, y) - f_\alpha^\pm(x, y) \right| \\
\leq \quad & \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}(A^2 + B^2) \right) \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} \widetilde{L}_{jk}(g_0; x, y) - g_0(x, y) \right| \\
& + \frac{2M}{\delta^2} \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_3; x, y) - g_3(x, y) \right| \\
& + \frac{4MA}{\delta^2} \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_1; x, y) - g_1(x, y) \right| \\
& + \frac{4MB}{\delta^2} \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_2; x, y) - g_2(x, y) \right|,
\end{aligned}$$

where $A = \max\{|x|\}$, $B = \max\{|y|\}$. Taking supremum over $(x, y) \in K$, for $\alpha \in [0, 1]$ we obtain

$$\left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^\pm) - f_\alpha^\pm \right\| \leq \varepsilon + C \sum_{i=0}^{3} \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_i) - g_i \right\| \tag{18}$$

where

$$C = \max\left\{ \left(\varepsilon + M + \frac{2M}{\delta^2}(A^2 + B^2) \right), \frac{2M}{\delta^2}, \frac{4MA}{\delta^2}, \frac{4MB}{\delta^2} \right\}.$$

From definition of D^* we have

$$\begin{aligned} D^*\left(\frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k L_{jk}(f), f \right) &= \sup_{(x,y)\in K} D\left(\frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k L_{jk}(f; x, y), f(x, y) \right) \\ &= \sup_{(x,y)\in K} \sup_{\alpha\in[0,1]} \max\left\{ \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^-; x, y) - f_\alpha^-(x, y) \right|, \right. \\ &\qquad \left. \left| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^+; x, y) - f_\alpha^+(x, y) \right| \right\} \\ &= \sup_{\alpha\in[0,1]} \max\left\{ \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^-) - f_\alpha^- \right\|, \right. \\ &\qquad \left. \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(f_\alpha^+) - f_\alpha^+ \right\| \right\}. \end{aligned}$$

Combining the above equality with (18), we have

$$D^*\left(\frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k L_{jk}(f), f \right) \leq \varepsilon + C \sum_{i=0}^{3} \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_i) - g_i \right\|. \tag{19}$$

Now, for $\varepsilon' > 0$, take $\varepsilon > 0$ fulfiling $0 < \varepsilon < \varepsilon'$ and also set the following sets:

$$\begin{aligned} H: &= \left\{ (m, n) \in \mathbb{N}^2 : D^*\left(\frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k L_{jk}(f), f \right) \geq \varepsilon' \right\}, \\ H_i: &= \left\{ (m, n) \in \mathbb{N}^2 : \left\| \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k \widetilde{L}_{jk}(g_i) - g_i \right\| \geq \frac{\varepsilon' - \varepsilon}{4C} \right\} \; i = 0, 1, 2, 3. \end{aligned}$$

Then (19) yields $H \subseteq H_0 \cup H_1 \cup H_2 \cup H_3$. By taking this inclusion and (15) together into consideration, we have (16), which completes the proof. □

Example 4.2. *Let*

$$B_{mn}(f; x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n} \right) \binom{m}{j} x^j (1 - x)^{m-j} \binom{n}{k} y^k (1 - y)^{n-k} \tag{20}$$

where $(x, y) \in K = [0, 1] \times [0, 1]$, $f \in C_{\mathcal{F}}(K)$, $(m, n) \in \mathbb{N}$. *Consider*

$$L_{mn}(f; x, y) = (1/2 + x_{mn}) B_{mn}(f; x, y) \tag{21}$$

where (x_{mn}) *is defined by*

$$x_{mn} = \begin{cases} 1, & j \text{ is odd, for all } k, \\ 0, & \text{otherwise.} \end{cases}$$

For each $m, n \in \mathbb{N}$, L_{mn} *is a positive linear operator on* $C_{\mathcal{F}}(K)$. *It is simple matter to see* $x_{mn} \xrightarrow{st} 1/2\ (C, 1, 1)$. *But* (x_{mn}) *is not statistically convergent. For all* $\alpha \in [0, 1]$ *we have*

$$\begin{aligned} \{L_{mn}(f; x, y)\}_{\alpha}^{\pm} &= \widetilde{L}_{mn}(f_{\alpha}^{\pm}; x, y) \\ &= (1/2 + x_{mn}) \sum_{j=0}^{m} \sum_{k=0}^{n} f_{\alpha}^{\pm}\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m}{j} x^{j}(1-x)^{m-j} \binom{n}{k} y^{k}(1-y)^{n-k}. \end{aligned}$$

It is evident that $\{L_{mn}\}$ *satisfies (15). Thus, we obtain*

$$st - \lim_{m,n\to\infty} D^{*}\left(\frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} L_{jk}(f), f\right) = 0.$$

Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions to improve the quality and readability of the paper.

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