Filomat 35:14 (2021), 4789–4809
https://doi.org/10.2298/FIL2114789Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Associated GBS Operators of Szász-Mirakjan Type Operators

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Abstract. In this article, the approximation properties of bivariate Szász-Mirakjan type operators are studied for the function of two variables and rate of convergence of the bivariate operators is determined in terms of total and partial modulus of continuity. An associated GBS (Generalized Boolean Sum)-form of the bivariate Szász-Mirakjan type operators is considered for the function of two variables to find an approximation of B-continuous and B-differentiable function in the Bögel's space. Further, the degree of approximation of the GBS type operators is found in terms of mixed modulus of smoothness and functions belonging to the Lipschitz class as well as a pioneering result is obtained in terms of Peetre K-functional. Finally, the rate of convergence of the bivariate Szász-Mirakjan type operators and the associated GBS type operators are examined through graphical representation for the finite and infinite sum which shows that the rate of convergence of the associated GBS type operators is better than the bivariate Szász-Mirakjan type operators and also a comparison is taken place for the bivariate operators with bivariate Kantorovich operators.

1. Introduction

Approximation properties form an integral part in the study of approximation theory that includes convergence, rate of convergence, the order of approximation etc. Applications and convergence based discussion of the linear positive operators defined over different types of interval (finite or infinite) on $\mathbb{R}_+$, have been discussed by many researchers. In 1912, first of all, Bernstein proposed an operator, so-called Bernstein operator of one variable which approximates the functions defined over a finite interval $[0,1]$.

In the study of [1–4], it is found that the Bernstein operators have been converted into bivariate Bernstein operators for function of two variables over $[0,1]\times[0,1]$ with their graphical representation in the study of the approximation properties for the function of two variables.

Many results related to approximations theory have also been discussed by many authors [5–9]. Despite these, if we move towards the operators defined over an infinite interval, then we look at the Szász-Mirakjan operators, which were introduced and studied by Mirakjan and Szász [10, 11] independently and so many work done in a bivariate direction of these operators to generalize and check the behavior of the operators

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A36
Keywords. Szász-Mirakjan operators, modulus of continuity, Bögel space, GBS (Generalized Boolean Sum) operators
Received: 21 November 2020; Revised: 26 August 2021; Accepted: 13 September 2021
Communicated by Miodrag Spalević
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for the function of two variables. Later on Szász-Mirakjan operators have been discussed theoretically, numerically as well graphically by many authors [12–15] using bivariate extension for approximation of the functions of two variables.

Similarly, for the bivariate operators, one more property has been studied in Bögel space and that is the property of generalized boolean sum of the bivariate operators, so called GBS-type operators while the functions are considered to be B-continuous. In 1934 and in 1935, Bögel [16, 17] introduced Bögel space, after that Dobrescu and Matei [18], estimated the rate of convergence of associated GBS-type operators of the bivariate Bernstein operators in the Bögel space. In 1988, Badea et al. [19] gave a quantitative variant of Korovkin type theorem for B-continuous function and estimated the degree of approximation by certain linear positive operators. After that, in 1991, quantitative and non-quantitative Korovkin type theorem was proved by Badea and Cottin [20] in the Bögel space. On other hand, the approximation properties of bivariate Bernstein type operators and their associated GBS operators have been examined by many researchers (see [21–25]).

In 2015, Bărbosu and Muraru [26] established some pioneering results through the associated GBS-type operators of Bernstein-Schurer-Stancu type operators using q-integers. Bărbosu et al. [27] introduced GBS-Durrmeyer type operators based on q-integers. In 2016, Agrawal and Ispir [28] estimated the degree of approximation of the Chlodowsky-Szász-Charlier type operators for the function of two variables.

Yadav et al. [29] proposed bivariate Szász-Mirakjan type operators for the function of two variables. They studied the approximation properties as well as rate of convergence of proposed bivariate operators in polynomial weighted spaces and obtained a Voronovskaya type theorem as well as discussed simultaneous approximation property. The bivariate Szász-Mirakjan type operators are considered for continuous and bounded functions on $[0,\infty)\times[0,\infty)$ as below:

$$\hat{Y}_{m,n,a}(f;x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s_{m,n}^{a}(x,y)\, f\left(\frac{k_1}{m},\frac{k_2}{n}\right), \tag{1}$$

where $s_{m,n}^{a}(x,y) = a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)} \frac{x^{k_1} y^{k_2} (\log a)^{k_1+k_2}}{(-1+a^{\frac{1}{m}})^{k_1}(-1+a^{\frac{1}{n}})^{k_2} k_1! k_2!}$, $m,n\in\mathbb{N}$, $(x,y)\in X=[0,\infty)\times[0,\infty)$.

Remark 1.1. *For all $m,n\in\mathbb{N}$, above operator (1) reproduces the functions 1 and a^{x+y} ($a>1$ fixed), for all $x,y\in[0,\infty)$.*

The main purpose of this article is to investigate some results related to bivariate operators (1), like order of approximation in terms of total modulus of continuity and partial modulus of continuity. For further study, the bivariate operators (1) are generalized into GBS (Generalized boolean Sum) form to determine the better rate of convergence than proposed bivariate operators and to establish the convergence properties of the GBS-type operators in the Bögel space with some approximations theorems in terms of mixed modulus of smoothness with the aid of Lipschitz classes. The graphical and numerical approaches are presented to support the approximation results and a comparison of bivariate operator (1) with its associated GBS-type operator in numerical sense is presented well. The best part of the this article is that, the graphical representation is shown for finite sum to determine the accuracy of the rate of convergence in its convergence behaviour. Finally, we have shown the comparison result of the bivariate operators (1) with bivariate Kantorovich operators of Szász-Mirakjan operators as well as comparison result of the associated GBS-type operator with the GBS operator of Mirakjan-Favard-Szász to check the accuracy of the rate of convergence in terms of its numerical values.

For our main results, we need some basic lemmas. Consider the function $e_{ij}=x^i y^j$ such that $i,j\in\{0,1\}$ and $i+j\leq 2$. Then the following lemma holds:

Lemma 1.2. *Let $x, y \geq 0$ and for each $m, n \in \mathbb{N}$. Then the following results hold:*

$$\hat{Y}_{m,n,a}(e_{10}; x, y) = \frac{x\log(a)}{m\left(a^{\frac{1}{m}} - 1\right)} \tag{2}$$

$$\hat{Y}_{m,n,a}(e_{01}; x, y) = \frac{y\log(a)}{n\left(a^{\frac{1}{n}} - 1\right)} \tag{3}$$

$$\hat{Y}_{m,n,a}(e_{20}; x, y) = \frac{x\log(a)\left(a^{\frac{1}{m}} + x\log(a) - 1\right)}{m^2\left(a^{\frac{1}{m}} - 1\right)^2} \tag{4}$$

$$\hat{Y}_{m,n,a}(e_{02}; x, y) = \frac{y\log(a)\left(a^{\frac{1}{n}} + y\log(a) - 1\right)}{n^2\left(a^{\frac{1}{n}} - 1\right)^2}. \tag{5}$$

Proof. Here, we have $x, y \geq 0$ and $m, n \in \mathbb{N}$, then

$$\begin{aligned}
\hat{Y}_{m,n,a}(e_{10}; x, y) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x, y)\frac{k_1}{m} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)} \frac{x^{k_1} y^{k_2} (\log a)^{k_1+k_2}}{(-1+a^{\frac{1}{m}})^{k_1}(-1+a^{\frac{1}{n}})^{k_2} k_1! k_2!}\frac{k_1}{m} \\
&= \frac{a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)}}{m} \sum_{k_1=1}^{\infty} \frac{x^{k_1}(\log a)^{k_1}}{(-1+a^{\frac{1}{m}})^{k_1}(k_1-1)!} \cdot \sum_{k_2=0}^{\infty} \frac{y^{k_2}(\log a)^{k_2}}{(-1+a^{\frac{1}{n}})^{k_2} k_2!} \\
&= \frac{a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)}}{m} \cdot \frac{x\log a}{-1+a^{\frac{1}{m}}} \sum_{k_1=1}^{\infty} \frac{x^{k_1-1}(\log a)^{k_1-1}}{(-1+a^{\frac{1}{m}})^{k_1-1}(k_1-1)!} \cdot \left(e^{\log a}\right)^{\frac{y}{-1+a^{\frac{1}{n}}}}. \\
&= \frac{x\log a}{\left(-1+a^{\frac{1}{m}}\right)m} \\
\hat{Y}_{m,n,a}(e_{20}; x, y) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x, y)\left(\frac{k_1}{m}\right)^2 \\
&= a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)} \sum_{k_1=2}^{\infty} \frac{k_1 x^{k_1}(\log a)^{k_1}}{(-1+a^{\frac{1}{m}})^{k_1}(k_1-1)(k_1-2)!} \cdot \sum_{k_2=0}^{\infty} \frac{y^{k_2}(\log a)^{k_2}}{(-1+a^{\frac{1}{n}})^{k_2} k_2!} \\
&= a^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)} \frac{1}{m^2} \sum_{k_1=2}^{\infty} \frac{x^{k_1}(\log a)^{k_1}}{(-1+a^{\frac{1}{m}})^{k_1}(k_1-2)!}\left(1 + \frac{1}{k_1-1}\right) \\
&= \left(\frac{x\log a}{\left(-1+a^{\frac{1}{m}}\right)m}\right)^2 + \frac{x\log a}{\left(-1+a^{\frac{1}{m}}\right)m^2}.
\end{aligned}$$

Similarly, we can prove other results. □

Lemma 1.3. *For every $x, y \in X = [0, \infty) \times [0, \infty)$ and $m, n \in \mathbb{N}$, it gives the following results:*

$$
\begin{aligned}
1.\ \hat{Y}_{m,n,a}((t-x);x,y) &= -\frac{x\left(ma^{\frac{1}{m}}-\log(a)-m\right)}{m\left(a^{\frac{1}{m}}-1\right)}\\
2.\ \hat{Y}_{m,n,a}((s-y);x,y) &= -\frac{y\left(na^{\frac{1}{n}}-\log(a)-n\right)}{n\left(a^{\frac{1}{n}}-1\right)}\\
3.\ \hat{Y}_{m,n,a}((t-x)^2;x,y) &= \frac{x\left(m^2x\left(a^{\frac{1}{m}}-1\right)^2-\left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1)+x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2}\\
4.\ \hat{Y}_{m,n,a}((s-y)^2;x,y) &= \frac{y\left(n^2y\left(a^{\frac{1}{n}}-1\right)^2-\left(a^{\frac{1}{n}}-1\right)\log(a)(2ny-1)+y(\log a)^2\right)}{n^2\left(a^{\frac{1}{n}}-1\right)^2}\\
5.\ \hat{Y}_{m,n,a}((t-x)^4;x,y) &= \frac{x}{m^4\left(a^{\frac{1}{m}}-1\right)^4}\left\{m^4x^3\left(a^{\frac{1}{m}}-1\right)^4-\left(a^{\frac{1}{m}}-1\right)^3(-1+4mx-6m^2x^2+4m^3x^3)\log a\right.\\
&\quad+\left(a^{\frac{1}{m}}-1\right)^2x(7-12mx+6m^2x^2)(\log a)^2\\
&\quad\left.-2\left(a^{\frac{1}{m}}-1\right)x^2(-3+2mx)(\log a)^3+x^3(\log a)^4\right\}\\
6.\ \hat{Y}_{m,n,a}((s-y)^4;x,y) &= \frac{y}{n^4\left(a^{\frac{1}{n}}-1\right)^4}\left\{n^4y^3\left(a^{\frac{1}{n}}-1\right)^4-\left(a^{\frac{1}{n}}-1\right)^3(-1+4ny-6n^2y^2+4n^3y^3)\log a\right.\\
&\quad+\left(a^{\frac{1}{n}}-1\right)^2y(7-12ny+6n^2y^2)(\log a)^2\\
&\quad\left.-2\left(a^{\frac{1}{n}}-1\right)y^2(-3+2ny)(\log a)^3+y^3(\log a)^4\right\}.
\end{aligned}
$$

Proof. Using Lemma 1.2, for every $x, y \in X$ and for all $m, n \in \mathbb{N}$, we have

$$
\begin{aligned}
1.\ \hat{Y}_{m,n,a}((t-x);x,y) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x,y)\left(\frac{k_1}{m}-x\right)\\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x,y)\frac{k_1}{m}-x\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x,y)\\
&= \frac{x\log a}{\left(-1+a^{\frac{1}{m}}\right)m}-xa^{\left(\frac{-x}{-1+a^{\frac{1}{m}}}\right)}a^{\left(\frac{-y}{-1+a^{\frac{1}{n}}}\right)}\left(e^{\log a}\right)^{\frac{x}{-1+a^{\frac{1}{m}}}}\left(e^{\log a}\right)^{\frac{y}{-1+a^{\frac{1}{n}}}}\\
&= \frac{x\log a}{\left(-1+a^{\frac{1}{m}}\right)m}-x,\\
3.\ \hat{Y}_{m,n,a}((t-x)^2;x,y) &= \hat{Y}_{m,n,a}((t^2-2tx+x^2);x,y)\\
&= \hat{Y}_{m,n,a}(e_{20};x,y)-2x\hat{Y}_{m,n,a}(e_{10};x,y)+x^2\\
&= \frac{x\log(a)\left(a^{\frac{1}{m}}+x\log(a)-1\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2}-2x\left(\frac{x\log(a)}{m\left(a^{\frac{1}{m}}-1\right)}\right)-2x\\
&= \frac{x\left(m^2x\left(a^{\frac{1}{m}}-1\right)^2-\left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1)+x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2}.
\end{aligned}
$$

Similarly, other equalities can be proved. □

Lemma 1.4. *For all $x, y \geq 0$, the following inequalities hold true:*

$$\begin{aligned} \hat{\Upsilon}_{m,n,a}((t-x)^2; x, y) &\leq \frac{x(x+1)}{m} = \delta_m'^2(x), \\ \hat{\Upsilon}_{m,n,a}((s-y)^2; x, y) &\leq \frac{y(y+1)}{n} = \delta_n'^2(y). \end{aligned}$$

Proof. For all $m, n \in \mathbb{N}$ and $x \geq 0$, we have

$$\begin{aligned} \hat{\Upsilon}_{m,n,a}((t-x)^2; x, y) &= \frac{x\left(m^2 x\left(a^{\frac{1}{m}}-1\right)^2 - \left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1) + x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2} \\ &= x\left(x\left(\frac{\log a}{m\left(a^{\frac{1}{m}}-1\right)}\right)^2 - \frac{2x\log a}{m\left(a^{\frac{1}{m}}-1\right)} + x + \frac{\log a}{m^2\left(a^{\frac{1}{m}}-1\right)}\right) \\ &= x\left(x\left(\frac{\log a}{m\left(a^{\frac{1}{m}}-1\right)} - 1\right)^2 + \frac{\log a}{m^2\left(a^{\frac{1}{m}}-1\right)}\right) \\ &\leq x\left(\frac{x}{m} + \frac{1}{m}\right) = \frac{x(x+1)}{m}. \end{aligned}$$

Similarly, other inequality can be proved. □

Remark 1.5. *For all $(x, y) \in [0, c] \times [0, d]$, where $0 \leq x \leq c$ and $0 \leq y \leq d$, we have*

$$\hat{\Upsilon}_{m,n,a}((t-x)^2; x, y) \leq \frac{c(c+1)}{m} = \frac{\lambda_x}{m}, \tag{6}$$

$$\hat{\Upsilon}_{m,n,a}((s-y)^2; x, y) \leq \frac{d(d+1)}{n} = \frac{\lambda_y}{n}, \tag{7}$$

where λ_x, λ_y are positive constants.

Proof. Using Lemma 1.4, we can obtain the required results. □

Lemma 1.6. *For all $x, y \in [0, c] \times [0, d]$ and $m, n \in \mathbb{N}$, following inequalities hold true*

$$\hat{\Upsilon}_{m,n,a}((t-x)^4; x, y) \leq \frac{M_x}{m^2}, \tag{8}$$

$$\hat{\Upsilon}_{m,n,a}((s-y)^4; x, y) \leq \frac{M_y}{n^2}, \tag{9}$$

where M_x, M_y are positive constants.

Proof. By Lemma 1.3, we have

$$\begin{aligned}
\hat{\Upsilon}_{m,n,a}((t-x)^4;x,y) &= x\left(\frac{\log a}{m^4\left(a^{\frac{1}{m}}-1\right)}\right)+x^2\frac{\log a}{m\left(a^{\frac{1}{m}}-1\right)}\left(\frac{7\log a}{m^3\left(a^{\frac{1}{m}}-1\right)}-\frac{4}{m^2}\right)\\
&\quad+\frac{6x^3\log a}{m\left(a^{\frac{1}{m}}-1\right)}\left(\frac{1}{m}-\frac{2\log a}{m^2\left(a^{\frac{1}{m}}-1\right)}+\frac{(\log a)^2}{m^3\left(a^{\frac{1}{m}}-1\right)^2}\right)\\
&\quad+x^4\left(1-\frac{4\log a}{m\left(a^{\frac{1}{m}}-1\right)}+\frac{6(\log a)^2}{m^2\left(a^{\frac{1}{m}}-1\right)^2}-\frac{4(\log a)^3}{m^3\left(a^{\frac{1}{m}}-1\right)^3}+\frac{(\log a)^4}{m^4\left(a^{\frac{1}{m}}-1\right)^4}\right)\\
&\leq \frac{x}{n^3}+\frac{x^2}{n^2}\left(\frac{7\log a}{m\left(a^{\frac{1}{m}}-1\right)}-4\right)+\frac{6x^3}{n}\left(\frac{\log a}{m\left(a^{\frac{1}{m}}-1\right)}-1\right)^2+x^4\left(\frac{\log a}{m\left(a^{\frac{1}{m}}-1\right)}-1\right)^4\\
&\leq \frac{x}{m^3}+\frac{7x^2}{m^2}\left(\frac{1}{m}-\frac{3}{7}\right)+\frac{6x^3}{m^2}+\frac{x^4}{m^3}\\
&= \frac{1}{m^3}(x^4+7x^2+x)+\frac{1}{m^2}(3x^2+6x^3)\\
&\leq \frac{1}{m^2}(x^4+10x^2+6x^3+x)\\
&\leq \frac{1}{m^2}(c^4+10c^2+6c^3+c)=\frac{M_x}{m^2}.
\end{aligned}$$

Similarly, it can be proved that

$$\hat{\Upsilon}_{m,n,a}((s-y)^4;x,y)\leq\frac{M_y}{n^2}.$$

□

2. Basic properties of the bivariate operators

For determining the rate of convergence of the bivariate operators defined by (1) in terms of modulus of continuity, here it is defined the modulus of continuity. Let $f(x,y)\in C_B(X=[0,\infty)\times[0,\infty))$, be the space of all continuous and bounded function defined on $X=[0,\infty)\times[0,\infty)$. Then the total (complete) modulus of continuity for the function of two variables can be defined as:

$$\omega(f,\delta)=\sup\{|f(t,s)-f(x,y)|:\sqrt{(t-x)^2+(s-y)^2}\leq\delta,\ (t,s)\in X,\ \delta>0\} \tag{10}$$

and the partial modulus of continuity can be defined as [30]:

$$\begin{aligned}
\omega_1(f,\delta) &= \sup\{|f(u_1,y)-f(u_2,y)|:|u_1-u_2|\leq\delta,\ \delta>0\}, &(11)\\
\omega_2(f,\delta) &= \sup\{|f(x,v_1)-f(x,v_2)|:|v_1-v_2|\leq\delta,\ \delta>0\}. &(12)
\end{aligned}$$

Following theorem will show the rate of convergence of the bivariate operators (1) with the help of modulus of continuity.

Theorem 2.1. *If bivariate operators $\hat{\Upsilon}_{m,n,a}(f;x,y)$ defined by (1) are linear and positive, then the following relations hold:*

$$\begin{aligned}
|\hat{\Upsilon}_{m,n,a}(f;x,y)-f(x,y)| &\leq 2\omega(f;\delta_{m,n}), &(13)\\
|\hat{\Upsilon}_{m,n,a}(f;x,y)-f(x,y)| &\leq 2\{\omega_1(f,\delta_m)+\omega_2(f,\delta_n)\}, &(14)
\end{aligned}$$

where ω is the total modulus of continuity and ω_1,ω_2 are the partial modulus of continuity with respect to x,y respectively.

Proof. Using the definition of modulus of continuity, we can write

$$\begin{aligned}
|\hat{Y}_{m,n,a}(f;x,y)-f(x,y)| &\leq \hat{Y}_{m,n,a}(|f(t,s)-f(x,y)|;x,y)) \\
&\leq \hat{Y}_{m,n,a}(\omega(\sqrt{(t-x)^2+(s-y)^2};x,y)) \\
&\leq \omega(f;\delta)\left(1+\frac{1}{\delta}(\hat{Y}_{m,n,a}(\sqrt{(t-x)^2+(s-y)^2};x,y))\right) \\
&\leq \omega(f;\delta)\left(1+\frac{1}{\delta}\{\hat{Y}_{m,n,a}((t-x)^2+(s-y)^2;x,y)\}^{\frac{1}{2}}\right) \\
&= \omega(f;\delta)\left\{1+\frac{1}{\delta}\left(\frac{x\left(m^2x\left(a^{\frac{1}{m}}-1\right)^2-\left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1)+x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2}\right.\right. \\
&\quad \left.\left.+\frac{y\left(n^2y\left(a^{\frac{1}{n}}-1\right)^2-\left(a^{\frac{1}{n}}-1\right)\log(a)(2ny-1)+y(\log a)^2\right)}{n^2\left(a^{\frac{1}{n}}-1\right)^2}\right)^{\frac{1}{2}}\right\},
\end{aligned}$$

upon considering

$$\begin{aligned}
\delta_{m,n} &= \left\{1+\frac{1}{\delta}\left(\frac{x\left(m^2x\left(a^{\frac{1}{m}}-1\right)^2-\left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1)+x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2}\right.\right. \\
&\quad \left.\left.+\frac{y\left(n^2y\left(a^{\frac{1}{n}}-1\right)^2-\left(a^{\frac{1}{n}}-1\right)\log(a)(2ny-1)+y(\log a)^2\right)}{n^2\left(a^{\frac{1}{n}}-1\right)^2}\right)^{\frac{1}{2}}\right\}=\delta,
\end{aligned}$$

next one step will give the required result.
Now to prove the second part of this theorem, we use the properties (11), (12) and with the help of Cauchy-Schwartz inequality, we get:

$$\begin{aligned}
|\hat{Y}_{m,n,a}(f;x,y)-f(x,y)| &\leq \hat{Y}_{m,n,a}(|f(t,s)-f(x,y)|;x,y)) \\
&\leq \hat{Y}_{m,n,a}(|f(t,s)-f(t,y)|;x,y))+\hat{Y}_{m,n,a}(|f(t,y)-f(x,y)|;x,y)) \\
&\leq \omega_2(f,\delta_m)\left(1+\frac{1}{\delta_m}(\hat{Y}_{m,n,a}(\sqrt{(s-y)^2};x,y))^{\frac{1}{2}}\right) \\
&\quad +\omega_1(f,\delta_n)\left(1+\frac{1}{\delta_n}(\hat{Y}_{m,n,a}(\sqrt{(t-x)^2};x,y))^{\frac{1}{2}}\right), \qquad (15)
\end{aligned}$$

where

$$\delta_m = \frac{x\left(m^2x\left(a^{\frac{1}{m}}-1\right)^2-\left(a^{\frac{1}{m}}-1\right)\log(a)(2mx-1)+x(\log a)^2\right)}{m^2\left(a^{\frac{1}{m}}-1\right)^2} \qquad (16)$$

$$\delta_n = \frac{y\left(n^2y\left(a^{\frac{1}{n}}-1\right)^2-\left(a^{\frac{1}{n}}-1\right)\log(a)(2ny-1)+y(\log a)^2\right)}{n^2\left(a^{\frac{1}{n}}-1\right)^2} \qquad (17)$$

hence, by using Inequality 15, the required result can be obtained. □

2.1. Some basic definitions for associated GBS (Generalized Boolean Sum) operators

In recent years, the study of generalized Boolean sum (GBS) operators of certain linear positive operators is an interesting topic in approximation theory and function theory. In order to make analysis in multidimensional spaces, Karl Bögel introduced the concepts of B-continuous and B-differentiable function in [16, 17]. In [31], the authors discussed some significance role of the Bögel space. They proved that the space of all bounded Bögel functions is isometrically isometric with the completion of the blending function space with respect to suitable norm. Also the main importance of the Bögel space is that the functions which are not continuous in general but are B-continuous and that can also be approximated by the operators.

In this subsection, some basic definitions are defined for associated GBS-type operators in the Bögel space and their related properties are discussed.

Definition 2.2. ***B-Continuous:*** *Consider two compact intervals $\mathfrak{A}_g, \mathfrak{A} \subset \mathbb{R}$, a function $f : \mathfrak{A}_g \times \mathfrak{A} \to \mathbb{R}$ is said to be B-continuous function at a point $(u_0, v_0) \in \mathfrak{A}_g \times \mathfrak{A}$, if*

$$\lim_{(u,v)\to(u_0,v_0)} \Delta f((u,v),(u_0,v_0)) = 0, \tag{18}$$

where $\Delta f((u,v),(u_0,v_0)) = f(u,v) - f(u,v_0) - f(u_0,v) + f(u_0,v_0)$ and the set of all B-continuous function is denoted by $C_b(\mathfrak{A}_g \times \mathfrak{A})$.

Definition 2.3. ***B-Bounded:*** *A real valued function f defined on $\mathfrak{A}_g \times \mathfrak{A}$ is said to be B-Bounded, if there exists a positive constant $\mathcal{M}$ such that*

$$\Delta f((u,v),(u_0,v_0)) \leq \mathcal{M}, \tag{19}$$

denoted by $B_b(\mathfrak{A}_g \times \mathfrak{A})$.

Definition 2.4. ***B-Differentiable:*** *A function f is called B-Differentiable iff*

$$D_B f(u_0,v_0) = \lim_{(u,v)\to(u_0,v_0)} \frac{\Delta f((u,v),(u_0,v_0))}{(u-u_0)(v-v_0)}, \tag{20}$$

provided the limit exists and finite where the set of all B-differentiable functions is denoted by $D_b(\mathfrak{A}_g \times \mathfrak{A})$. For more details see [16, 17].

Motivated by cited papers in introduction part, here, we define the associated GBS-type operators of the defined bivariate operators (1) to investigate their approximation properties in the Bögel space. The main motive of this part is to determine the convergence results of the GBS-type operators defined by (21) along with their properties by theoretical, numerical as well as graphical sense. So, before the discussion of their properties, first we construct here the GBS-type operators of the bivariate operators (1).

Consider two compact intervals $\mathfrak{A}_g, \mathfrak{A} \subset \mathbb{R}$ and for any point $(x,y) \in \mathfrak{A}_g \times \mathfrak{A}$, the Boolean sum of the function $f : \mathfrak{A}_g \times \mathfrak{A} \to \mathbb{R}$ can be defined as $\Delta f((x,y),(t,s)) = f(x,y) - f(x,t) - f(s,y) + f(t,s)$ at a point $(t,s) \in \mathfrak{A}_g \times \mathfrak{A}$. Then the associated GBS (Generalized Boolean Sum)-type operators of $\hat{Y}_{m,n,a}(f;x,y)$ can be expressed as

$$\begin{aligned} B\hat{Y}^a_{m,n}(f;x,y) &= \hat{Y}_{m,n,a}(f(x,s) + f(t,y) - f(t,s)) \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x,y)\left(f\left(x,\frac{k_2}{n}\right) + f\left(\frac{k_1}{m},y\right) - f\left(\frac{k_1}{m},\frac{k_2}{n}\right)\right), \end{aligned} \tag{21}$$

where $f \in C_b(X_b = [0,c] \times [0,d])$.

2.2. Degree of the approximation of the GBS-type operators

In this subsection, we discuss the rate of convergence of the GBS-type operators with the help of modulus of smoothness in a Bögel space, and get a relation using the mixed modulus of smoothness. Now, to define the modulus of smoothness, we assume that the function $f \in C_b(X_b = [0,c] \times [0,d])$. The property of mixed modulus of smoothness is same as the modulus of continuity, which can be defined as

$$\omega_B(f;\delta_1,\delta_2) = \sup\{|\Delta f(t,s;x,y)| : |t-x| < \delta_1, |s-y| < \delta_2, \ (x,y),(t,s) \in X_b = [0,c] \times [0,d]\}, \tag{22}$$

for any $(\delta_1,\delta_2) \in X = [0,\infty) \times [0,\infty)$ and having property

$$\omega_B(f;\delta_m,\delta_n) \to 0, \text{ as } m,n \to \infty. \tag{23}$$

Remark 2.5. *The property of the modulus of smoothness can be defined as:*

$$\omega_B(f;\mu_1\delta_1,\mu_2\delta_2) = (1+\mu_1)(1+\mu_2)\omega_B(f;\delta_1,\delta_2), \quad \mu_1,\mu_2 > 0. \tag{24}$$

Theorem 2.6. *Let $f \in C_b(X_b)$ and $B\hat{Y}^a_{m,n}(f;x,y)$ be linear positive operators defined by (21). Then the following inequality holds:*

$$|B\hat{Y}^a_{m,n}f)(x,y) - f(x,y)| \le 4\omega_B(f;\delta'_m,\delta'_n). \tag{25}$$

Proof. With the help of Remark 2.5, one can write as:

$$\begin{aligned} |\Delta f(t,s;x,y)| &\le \omega_B(f;\delta_1,\delta_2) \\ &\le \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_B(f;\delta_1,\delta_2), \ \ \delta_1,\delta_2 \ge 0. \end{aligned}$$

Using the property of the difference function $\Delta f(t,s;x,y)$ and applying the operators (1), we get

$$B\hat{Y}^a_{m,n}(f;x,y) = f(x,y)\hat{Y}_{m,n,a}(1,x,y) - \hat{Y}_{m,n,a}(\Delta f(t,s;x,y),x,y), \tag{26}$$

Using Cauchy-Schwartz inequality in (26), we obtain

$$\begin{aligned} |B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)| &\le \hat{Y}_{m,n,a}(|\Delta f(t,s;x,y)|;x,y) \\ &\le \left(\hat{Y}_{m,n,a}(e_{00};x,y) + \frac{1}{\delta_1}\hat{Y}_{m,n,a}(|t-x|;x,y)\right) \\ &\quad \times\left(\hat{Y}_{m,n,a}(e_{00};x,y) + \frac{1}{\delta_2}\hat{Y}_{m,n,a}(|s-y|;x,y)\right)\omega_B(f;\delta_1,\delta_2) \\ &\le \left(1 + \frac{1}{\delta_1}\sqrt{\hat{Y}_{m,n,a}((t-x)^2;x,y)} + \frac{1}{\delta_2}\sqrt{\hat{Y}_{m,n,a}((s-y)^2;x,y)}\right. \\ &\quad \left. + \frac{1}{\delta_1\delta_2}\sqrt{\hat{Y}_{m,n,a}((t-x)^2;x,y)}\sqrt{\hat{Y}_{m,n,a}((s-y)^2;x,y)}\right)\omega_B(f;\delta_1,\delta_2). \end{aligned}$$

Now by using Lemma 1.4, and choosing $\delta_1 = \delta'_m, \delta_2 = \delta'_n$, the desired results can be obtained. □

Next we will find the degree of approximation of the GBS-type operators defined by (21), by means of B-continuous function belonging to the Lipschitz class and it can be defined as:

$$\mathrm{Lip}_M(\mu_1,\mu_2) = \{f \in C_b(X_b) : |\Delta f((u,v),(u_0,v_0))| \le M|u-u_0|^{\mu_1}|v-v_0|^{\mu_2}, \ \mu_1,\mu_2 \in (0,1]\}, \tag{27}$$

where $(u,v),(u_0,v_0) \in X_b$ and $M > 0$.

Theorem 2.7. *If $f \in Lip_M(\mu_1, \mu_2)$, then there exists a positive constant M, such that*

$$|B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)| \leq M\delta'^{\frac{\mu_1}{2}}_m \delta'^{\frac{\mu_2}{2}}_n, \tag{28}$$

where $\delta'_m = \sqrt{\frac{x(x+1)}{m}}$, $\delta'_n = \sqrt{\frac{y(y+1)}{n}}$.

Proof. By using the linearity property of GBS-type operators (21) and by definition of $B\hat{Y}^a_{m,n}(f;x,y)$, we can write

$$\begin{aligned} |B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)| &\leq \hat{Y}_{m,n,a}(|\Delta f((t,s),(x,y))|;x,y) \\ &\leq M\hat{Y}_{m,n,a}(|u-u_0|^{\mu_1}|v-v_0|^{\mu_2};x,y) \\ &= M\hat{Y}_{m,n,a}(|t-x|^{\mu_1};x,y)\hat{Y}_{m,n,a}(|s-y|^{\mu_2};x,y), \end{aligned}$$

Using Hölder's inequality with $l_1 = \frac{2}{\mu_1}$, $r_1 = \frac{2}{2-\mu_1}$ and $l_2 = \frac{2}{\mu_2}$, $r_2 = \frac{2}{2-\mu_2}$, in the next step, the required result can be obtained as

$$\begin{aligned} |B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)| &\leq M(\hat{Y}_{m,n,a}((t-x)^2;x,y))^{\frac{\mu_1}{2}}(\hat{Y}_{m,n,a}((s-y)^2;x,y))^{\frac{\mu_2}{2}} \\ &\leq M\delta'^{\frac{\mu_1}{2}}_m \delta'^{\frac{\mu_2}{2}}_n. \end{aligned}$$

Hence proved. □

Next, the rate of convergence of associated GBS-type operators can be obtained, when the function is B-differentiable, and it is defined by (20).

Lemma 2.8. *For any $x, y \geq 0$ and for all $m, n \in \mathbb{N}$, we have*

$$\hat{Y}_{m,n,a}((\cdot - x)^{2i}(\star - y)^{2j};x,y) = \hat{Y}_{m,n,a}((\cdot - x)^{2i});x,y)\hat{Y}_{m,n,a}(\star - y)^{2j};x,y), \quad \forall\, i,j \in \mathbb{N}\cup\{0\}.$$

Proof. Given that $x, y \geq 0$ and $n, m \in \mathbb{N}$ then, we have

$$\begin{aligned} \hat{Y}_{m,n,a}((\cdot - x)^{2i}(\star - y)^{2j};x,y) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,n}(x,y)\left(\frac{k_1}{m} - x\right)^{2i}\left(\frac{k_2}{n} - y\right)^{2j} \\ &= \sum_{k_1=0}^{\infty} s^a_m(x,y)\left(\frac{k_1}{m} - x\right)^{2i}\sum_{k_2=0}^{\infty} s^a_m(x,y)\left(\frac{k_2}{n} - y\right)^{2j} \\ &= \hat{Y}_{m,n,a}((\cdot - x)^{2i});x,y)\hat{Y}_{m,n,a}(\star - y)^{2j};x,y). \end{aligned}$$

Hence proved. □

Theorem 2.9. *If $f \in D_b(X_b)$ and $D_B f \in B_b(X_b)$, then there exists a positive constant M_1, such that*

$$|B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)| \leq \frac{M_4}{\sqrt{mn}}\left\{3M_3\|D_B f\| + \omega_B\left(D_B f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right\} \tag{29}$$

Proof. Using mean value theorem for B-differentiable functions, it can be written as

$$D_B f(\beta,\gamma) = \frac{\Delta f((t,s),(x,y))}{(t-x)(s-y)}, \quad \text{where } \beta \in (t,x);\ \gamma \in (s,y). \tag{30}$$

By using the property of $\Delta f((t,s),(x,y))$, it gives:

$$D_B f(\beta,\gamma) = \Delta D_B f((\beta,\gamma),(x,y)) + D_B f(x,\gamma) + D_B f(\beta,y) - D_B f(x,y). \tag{31}$$

Since $D_B f \in B_b(X_b)$, so by using (31) and (30), we get

$$\begin{aligned}
|\hat{\Upsilon}_{m,n,a}(\Delta f((t,s),(x,y));x,y)| &= |\hat{\Upsilon}_{m,n,a}((t-x)(s-y)D_B f(\beta,\gamma));x,y)| \\
&\leq \hat{\Upsilon}_{m,n,a}(|(t-x)||(s-y)||\Delta D_B f(\beta,\gamma),(x,y))|;x,y) \\
&\quad + \hat{\Upsilon}_{m,n,a}(|t-x||s-y|(|D_B f(x,\gamma)| + |D_B f(\beta,y)| - |D_B f(x,y)|);x,y) \\
&\leq \hat{\Upsilon}_{m,n,a}(|t-x||y-s|\omega_B(D_B f;|\beta-x|,|\gamma-y|);x,y) \\
&\quad + 3\|D_B f\|\hat{\Upsilon}_{m,n,a}(|t-x||y-s|;x,y),
\end{aligned}$$

as $\beta \in (x,t)$ and $\gamma \in (y,s)$ (already assumed) and with the property of modulus, for $h_m, h_n > 0$, we have

$$\begin{aligned}
\omega_B(D_B f;|\beta-x|,|\gamma-y|) &\leq \omega_B(D_B f;|t-x|,|s-y|) \\
&\leq \left(1+\frac{|t-x|}{h_m}\right)\left(1+\frac{|s-y|}{h_n}\right)\omega_B(D_B f;h_m,h_n).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\hat{\Upsilon}_{m,n,a}(\Delta f((t,s),(x,y));x,y)| &\leq \hat{\Upsilon}_{m,n,a}\left(|t-x||y-s|\left(\left(1+\frac{|t-x|}{h_m}\right)\left(1+\frac{|s-y|}{h_n}\right)\omega_B(D_B f;h_m,h_n)\right);x,y\right) \\
&\quad + 3\|D_B f\|\hat{\Upsilon}_{m,n,a}(|t-x||y-s|;x,y). \qquad (32)
\end{aligned}$$

Since

$$|B\hat{\Upsilon}^{a}_{m,n}(f;x,y) - f(x,y)| \leq \hat{\Upsilon}_{m,n,a}(|\Delta f((t,s),(x,y))|;x,y), \qquad (33)$$

Using Inequalities 32, 33 and with the help of Cauchy-Schwartz inequality, we get

$$\begin{aligned}
|B\hat{\Upsilon}^{a}_{m,n}(f;x,y) - f(x,y)| &\leq \Bigg\{\left(\hat{\Upsilon}_{m,n,a}((t-x)^2(s-y)^2;x,y)\right)^{\frac{1}{2}} + h_m^{-1}\left(\hat{\Upsilon}_{m,n,a}((t-x)^4(s-y)^2;x,y)\right)^{\frac{1}{2}} \\
&\quad + h_n^{-1}\left(\hat{\Upsilon}_{m,n,a}((t-x)^2(s-y)^4;x,y)\right)^{\frac{1}{2}} \\
&\quad + h_m^{-1}h_n^{-1}\left(\hat{\Upsilon}_{m,n,a}((t-x)^4(s-y)^4;x,y)\right)^{\frac{1}{2}}\Bigg\}\omega_B(D_B f;h_m,h_n) \\
&\quad + 3\|D_B f\|\left(\hat{\Upsilon}_{m,n,a}((t-x)^2(s-y)^2;x,y)\right)^{\frac{1}{2}} \\
&= \Bigg\{\sqrt{\hat{\Upsilon}_{m,n,a}((t-x)^2;x,y)}\sqrt{\hat{\Upsilon}_{m,n,a}((s-y)^2;x,y)} \\
&\quad + h_m^{-1}\sqrt{\hat{\Upsilon}_{m,n,a}((t-x)^4;x,y)}\sqrt{\hat{\Upsilon}_{m,n,a}((s-y)^2;x,y)} \\
&\quad + h_n^{-1}\sqrt{\hat{\Upsilon}_{m,n,a}((t-x)^2;x,y)}\sqrt{\hat{\Upsilon}_{m,n,a}((s-y)^4;x,y)} \\
&\quad + h_m^{-1}h_n^{-1}\sqrt{\hat{\Upsilon}_{m,n,a}((t-x)^4;x,y)}\sqrt{\hat{\Upsilon}_{m,n,a}((s-y)^4;x,y)}\Bigg\}\omega_B(D_B f;h_m,h_n) \\
&\quad + 3\|D_B f\|\sqrt{\hat{\Upsilon}_{m,n,a}((t-x)^2;x,y)}\sqrt{\hat{\Upsilon}_{m,n,a}((s-y)^2;x,y)}.
\end{aligned}$$

Now using Inequalities 6, 7 and Lemma 1.6, we have

$$\begin{aligned}
|B\hat{Y}^{a}_{m,n}(f;x,y)-f(x,y)| &\leq \left\{\sqrt{\frac{\lambda_x}{m}}\sqrt{\frac{\lambda_y}{n}}+h_m^{-1}\sqrt{\frac{M_x}{m^2}}\sqrt{\frac{\lambda_y}{n}}+h_n^{-1}\sqrt{\frac{\lambda_x}{m}}\sqrt{\frac{M_x}{m^2}}\right.\\
&\left.+h_m^{-1}h_n^{-1}\sqrt{\frac{M_x}{m^2}}\sqrt{\frac{M_y}{n^2}}\right\}\omega_B(D_Bf;h_m,h_n)\\
&+3\|D_Bf\|\sqrt{\frac{\lambda_x}{m}}\sqrt{\frac{\lambda_y}{n}},
\end{aligned}$$

By considering $h_m^{-1}=\frac{1}{\sqrt{m}}$ and $h_n^{-1}=\frac{1}{\sqrt{n}}$, one can write

$$\begin{aligned}
|B\hat{Y}^{a}_{m,n}(f;x,y)-f(x,y)| &\leq \frac{1}{\sqrt{mn}}\left\{\left(\sqrt{\lambda_x\lambda_y}+\sqrt{M_x\lambda_y}+\sqrt{\lambda_xM_y}+\sqrt{M_xM_y}\right)\omega_B\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right.\\
&\left.+3\|D_Bf\|\sqrt{\lambda_x\lambda_y}\right\}\\
&= \frac{1}{\sqrt{mn}}\left\{\left(\sqrt{\lambda_x}+\sqrt{M_x}\right)\left(\sqrt{\lambda_y}+\sqrt{M_y}\right)\omega_B\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)+3\|D_Bf\|\sqrt{\lambda_x\lambda_y}\right\}\\
&= \frac{1}{\sqrt{mn}}\left\{M_1M_2\omega_B\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)+3M_3\|D_Bf\|\right\},
\end{aligned}$$

where $M_1=\left(\sqrt{\lambda_x}+\sqrt{M_x}\right)$, $M_2=\left(\sqrt{\lambda_y}+\sqrt{M_y}\right)$ and $M_3=\sqrt{\lambda_x\lambda_y}$ and $M_4=\max\{M_1M_2,M_3\}$, Hence, above Inequality gives

$$|B\hat{Y}^{a}_{m,n}(f;x,y)-f(x,y)|\leq\frac{M_4}{\sqrt{mn}}\left\{3M_3\|D_Bf\|+\omega_B\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right\}. \tag{34}$$

Hence, the proof is completed. □

To improve the measure of smoothness, a mixed K-functional is introduced [32, 33] and it is defined by

$$K_B(f;x_1,x_2)=\{\|f-g_1-g_2-h\|+x_1\|D_B^{2,0}g_1\|+x_2\|D_B^{0,2}g_2\|+x_1x_2\|D_B^{2,2}h\|\}, \tag{35}$$

where $g_1\in D_B^{2,0}$, $g_2\in D_B^{0,2}$, $h\in D_B^{2,2}$ and $D_B^{i,j}$ represent the space of all functions $f\in C_B(X_b)$ for $0\leq i,j\leq 2$ having mixed partial derivatives $D_B^{\eta,\mu}f$ with $0\leq\eta\leq i$, $0\leq\mu\leq j$ defined by

$$D_xf(u,v)=D_B^{1,0}(f;u,v)=\lim_{x\to u}\frac{\Delta_xf([u,x];v)}{x-u}, \tag{36}$$

$$D_yf(u,v)=D_B^{0,1}(f;u,v)=\lim_{y\to v}\frac{\Delta_yf(u;[v,y])}{y-v}, \tag{37}$$

$$D_yD_xf(u,v)=D_B^{0,1}D_B^{1,0}(f;u,v)=\lim_{y\to v}\frac{\Delta_y(\Delta_x)f(u;[v,y])}{y-v}, \tag{38}$$

$$D_xD_yf(u,v)=D_B^{1,0}D_B^{0,1}(f;u,v)=\lim_{x\to u}\frac{\Delta_x(\Delta_y)f([u,x];v)}{x-u}. \tag{39}$$

where $\Delta_xf([u,x];v)=f(x,v)-f(u,v)$, $\Delta_yf(u;[v,y])=f(u,y)-f(u,v)$.

Theorem 2.10. *Let $B\hat{Y}^{a}_{m,n}(f;x,y)$ be a GBS-type operator of $\hat{Y}_{m,n,a}(f;x,y)$ for all $x, y \in X_b = [0,c] \times [0,d]$ and for each function $f \in C_B(X_b)$ with $m, n \in \mathbb{N}$, we have*

$$|B\hat{Y}^{a}_{m,n}(f;x,y) - f(x,y)| \leq 2K_B\left(f, \frac{\lambda_x}{m}, \frac{\lambda_y}{n}\right). \tag{40}$$

Proof. With the help of Taylor's formula for the function $g_1 \in C_B^{2,0}(X_b)$, we obtain

$$g_1(t,s) - g_1(x,y) = (t-x)D_B^{1,0}g_1(x,y) + \int_x^t (t-\xi)D_B^{2,0}g_1(\xi,y)d\xi, \tag{41}$$

Upon using the linearity and positivity properties of $\hat{Y}_{m,n,a}$ operators, we can write

$$\begin{aligned}
|B\hat{Y}^{a}_{m,n}(g_1;x,y) - g_1(x,y)| &= \left|\hat{Y}_{m,n,a}\left(\int_x^t (t-\xi)[D_B^{2,0}g_1(\xi,y) - D_B^{2,0}g_1(\xi,s)]d\xi; x,y\right)\right| \\
&\leq \hat{Y}_{m,n,a}\left(\left|\int_x^t |(t-\xi)||D_B^{2,0}g_1(\xi,y) - D_B^{2,0}g_1(\xi,s)|d\xi; x,y\right|\right) \\
&\leq \|D_B^{2,0}g_1\|\hat{Y}_{m,n,a}((t-x)^2;x,y) < \|D_B^{2,0}g_1\|\frac{c}{m}.
\end{aligned}$$

Similarly for $g_2 \in D_B^{0,2}$, we get

$$|B\hat{Y}^{a}_{m,n}(g_2;x,y) - g_2(x,y)| < \|D_B^{0,2}g_2\|\frac{d}{n}.$$

For $h \in D_B^{2,2}$, we have

$$\begin{aligned}
h(t,s) - h(x,y) &= (t-x)D_B^{1,0}h(x,y) + (s-y)D_B^{0,1}h(x,y) + (t-x)(s-y)D_B^{1,1}h(x,y) \\
&\quad + \int_x^t (t-\xi)D_B^{2,0}h(\xi,y)d\xi + \int_y^s (s-\phi)D_B^{0,2}h(x,\phi)d\phi + \int_x^t (s-y)(t-\xi)D_B^{2,1}h(\xi,y)d\xi \\
&\quad + \int_y^s (t-x)(s-\phi)D_B^{1,2}h(x,\phi)d\phi + \int_x^t\int_y^s (t-\xi)(s-\phi)D_B^{2,2}h(\xi,\phi)d\xi d\phi.
\end{aligned} \tag{42}$$

Using integration by parts in above expression (42) and thereafter applying bivariate operators $\hat{Y}_{m,n,a}$ on the remaining terms of the given expression after some cancellation. Taking into account the definition of the GBS-type operators $B\hat{Y}^{a}_{m,n}$ and by using

$$B\hat{Y}^{a}_{m,n}((t-x);x,y) = 0, \; B\hat{Y}^{a}_{m,n}((s-y);x,y) = 0, \tag{43}$$

we get

$$
\begin{aligned}
|B\hat{Y}^{a}_{m,n}(h;x,y)-h(x,y)| &\leq \left|\hat{Y}_{m,n,a}\left(\int_x^t\int_y^s (t-\xi)(s-\phi)D_B^{2,2}h(\xi,\phi)d\xi d\phi;x,y\right)\right| \\
&\leq \hat{Y}_{m,n,a}\left(\int_x^t\int_y^s |(t-\xi)||(s-\phi)|\left|D_B^{2,2}h(\xi,\phi)\right|d\xi d\phi;x,y\right) \\
&\leq \frac{1}{4}\|D_B^{2,2}h\|\hat{Y}_{m,n,a}((t-x)^2(s-y)^2;x,y) \\
&\leq \|D_B^{2,2}h\|\frac{\lambda_x\lambda_y}{mn}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|B\hat{Y}^{a}_{m,n}(f;x,y)-f(x,y)| &\leq |(f-g_1-g_2-h)(x,y)| + \left|\left(g_1-B\hat{Y}^{a}_{m,n}g_1\right)(x,y)\right| + \left|\left(g_2-B\hat{Y}^{a}_{m,n}g_2\right)(x,y)\right| \\
&\quad + \left|\left(h-B\hat{Y}^{a}_{m,n}h\right)(x,y)\right| + \left|B\hat{Y}^{a}_{m,n}((f-g_1-g_2-h);x,y)\right| \\
&\leq 2\|f-g_1-g_2-h\| + \|D_B^{2,0}g_1\|\frac{\lambda_x}{m} + \|D_B^{0,2}g_2\|\frac{\lambda_y}{n} + \|D_B^{2,2}h\|\frac{\lambda_x\lambda_y}{mn},
\end{aligned}
$$

by taking infimum over for all $g_1 \in C_B^{2,0}$, $g_2 \in C_B^{0,2}$, $h \in C_B^{2,2}$, we get our desired result. □

3. Graphical approach and convergence based discussion

For validation of the results, GBS-type operators are compared with the bivariate operators (1) and the rate of convergence is examined for finite sum over the interval $[0,1]$ as well as over the interval $[0,\infty)$ through graphical representations along with their numerical approximation.

In this section, we discuss the behaviour of the operators with the function $f(x,y)$ for particular values of k_1, k_2 and for an infinite series (i.e. for $k_1 = 0,1,\cdots$ and $k_2 = 0,1,\cdots$). Also, check the behaviour of the operators (1) and (21) by comparison.

Example 3.1. *Consider the function $f(x,y) = x\sin\pi y$ (green). For the particular value of $m = n = 10$, $k_1 = 9 = k_2$, the corresponding operators are represented by $\hat{Y}_{10,10,a}(f;x,y)$(blue) and $B\hat{Y}^{a}_{10,10}(f;x,y)$(red) respectively. Upon considering the partitions as $x_0 = 0, x_1 = \frac{1}{10},\cdots,x_9 = \frac{9}{10}$ of $[0,1]$ and $y_0 = 0, y_1 = \frac{1}{10},\cdots,y_9 = \frac{9}{10}$ of $[0,1]$, the convergence approach of the operators $\hat{Y}_{m,n,a}(f;x,y)$ and $B\hat{Y}^{a}_{m,n}(f;x,y)$ to the function and their comparison are shown in Figure 1.*

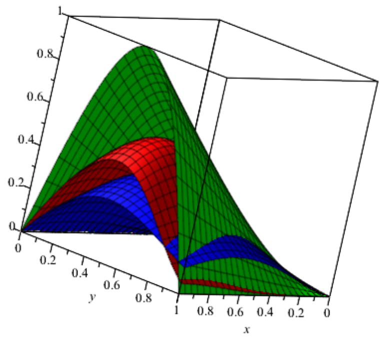

Figure 1: Comparison of the convergence approach of operators $\hat{Y}_{m,n,a}(f;x,y)$(blue) and $B\hat{Y}^{a}_{m,n}(f;x,y)$(red) to the function $f(x,y)$(green).

Now, we choose less numbers of partitions for the same function and for the same particular values of $m = n = 10$*.*

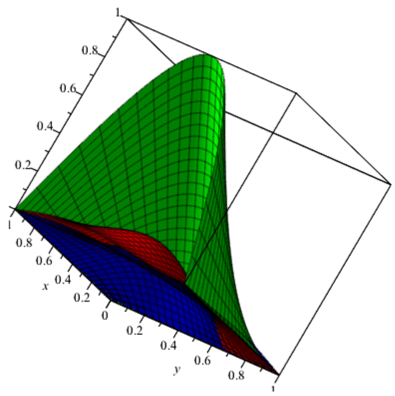

Figure 2: Comparison of the convergence approach of bivariate operators $\hat{Y}_{m,n,a}(f;x,y)$ (blue) and $B\hat{Y}^{a}_{m,n}(f;x,y)$ (red) to the function $f(x,y)$ (green).

Here, we take the partitions within six terms like as $x_0 = 0, x_1 = \frac{1}{10}, \cdots, x_5 = \frac{5}{10}$ of $[0,1]$ and $y_0 = 0, y_1 = \frac{1}{10}, \cdots, y_5 = \frac{5}{10}$ of $[0,1]$ as shown in Figure 2. It can be seen from Figure 2 that the error gap between the function and operators are maximum in Figure 2 rather than in Figure 1.

Finally, it can be observed from Figures 1, 2 that the accuracy approach of the GBS-type operators (21) to the function $f(x,y)$ is better than the bivariate operators (1) but it depends on the number of partitions of $[0,1]$. By observing Figure 2 and Figure 1, it can be seen that for large number of partitions, i.e., as the length of the partition be small, the approximation is better as compared to less number of partitions of the interval, i.e, for larger length of partitions. It can also be concluded that the approach of the operators to the function will be good upon using large number of partitions as compared to less numbers of partitions for the same interval. On other the hand, the approach of the GBS-type operators (21) is better than the bivariate operators (1). So, finally we can say that the convergence rate of the GBS-type operators is better than the convergence rate of the bivariate operators in any case.

Remark: In general, if we consider $[x_0,x_1],[x_1,x_2],\cdots,[x_{i-1},x_i]$ and $[y_0,y_1],[y_1,y_2],\cdots,[y_{j-1},y_j]$, are the sub-intervals of the $[0,1]$, provided each x_i, y_j are the some form of $\frac{i}{m}, \frac{j}{n}$ respectively, where $i = 1,2,\cdots,k_1,\ j = 1,2,\cdots,k_2$ while $k_1 \leq m, k_2 \leq n$, then the following concluding remarks can be obtained.

Concluding Remark:

- If the number of sub-intervals are maximum i.e., the sub-length $x_i - x_{i-1}, y_j - y_{j-1}$ are small, then the approximation is good.
- If the number of sub-interval are minimum i.e., the sub-length $x_i - x_{i-1}, y_j - y_{j-1}$ are large, then the approximation is not good.

Note: In above both conditions, the approach of the GBS-type operators (21) to the function is better than the bivariate operators as defined by (1).

Example 3.2. *Consider a function defined by $f(x, y) = x \sin \pi y$(green). For the particular value of $m = n = 10$, the corresponding operators $\hat{Y}_{10,10,a}(f; x, y)$ and $B\hat{Y}^{a}_{10,10}(f; x, y)$ are shaded by blue and red colors respectively as given in Figure 3. Here, it can be seen the approximation of the function defined by the operators (1), (21) and the error determined by the GBS-type operators to the function is minimum than the bivariate operators (21).*

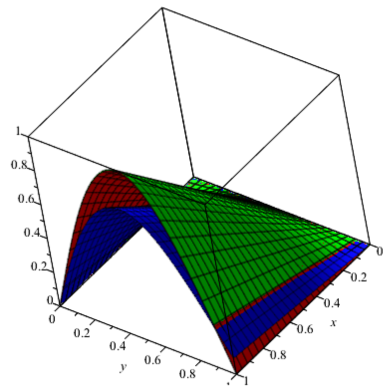

Figure 3: Comparison of the convergence for bivariate operators $\hat{Y}_{m,n,a}(f; x, y)$(blue) and $B\hat{Y}^{a}_{m,n}(f; x, y)$(red) to the function $f(x, y)$ (green).

Concluding result: From Figure 3, it can be concluded that the convergence behavior of the GBS-type operators defined by (21) is better than the bivariate operators defined by (1).

Example 3.3. *Consider a function $f(x, y) = \sin(x + y)$ (green). On choosing the value of $m = n = 10, 15$ for the GBS-type operators, the corresponding GBS operators can be represented as $B\hat{Y}^{2}_{10,10}(f; x, y)$(blue), $B\hat{Y}^{2}_{15,15}(f; x, y)$(yellow). It can be observed from Figure 4 that the error becomes smaller as the value of m and n increases.*

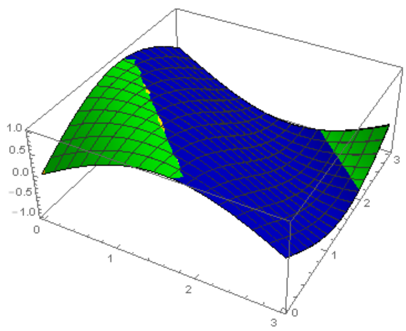

Figure 4: Convergence of GBS-type operators $B\hat{Y}^{a}_{m,m}(f; x, y)$ to the function $f(x, y)$.

From Figure 4, it can be seen the convergence behaviour of the GBS-type operators with the small value of the parameters (as $m = n = 10, 15$) whereas Figure 5 represents the convergence behaviour of the GBS-type operators

$B\hat{Y}^2_{15,15}(f;x,y)$ for $m = n = 15$ (yellow color) to the same function in more accurate form (which is unable to show in Figure 4 clearly).

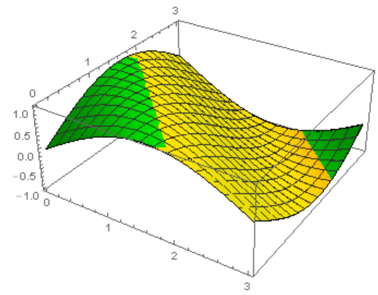

Figure 5: Convergence of GBS-type operator $B\hat{Y}^2_{15,15}(f;x,y)$(yellow) to the function $f(x,y)$(green).

Concluding result: *It can be concluded from the graphical representations of the operators $B\hat{Y}^a_{m,n}(f;x,y)$ and $\hat{Y}_{m,n,a}(f;x,y)$ that the rate of convergence of the GBS-type operators (21) is better than the bivariate operators (1).*

3.1. Numerical approach

Next, we discuss the absolute error of the GBS-type operators (21) as well as the bivariate operators (1) to the function $f(x,y)$ and compare these operators with their numerical errors at different points and for different values of m, n.

Let $G^a_{m,n}(f;x,y) = |B\hat{Y}^a_{m,n}(f;x,y) - f(x,y)|$ and $SY_{m,n,a}(f;x,y) = |\hat{Y}_{m,n,a}(f;x,y) - f(x,y)|$, then the given Table 1 represents the numerical approximations of the GBS-type operators (21) and bivariate operators (1). Here, function is $f(x,y) = \sin(x+y)$ and error bounds are presented at point $(x,y) = (0.1, 0.1)$.

Error in the approximation for GBS-type operators and bivariate operators to the function $f(x,y)$		
m=n	$SY_{m,n,a}(f;x,y)$	$G^a_{m,n}(f;x,y)$
10	0.00887548	0.0000358021
15	0.00589734	0.0000160308
25	0.0035283	5.80313×10^{-6}
50	0.00176016	1.45647×10^{-6}
100	0.000879051	3.64803×10^{-7}

Table 1: Comparison of GBS-type operators and bivariate operators to the function $f(x,y)$

Concluding remark: From Table 1, it can be concluded that the approximation by the GBS-type operators (21) to the function is better than the bivariate operators (1).

3.2. A comparison of bivariate operators $\hat{Y}_{m,n,a}(f;x,y)$ with bivariate Kantorovich operators

In this subsection, we show the graphical representation for the comparison of convergence of the bivariate operators (1) with the bivariate Kantorovich operators of Szász-Mirakjan. In 2006, Muraru [34] gave a quantitative approximation of Kantorovich-Sász bivariate operators, defined as

$$\hat{K}_{m,n}f : L_1([0,\infty)\times[0,\infty)) \to B([0,\infty)\times[0,\infty)),\ (m,n)\in\mathbb{N}\times\mathbb{N};$$

$$\hat{K}_{m,n}(f;x,y) = mne^{-mx-ny}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(mx)^{k_1}}{k_1!}\frac{(nx)^{k_2}}{k_2!}\int_{\frac{k_1}{m}}^{\frac{k_1+1}{m}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} f(u,v)\,dudv. \quad (44)$$

There are following computational examples, which represent the comparison.

Example 3.4. *Let the function* $f(x) = x^2y(x-1)\sin(2\pi y)$ *(green), for all* $0 \le x, y \le 2$ *and choose the value of* $m, n = 10$*, for which the bivariate operators (yellow) defined by (1) show the better rate of convergence than the Kantorovich-Szász bivariate operators* $\hat{K}_{m,n}(f; x, y)$*(red) defined by (44), graphical representation can be seen by the Figure 6.*

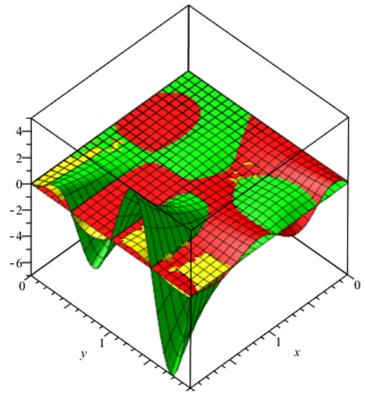

Figure 6: Comparison of the convergence of operators $\hat{Y}_{m,n,a}(f; x, y)$ (yellow) and $\hat{K}_{m,n}(f; x, y)$ (red) to the function $f(x)$ (green)

Example 3.5. *Consider the function* $f(x) = x^2y\cos(\pi y)$ *(green), for all* $0 \le x, y \le 4$ *and choose* $m, n = 10$*, for which the bivariate operators (yellow) defined by (1) present the better rate of convergence than the bivariate Kantorovich operators* $\hat{K}_{m,n}(f; x, y)$ *(red) defined by (44), the graphical representation is illustrated by Figure 7.*

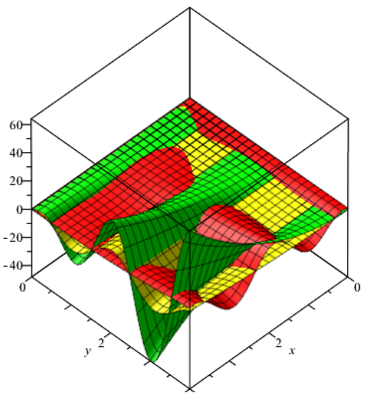

Figure 7: Comparison of the convergence of operators $\hat{Y}_{m,n,a}(f; x, y)$ (yellow) and $\hat{K}_{m,n}(f; x, y)$ (red) to the function $f(x)$ (green)

Example 3.6. *Let the function* $f(x) = y^2\cos(2\pi x)$ *(green), for all* $0 \le x, y \le 4$ *and consider* $m, n = 20$*, for which the graphical representation of the bivariate operators* $\hat{Y}_{m,n}(f; x, y)$ *(yellow) defined by (1) and the bivariate Kantorovich operators* $\hat{K}_{m,n}(f; x, y)$ *(red) defined by (44) is illustrated in Figure 8.*

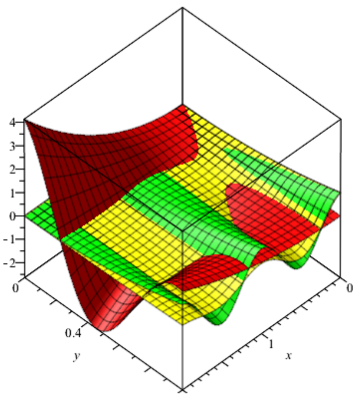

Figure 8: Comparison of the convergence of operators $\hat{\Upsilon}_{m,n,a}(f;x,y)$ (yellow) and $\hat{K}_{m,n}(f;x,y)$ (red) to the function $f(x)$ (green)

Concluding Remark: By above Figures 6-8, we can say that the rate of convergence of the bivariate operators $\hat{\Upsilon}_{m,n,a}(f;x,y)$ (1) is better than bivariate Kantorovich operators $\hat{K}_{m,n}(f;x,y)$ defined by (44).

3.3. *Comparison of associated GBS operators with GBS-type operators of an infinite sum*

In 2008, Pop [35] introduced an associated GBS-type operators of the linear positive operators defined by an infinite sum, so called GBS operators of Mirakjan-Favard-Szász and gave an approximation of the functions considered to be B-continuous and B-differentiable. The defined associated GBS operators can be stated as:

Let $m,n \in \mathbb{N}$, the operators $UL^{*}_{m,n} : E(I\times I) \to F(J\times J)$ are defined for any function $f \in E(I\times I)$ and for $(x,y) \in J\times J$ such that

$$UL^{*}_{m,n}(f;x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \psi_{m,k_1}(x)\psi_{n,k_2}(y)[f(x_{m,k_1},y) + f(x,x_{n,k_2}) - f(x_{m,k_1},x_{n,k_2})], \tag{45}$$

where $((x_{m,k_1})_{k_1\in\mathbb{N}_0})_{m\geq 1}, ((x_{n,k_2})_{k_2\in\mathbb{N}_0})_{n\geq 1}$ are the sequences of nodes and the functions $\psi_{m,k_1} : I \to \mathbb{R}, \psi_{n,k_2} : J \to \mathbb{R}$ with the properties, $\psi_{m,k_1} \geq 0, \psi_{n,k_2} \geq 0$, where $I,J \subset \mathbb{R}, I\cap J \neq \phi$.

Above GBS-modification operators (45) are the GBS-form of the operators L^{*}-type operators [35] and are given by:

$$L^{*}_{m,n}(f;x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \psi_{m,k_1}(x)\psi_{n,k_2}(y)f(x_{m,k_1},x_{n,k_2}), \quad (x,y) \in J\times J, \tag{46}$$

where $m,n \in \mathbb{N}$, $f \in E(I\times I)$ and $L^{*}_{m,n} : E(I\times I) \to F(J\times J)$.

For the particular case, Pop [35] determined the convergence properties for the GBS operators of Mirakjan-Favard-Szász. Here, if $\psi_{m,k_1}(x) = \frac{k_1}{m}, \psi_{n,k_2}(x) = \frac{k_2}{n}$ and $\psi_{m,k_1} = e^{-mx}\frac{(mx)^{k_1}}{k_1!}, \psi_{n,k_2} = e^{-ny}\frac{(ny)^{k_2}}{k_2!}$ then for $f \in C([0,\infty)\times[0,\infty))$, the above operators (45) can be reduced to GBS operators of Mirakjan-Favard-Szász, which can be defined as follows:

$$US^{*}_{m,n}(f;x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} e^{-mx-ny}\frac{(mx)^{k_1}}{k_1!}\frac{(ny)^{k_2}}{k_2!}\left[f\left(\frac{k_1}{m},y\right) + f\left(x,\frac{k_2}{n}\right) - f\left(\frac{k_1}{m},\frac{k_2}{n}\right)\right]. \tag{47}$$

This subsection is very crucial from the comparison point of view of the GBS type operators as defined by (21) with the GBS operators of the Mirakjan-Favard-Szász type (47), which is shown by the following examples.

Example 3.7. *Let the function be defined by $f(x,y)=e^{x+y}$(green). A comparison for the convergence of the GBS-type operators $B\hat{Y}^{a}_{m,n}(f;x,y)$ (red) with the GBS operators of Mirakjan-Favard-Szász $US^{*}_{m,n}(f;x,y)$ (black) to the function $f(x,y)$ is illustrated in Figure 9 for $m=n=2$. It can be observed that the GBS-type operators defined by (21) have a better rate of convergence than the GBS operators of Mirakjan-Favard-Szász as defined by (47).*

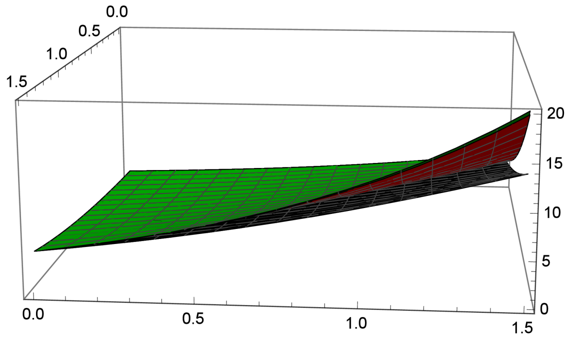

Figure 9: Comparison of the rate of convergence of GBS-type operators $B\hat{Y}^{a}_{m,m}(f;x,y)$ and GBS operators of Mirakjan-Favard-Szász to the function $f(x,y)$

Example 3.8. *For the same function $f(x,y)=e^{x+y}$ and at a certain point (x,y), the error estimation of the GBS-type operators $B\hat{Y}^{a}_{m,n}(f;x,y)$ and GBS operators of Mirakjan-Favard-Szász $US^{*}_{m,n}(f;x,y)$ has been computed in Table 2.*

Error in the approximation for $B\hat{Y}^{a}_{m,n}(f;x,y)$ and $US^{*}_{m,n}(f;x,y)$ to the function $f(x,y)$		
m=n	$\lvert US^{*}_{m,n}(f;x,y)-f(x,y)\rvert$	$\lvert B\hat{Y}^{a}_{m,n}(f;x,y)-f(x,y)\rvert$
10	3.28277×10^{-5}	3.00798×10^{-6}
20	7.91371×10^{-6}	7.35189×10^{-7}
50	1.23907×10^{-6}	1.16048×10^{-7}
100	3.07549×10^{-7}	2.88814×10^{-8}

Table 2: Comparison of GBS-type operators $B\hat{Y}^{a}_{m,n}(f;x,y)$ and GBS operators of Mirakjan-Favard-Szász $US^{*}_{m,n}(f;x,y)$ to the function $f(x,y)$

Concluding Remark: It can be concluded from Table 2 that the error arising in the approximation at a certain point by GBS-type operators defined by (21) to the function is much smaller than the GBS operators of Mirakjan-Favard-Szász as defined by (47). Hence our GBS-type operators have a better rate of convergence than the GBS operators of Mirakjan-Favard-Szász type.

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