Inequalities Related to Schatten Norm

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Abstract. In this paper, we investigate the known operator inequalities for the \(p\)-Schatten norm and obtain some refinements of these inequalities when parameters taking values in different regions. Let \(A_1, \ldots, A_n, B_1, \ldots, B_n \in B_p(H)\) such that \(\sum_{i,j=1}^{n} A_i^* B_j = 0\). Then \(p \geq 2\), \(p \leq \lambda\) and \(\mu \geq 2\),

\[ 2^{1/\mu - 1/4} n^{3/\mu - 1} \left( \sum_{i=1}^{n} ||A_i||_p^{2/\mu} + \sum_{i=1}^{n} ||B_i||_p^{2/\mu} \right)^{1/2} \leq \left( \sum_{i,j=1}^{n} ||A_i + B_j||_p^{1/\lambda} \right)^{\lambda/4} \]

For \(0 < p \leq 2\), \(p \geq \lambda > 0\) and \(0 < \mu \leq 2\), the inequalities are reversed. Moreover, we get some applications of our results.

1. Introduction

Let \(B(H)\) be the \(C^*\)-algebra of all bounded linear operators acting on a complex separable Hilbert space \(H\). \(|A| = (X^* X)^{1/2}\) denotes the absolute value of an operator \(A \in B(H)\). If \(A \in B(H)\) is compact, let \(\{s_j(A)\}_{j=1}^{\infty}\) be the sequence of decreasingly ordered singular values of \(A\). For \(0 < p < \infty\), let \(||A||_p = (tr|A|^p)^{1/p} = (\sum_{j=1}^{\infty} s_j^p(A))^{1/p}\), where \(tr\) is the usual trace function. This defines the Schatten \(p\)-norm (quasi-norm, resp.) for \(1 \leq p < \infty\) (\(0 < p < 1\), resp.) on the set

\[ B_p(H) = \{ X \in B(H) : ||X||_p < \infty \}, \]

which is called the \(p\)-Schatten class of \(B(H)\) (see [5]). The Schatten \(p\)-norms are unitarily invariant and when \(p = 1\), \(||A|| = tr|A|\) is called the trace norm of \(A\).
There are some classical Clarkson’s inequalities for the Schatten p-norms of operators in $B_p(H)$ (See [3]). If $A, B \in B_p(H)$, then
\[
2^{p-1}||A||_p^p + ||B||_p^p \leq ||A - B||_p^p + ||A + B||_p^p \leq 2(||A||_p^p + ||B||_p^p)
\] (1.1)
for $0 < p \leq 2$ and
\[
2(||A||_p^p + ||B||_p^p) \leq ||A - B||_p^p + ||A + B||_p^p \leq 2^{p-1}||A||_p^p + ||B||_p^p
\] (1.2)
for $2 \leq p < \infty$. For $p = 2$, by (1.1) and (1.2), we have
\[
||A - B||_2^2 + ||A + B||_2^2 = 2(||A||_2^2 + ||B||_2^2),
\]
which is called parallelogram law. When $p \neq 2$, the equality $2(||A||_p^p + ||B||_p^p) = ||A - B||_p^p + ||A + B||_p^p$ holds if and only if $A^*B = AB^* = 0$, or equivalently $R(A)$ and $R(B)$ are orthogonal. (See [3]).

Hirzallah, Kittaneh and Moslehian etc. have obtained some generalizations of (1.1) and (1.2) to $n$-tuples of operators and many different conclusions by using various methods such as complex interpolation method, concavity and convexity of certain functions, etc. (See [1, 6–10]).

Recently, some refinements of some $p$-Schatten inequalities have been given by Conde and Moslehian in [4].

**Theorem 1.1** ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^{n} A_i^*B_j = 0$, then for $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$,
\[
2^{1/2-1/\mu} n^{1-1/\mu} \left(\sum_{i=1}^{n} ||A_i||_p^\mu + \sum_{i=1}^{n} ||B_i||_p^\mu\right)^{1/\mu} \leq n^{1/2} \left(\sum_{i=1}^{n} ||A_i||_p^\mu\right)^{1/\mu} + \left(\sum_{i=1}^{n} ||B_i||_p^\mu\right)^{1/\mu}
\]
\[
\leq n^{2(1/p-1/\lambda)} \left(\sum_{i,j=1}^{n} ||A_i \pm B_j||_p^\lambda\right)^{1/\lambda}.
\] (1.3)
For $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$, the inequalities are reversed.

**Theorem 1.2** ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^{n} A_i^*B_j = 0$, then for $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$,
\[
n^{(1/2+1/p-1/\lambda)} \sum_{i,j=1}^{n} ||A_i \pm B_j||_p^\lambda \leq n^{1/2+1/p-1/\lambda} \left(\sum_{i=1}^{n} ((||A_i||_p^2 + ||B_i||_p^2)^{1/2})^{1/\lambda}\right).
\] (1.4)
For $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$, the inequality is reversed.

In this paper, motivated by the above conclusions, we consider some refinements of $p$-Schatten norm inequalities when $p$, $\lambda$ and $\mu$ taking values in different regions.

### 2. Main results

In this section we consider the $p$-Schatten norm inequalities of (1.3) and (1.4) when parameters taking values in different regions. We start our works with the following lemmas that we will use along the paper.

**Fact 1.** $M_s(\bar{x}) \leq M_{s'}(\bar{x})$ for $0 < s \leq s'$, where $M_s(\bar{x}) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^s\right)^{1/s}$, $\bar{x} = (x_1, \cdots, x_n)$ is an $n$-tuples of non-negative numbers.

**Fact 2.** $||T||_p^2 = ||T||_p^2$ for any $T \in B_p(H)$ with $p > 0$.

**Lemma 2.1** ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B(H)$ such that $\sum_{i,j=1}^{n} A_i^*B_j = 0$, then
\[
\sum_{i,j=1}^{n} |A_i \pm B_j|^2 = \sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2 \pm \sum_{i,j=1}^{n} A_i^*B_j + B_j^*A_i
\]
\[
= \sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2.
\] (2.1)
Lemma 2.2 ([2-3]). If $A_1, \cdots, A_n \in B_p(H)$ for some $p > 0$, and $A_1, \cdots, A_n$ are positive, then for $0 < p \leq 1$,\[
n^{p-1} \sum_{i=1}^{n} \|A_i\|_p^p \leq \left(\sum_{i=1}^{n} \|A_i\|_p\right)^p \leq \|\sum_{i=1}^{n} A_i\|_p^p \leq \sum_{i=1}^{n} \|A_i\|_p^p \tag{2.2}
\]
and for $1 \leq p < \infty$ the inequalities are reversed.

They are refinements of Lemma 2.1 in [7]. A commutative version of the previous lemma for scalars is the following:

Let $\bar{x} = (x_1, \ldots, x_n)$ be an $n$-tuples of non-negative numbers, then
\[
n^{p-1} \sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p \leq \sum_{i=1}^{n} x_i^p \tag{2.3}
\]
for $0 < p \leq 1$ and
\[
\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p \leq n^{p-1} \sum_{i=1}^{n} x_i^p \tag{2.4}
\]
for $1 \leq p < \infty$.

Theorem 2.3. Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\sum_{j=1}^{n} A_j^* B_j = 0$. Then for $p \geq 2$, $p \leq \lambda$ and $\mu \geq 2$,\[
2^{1/p-\mu/4} n^{3/p-\mu/4-1/2} \left(\sum_{i=1}^{n} \|A_i\|_p^{4/\mu} + \sum_{i=1}^{n} \|B_i\|_p^{4/\mu}\right)^{\mu/4} \leq n^{2/(p-1/2)} \sum_{i=1}^{n} |A_i|^2 + \sum_{i=1}^{n} |B_i|^2 \|\|_{p/2}^{1/2} \\
\leq n^{2/(p-1/2)} \left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\lambda\right)^{1/\lambda}.
\]

For $0 < p \leq 2$, $p \geq \lambda > 0$ and $0 < \mu \leq 2$, the inequalities are reversed.

**Proof.** Let $p \geq 2$, $p \leq \lambda$, $\mu \geq 2$. It follows from $M_p(\bar{x}) \leq M_2(\bar{x})$ that\[
n^{2/(p-1/2)} \left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\lambda\right)^{1/\lambda} = n^{2/p} \left(\frac{1}{n^2} \sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\lambda\right)^{1/\lambda} \\
\geq \left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\lambda\right)^{1/p}.
\]

Applying the well-known fact that $\|T\|_p^2 = \|\|T\|\|_{p/2}$ for any $T \in B_p(H)$ with $p > 0$ and Lemma 2.1 and Lemma 2.2, we get\[
\left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\lambda\right)^{1/p} \leq \left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_{p/2}^2\right)^{1/p} \\
\geq \left(\sum_{i,j=1}^{n} \|A_i \pm B_j\|_{p/2}^2\right)^{1/p} \\
= \left(\sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2 \|\|_{p/2}^2\right)^{1/p} \\
= n^{2/(p-1/2)} \left(\sum_{i=1}^{n} |A_i|^2 + \sum_{i=1}^{n} |B_i|^2 \|\|_{p/2}^2\right).
Using Lemma 2.2, (2.3) and the concavity of the function \( f(x) = x^\alpha \) on \([0, +\infty)\) for \(0 < \alpha \leq 1\), we obtain

\[
\begin{align*}
&n^{2/p-1/2} \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \|p/2\|^{1/2} \\
&= n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \\
&\geq n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \quad \text{by Lemma 2.2} \\
&\geq n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \quad \text{by (2.3)} \\
&= n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \\
&\geq n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \\
&= n^{2/p-1/2} \left( \|B_1\|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \\
&= \|B_1\|^2 \|p/2\|^{1/2} \mu^{1/4}.
\end{align*}
\]

Let \(0 < p \leq 2\), \(p \geq \lambda\) and \(0 < \mu \leq 2\). We can prove the inequalities by the same ways.

**Corollary 2.4.** Let \(A_1, \ldots, A_n, B_1, \ldots, B_n \in B_p(H)\) such that \(\sum_{i=1}^n A_i^* B_i = 0\). Then for \(p \geq 2\),

\[
2^{1/p-1/4} n^{3/p-1/4} \left( \sum_{i=1}^n |A_i|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \leq n^{2/p-1/2} \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \|p/2\|^{1/2} \\
\leq \left( \sum_{i,j=1}^n \|A_i \pm B_j\|^p \right)^{1/p}.
\]

For \(0 < p \leq 2\), the inequalities are reversed.

**Proof.** Motivated by Theorem 2.3, let \(\lambda = \mu = p\).

**Corollary 2.5.** Let \(A_1, \ldots, A_n \in B_p(H)\) such that \(\sum_{i=1}^n A_i = 0\). Then for \(p \geq 2\)

\[
2^{1/p} n^{3/p-1/4} \left( \sum_{i=1}^n |A_i|^2 \|p/2\|^{1/2} \right) \mu^{1/4} \leq n^{2/p-1/2} \sum_{i=1}^n |A_i|^2 \|p/2\|^{1/2} \leq \left( \sum_{i,j=1}^n \|A_i \pm A_j\|^p \right)^{1/p}.
\]

For \(0 < p \leq 2\), the inequalities are reversed.

**Proof.** \(\sum_{i=1}^n A_i = 0\) implies that \(\sum_{i,j=1}^n A_i^* A_j = 0\). The statement is a consequence of Corollary 2.4.

**Theorem 2.6.** Let \(A_1, \ldots, A_n, B_1, \ldots, B_n \in B_p(H)\) such that \(\sum_{i=1}^n A_i^* B_i = 0\). Then for \(p \geq 2\), \(p \leq \lambda\) and \(\mu \geq 2\),

\[
\frac{n}{\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|^p} \leq \frac{n^{1/2} \sum_{i=1}^n \|A_i\|^2 + \|B_i\|^2} \leq \left( \sum_{i,j=1}^n \|A_i \pm B_j\|^p \right)^{1/\lambda}.
\]

For \(0 < p \leq 2\), \(p \geq \lambda > 0\) and \(0 < \mu \leq 2\), the inequality is reversed.
**Proof.** We suppose that $p \geq 2$, $p \leq \lambda$ and $\mu \geq 2$. Then by Lemma 2.2 and the convexity of the function $f(x) = x^2$ on $[0, +\infty)$ for $\alpha \geq 1$

$$n(1/n^2 \sum_{i,j=1}^{n} [||A_i \pm B_j||_p^{\mu/\lambda}])^{1/\mu} = n(1/n^2 \sum_{i,j=1}^{n} [||A_i \pm B_j||_{p/2}^{\mu/2}])^{1/\mu} \geq \sum_{i,j=1}^{n} [||A_i \pm B_j||_{p/2}^{\mu/2}]^{1/\mu} \geq \parallel \sum_{i,j=1}^{n} [||A_i \pm B_j||_{p/2}^{\mu/2}] \parallel_{\lambda/\mu} \geq n^{1/2} \sum_{i=1}^{n} (||A_i||_{p/2}^{\mu/2} + ||B_i||_{p/2}^{\mu/2})^{1/p} \geq n^{1/2} \sum_{i=1}^{n} (||A_i||_{p/2}^{\mu/2} + ||B_i||_{p/2}^{\mu/2})^{1/\lambda} \geq n^{1/2} \sum_{i=1}^{n} (||A_i||_{p/2}^{\mu/2} + ||B_i||_{p/2}^{\mu/2})^{1/\lambda}$$

Let $0 < p \leq 2$, $p \geq \lambda > 0$ and $0 < \mu \leq 2$. We can prove the inequality by the same ways.

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**References**


