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Alternative Criteria for Boundedness of One Class of Integral Operators in Lebesgue Spaces

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Abstract. The paper discusses weighted Hardy type inequalities for a certain class of Volterra type integral operators with kernels. Criteria for the validity of the presented inequalities are found, which are different from the criteria obtained earlier.

1. Introduction

Let $I = (0, \infty)$, $1 < p, q < \infty$, $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. Let u and v be positive functions locally integrable on the interval I. Suppose that $v^{-p'}$ is also locally integrable on the interval I. Let $L_{p,v} \equiv L_p(v, I)$ be a weighted Lebesgue space such that $\|f\|_{p,v} = \|vf\|_p < \infty$, where $\|\cdot\|_p$ is the standard norm of the space $L_p(I)$ for $1 < p < \infty$.

We consider the following weighted inequality

$$\|\mathcal{K}f\|_{q,u} \le C\|f\|_{p,v} \tag{1}$$

for the integral operator

$$\mathcal{K}f(x) = \int_0^x K(x,s)f(s)ds, \quad x > 0. \tag{2}$$

In the case $K(\cdot,\cdot) \equiv 1$, operator (2) is the Hardy operator and inequality (1) is the classical weighted Hardy inequality, which was investigated for all relations between the parameters p and q. The most famous result for inequality (1) with the Hardy operator is its characterizations in the case $1 \le p = q < \infty$ given in the paper [8] by B. Muckenhoupt and known as "Muckenhoupt conditions". The case $1 \le p \le q < \infty$ was independently investigated by V. Kokilashvili in [4], J.S. Bradley in [3], by V.G. Maz'ya in [6] and by K.F. Andersen, B. Muckenhoupt in [1]. The case $1 \le q < p < \infty$ was considered by V.G. Maz'ya and A.L. Rozin (see [7]). The case $0 < q < 1 < p < \infty$ was considered by G. Sinnamon in [13]. The remaining case $0 < q < 1 = p$ was studied by G. Sinnamon, V.D. Stepanov in [14], where one can also find an elementary

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proof of the case $0 < q < p$, $1 < p < \infty$. This direction was further developed by V.D. Stepanov in the works [15]-[17], when in (1) the Hardy operator was replaced by the Riemann-Liouville operator. Concerning a more detailed history of the development of Hardy type inequalities we refer to the monograph by A. Kufner, L. Maligranda, L.-E. Persson [5] and the references therein. This monograph indicates that the entire history of Hardy type inequalities shows that several groups of mathematicians independently obtained the same or equivalent results at the same time.

A more general operator than the Riemann-Liouville operator is operator (2) with kernel $K(\cdot,\cdot)$ satisfying Oinarov's condition: $K(x,s) \geq 0$ for $x \geq s > 0$ and there exists a number $d > 1$ such that

$$\frac{1}{d}\Big(K(x,t) + K(t,s)\Big) \leq K(x,s) \leq d\Big(K(x,t) + K(t,s)\Big), \tag{3}$$

where $x \geq t \geq s > 0$. These operators have been extensively studied by many authors, and important results have been obtained.

For $1 < p \leq q < \infty$ necessary and sufficient conditions on the weight functions u and v for the validity of (1) with $K(\cdot,\cdot)$ satisfying (3) were independently obtained by R. Oinarov in [9] and by S. Bloom, R. Kerman in [2], but they are different. Thus, in [9] the characterizations have the form

$$\sup_{z>0}\left(\int_z^\infty u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z K^{p'}(z,s)v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

$$\sup_{z>0}\left(\int_z^\infty K^q(t,z)u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

while the characterizations obtained in [2] have the form

$$\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(t,s)u^q(t)dt\right)^{p'} ds \leq C\left(\int_z^\infty u^q(t)dt\right)^{\frac{p'}{q'}},$$

$$\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K^q(t,s)u^q(t)dt\right)^{p'} ds \leq C\left(\int_z^\infty K^q(t,z)u^q(t)dt\right)^{\frac{p'}{q'}},$$

for almost all $z > 0$, i.e., we have alternative characterizations for the validity of inequality (1).

Obtaining alternative criteria for the boundedness of any operators plays an important role in operator theory, since different criteria are suitable for different problems. In the paper [19] by V.D. Stepanov, E.P. Ushakova various alternative criteria were obtained for the validity of inequality (1) with integral operators satisfying Oinarov's condition. Moreover, the paper [19] completely reveals the needs for alternative criteria.

In the papers [11] and [12] by R. Oinarov the classes $O_n^{\pm}$, $n > 0$, of kernels $K(\cdot,\cdot)$ were introduced and inequality (1) was characterized for operators with kernels from these classes. The class O_1^+ consists of kernels $K(\cdot,\cdot)$ satisfying the following condition: $K(x,s)$ does not decrease in the first argument and it is non-negative for $x \geq s > 0$ and there exist non-negative functions $G(\cdot,\cdot)$, $\rho(\cdot)$ and a number $d > 1$ such that $G(x,s) \geq 0$ for $x \geq s > 0$, $\rho(s) > 0$ for $s > 0$ and

$$\frac{1}{d}\Big(G(x,t)\rho(s) + K(t,s)\Big) \leq K(x,s) \leq d\Big(G(x,t)\rho(s) + K(t,s)\Big), \tag{4}$$

where $x \geq t \geq s > 0$. The class O_1^- contains kernels $K(\cdot,\cdot)$ satisfying the following condition: $K(x,s)$ does not increase in the second argument and it is non-negative for $x \geq s > 0$ and there exist non-negative functions

$Q(\cdot,\cdot)$, $\omega(\cdot)$ and a number $d > 1$ such that $Q(t,s) \geq 0$ for $t \geq s > 0$, $\omega(x) > 0$ for $x > 0$ and

$$\frac{1}{d}\Big(K(x,t) + \omega(x)Q(t,s)\Big) \leq K(x,s) \leq d\Big(K(x,t) + \omega(x)Q(t,s)\Big), \tag{5}$$

where $x \geq t \geq s > 0$.

In the paper [11] it was shown that the function $G(x,t)$ is equivalent to the function $\inf\limits_{s\leq t} \frac{K(x,s)}{\rho(s)}$ and $Q(t,s)$ is equivalent to the function $\inf\limits_{t\leq x} \frac{K(x,s)}{\omega(x)}$. Hence, we can assume that the functions $K(x,s)$, $G(x,t)$ and $Q(t,s)$ are non-decreasing in the first argument and non-increasing in the second argument.

Let us note that conditions (4) and (5) are less restrictive than Oinarov's condition. It is easy to see that the classes O_1^+ and O_1^- are wider than the class of kernels satisfying Oinarov's condition, since (3) is a partial case of condition (4) for $\rho(s) \equiv 1$, $G(x,t) \equiv K(x,t)$ and condition (5) for $\omega(x) \equiv 1$, $Q(t,s) \equiv K(t,s)$, i.e., the kernels satisfying (3) belong to the class $O_1^+ \cap O_1^-$.

The aim of this paper is to find characterizations for the validity of inequality (1) with kernels of operator (2) belonging to the classes O_1^+ and O_1^-, but alternative to those found in the work [11].

The symbol $A \ll B$ means $A \leq cB$ with some constant $c > 0$, depending on the parameters p and q. Moreover, notation $A \approx B$ means $A \ll B \ll A$, while $\chi_{(a,b)}(\cdot)$ stands for the characteristic function of the interval (a,b).

2. Main results for class O_1^+

The following theorem was proved in the work [11].

Theorem A. *Let $1 < p \leq q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (2) belong to the class O_1^+. Then inequality (1) holds if and only if*

$$A_1^+ = \sup_{z>0}\left(\int_z^\infty u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z K^{p'}(z,s)v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

$$A_2^+ = \sup_{z>0}\left(\int_z^\infty G^q(t,z)u^q(t)dt\right)^{\frac{1}{q}}\left(\int_0^z \rho^{p'}(s)v^{-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

in addition, $C \approx \max\{A_1^+, A_2^+\}$, where C is the best constant in (1).

Let us now present alterative characterizations for the validity of inequality (1) under the conditions of Theorem A.

Theorem 2.1. *Let $1 < p \leq q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (2) belong to the class O_1^+. Then inequality (1) holds if and only if*

$$B_1^+ = \sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(t,s)u^q(t)dt\right)^{p'} ds\right)^{\frac{1}{p'}}\left(\int_z^\infty u^q(t)dt\right)^{-\frac{1}{q'}} < \infty,$$

$$B_2^+ = \sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(t,s)G^{q-1}(t,s)u^q(t)dt\right)^{p'} ds\right)^{\frac{1}{p'}}\left(\int_z^\infty G^q(t,z)u^q(t)dt\right)^{-\frac{1}{q'}} < \infty,$$

in addition, $C \approx \max\{B_1^+, B_2^+\}$, where C is the best constant in (1).

Theorem 2.1 will be proved by the method of the paper [2]. For the proof we need the following lemma. Assume that

$$E^+ = \int_0^\infty u^q(x)\left(\int_0^x K(x,s)f(s)ds\right)^q dx,$$

$$E_1^+ = \int_0^\infty f(s)\left(\int_0^s \rho(t)f(t)dt\right)^{q-1}\int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dxds,$$

$$E_2^+ = \int_0^\infty f(s)\left(\int_0^s K(s,t)f(t)dt\right)^{q-1}\int_s^\infty K(x,s)u^q(x)dxds.$$

Lemma 2.2. *Let $1 < q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (2) belong to the class O_1^+. Then*

$$E^+ \approx E_1^+ + E_2^+. \tag{6}$$

Proof. Using (4) and changing the order of integration, we get

$$E^+ = \int_0^\infty u^q(x)\left(\int_0^x K(x,s)f(s)ds\right)^q dx = q\int_0^\infty u^q(x)\int_0^x K(x,t)f(t)\left(\int_0^t K(x,s)f(s)ds\right)^{q-1} dtdx$$

$$\approx q\int_0^\infty u^q(x)\int_0^x K(x,t)G^{q-1}(x,t)f(t)\left(\int_0^t \rho(s)f(s)ds\right)^{q-1} dtdx$$

$$+ q\int_0^\infty u^q(x)\int_0^x K(x,t)f(t)\left(\int_0^t K(t,s)f(s)ds\right)^{q-1} dtdx$$

$$= q\int_0^\infty f(t)\left(\int_0^t \rho(s)f(s)ds\right)^{q-1}\int_t^\infty K(x,t)G^{q-1}(x,t)u^q(x)dxdt$$

$$+ q\int_0^\infty f(t)\left(\int_0^t K(t,s)f(s)ds\right)^{q-1}\int_t^\infty K(x,t)u^q(x)dxdt = q(E_1^+ + E_2^+).$$

□

Proof of Theorem 2.1. Necessity. Let inequality (1) hold. Then the dual inequality

$$\left(\int_0^\infty v^{-p'}(s)\left(\int_s^\infty K(x,s)g(x)dx\right)^{p'} ds\right)^{\frac{1}{p'}} \le C\left(\int_0^\infty |u^{-1}(t)g(t)|^{q'}dt\right)^{\frac{1}{q'}}, \ g \ge 0, \tag{7}$$

also holds. From the validity of (1) it follows that $u^q \in L_1(z,\infty)$ for any $z > 0$. Then $u^q \in L_{q',u^{-1}}(z,\infty)$. Therefore, assuming $g(x) = \chi_{(z,\infty)}(x)u^q(x)$ in (7), we have

$$\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(x,s)u^q(x)dx\right)^{p'} ds\right)^{\frac{1}{p'}} \le C\left(\int_z^\infty u^q(t)dt\right)^{\frac{1}{q'}}$$

for any $z > 0$. This gives

$$B_1^+ = \sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(x,s)u^q(x)dx\right)^{p'} ds\right)^{\frac{1}{p'}}\left(\int_z^\infty u^q(t)dt\right)^{-\frac{1}{q'}} \le C. \tag{8}$$

From the validity of (1) and from (6) we get

$$\int_0^\infty f(s)\left(\int_0^s \rho(t)f(t)dt\right)^{q-1}\int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dx \ll C^q\left(\int_0^\infty |v(t)f(t)|^p dt\right)^{\frac{q}{p}}.$$

The latter gives

$$\rho(s)\int_s^\infty G^q(x,s)u^q(x)dx \ll \int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dx < \infty$$

for almost all $s > 0$. It is easy to see that the function $g(x) = \chi_{(z,\infty)}(x)G^{q-1}(x,z)u^q(x)$ belongs to $L_{q',u^{-1}}(z,\infty)$. Hence, replacing the function $g(x)$ in (7), we have

$$\left(\int_z^\infty v^{-p'}(t)\left(\int_t^\infty K(x,t)G^{q-1}(x,z)u^q(x)dx\right)^{p'} dt\right)^{\frac{1}{p'}} \le C\left(\int_z^\infty G^q(x,z)u^q(x)dx\right)^{\frac{1}{q'}}.$$

Since $G(x,z) \ge G(x,t)$ for $t \ge z$, from the last we have

$$B_2^+ = \sup_{z>0}\left(\int_z^\infty v^{-p'}(t)\left(\int_t^\infty K(x,t)G^{q-1}(x,t)u^q(x)dx\right)^{p'} dt\right)^{\frac{1}{p'}}\left(\int_z^\infty G^q(x,z)u^q(x)dx\right)^{-\frac{1}{q'}} \ll C. \tag{9}$$

The proof of necessity is complete.

Sufficiency. Let $\max\{B_1^+, B_2^+\} < \infty$. We use Hölder's inequality and get

$$E_1^+ = \int_0^\infty f(s)\left(\int_0^s \rho(t)f(t)dt\right)^{q-1}\int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dxds \le \|vf\|_p J_1^+, \tag{10}$$

where

$$J_1^+ = \left(\int_0^\infty v^{-p'}(s)\left(\int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dx\right)^{p'}\left(\int_0^s \rho(t)f(t)dt\right)^{p'(q-1)} ds\right)^{\frac{1}{p'}}.$$

Let us estimate J_1^+. First, using the integration by parts, we have

$$J_1^+ = \left(\int_0^\infty\int_z^\infty v^{-p'}(s)\left(\int_s^\infty K(x,s)G^{q-1}(x,s)u^q(x)dx\right)^{p'} ds\, d\left(\int_0^z \rho(t)f(t)dt\right)^{p'(q-1)}\right)^{\frac{1}{p'}}.$$

Then, using the finiteness of $B_2^+ < \infty$, applying Minkowski's inequality and integrating by parts, we obtain

$$J_1^+ \le B_2^+\left(\int_0^\infty\left(\int_z^\infty G^q(x,z)u^q(x)dx\right)^{\frac{p'}{q'}} d\left(\int_0^z \rho(t)f(t)dt\right)^{p'(q-1)}\right)^{\frac{1}{p'}}$$

$$= B_2^+\left[\left(\int_0^\infty\left(\int_z^\infty d\left(-\int_y^\infty G^q(x,y)u^q(x)dx\right)\right)^{\frac{p'}{q'}} d\left(\int_0^z \rho(t)f(t)dt\right)^{p'(q-1)}\right)^{\frac{q'}{p'}}\right]^{\frac{1}{q'}}$$

$$\leq B_2^+ \left[\int_0^\infty d\left(-\int_y^\infty G^q(x,y)u^q(x)dx \right) \left(\int_0^y d\left(\int_0^z \rho(t)f(t)dt \right)^{p'(q-1)} \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$= B_2^+ \left(\int_0^\infty d\left(-\int_y^\infty G^q(x,y)u^q(x)dx \right) \left(\int_0^y \rho(t)f(t)dt \right)^q \right)^{\frac{1}{q'}}$$

$$\approx B_2^+ \left(\int_0^\infty f(y) \left(\int_0^y \rho(t)f(t)dt \right)^{q-1} \int_y^\infty G^q(x,y)\rho(y)u^q(x)dxdy \right)^{\frac{1}{q'}}.$$

Taking into account that $K(x,y) \gg G(x,y)\rho(y)$, we have

$$J_1^+ \ll B_2^+ \left(\int_0^\infty f(y) \left(\int_0^y \rho(t)f(t)dt \right)^{q-1} \int_y^\infty K(x,y)G^{q-1}(x,y)u^q(x)dxdy \right)^{\frac{1}{q'}} = B_2^+(E_1^+)^{\frac{1}{q'}}. \tag{11}$$

From (6), (10) and (11) it follows that

$$E_1^+ \ll B_2^+(E^+)^{\frac{1}{q'}} \|vf\|_p. \tag{12}$$

Let us estimate E_2^+. First, using Hölder's inequality, we get

$$E_2^+ = \int_0^\infty f(s) \left(\int_0^s K(s,t)f(t)dt \right)^{q-1} \int_s^\infty K(x,s)u^q(x)dx \leq \|vf\|_p J_2^+, \tag{13}$$

where

$$J_2^+ = \left(\int_0^\infty v^{-p'}(s) \left(\int_s^\infty K(x,s)u^q(x)dx \right)^{p'} \left(\int_0^s K(s,t)f(t)dt \right)^{p'(q-1)} ds \right)^{\frac{1}{p'}}.$$

Now, we estimate J_2^+ by the same steps used to estimate J_1^+.

$$J_2^+ = \left(\int_0^\infty \int_z^\infty v^{-p'}(s) \left(\int_s^\infty K(x,s)u^q(x)dx \right)^{p'} ds\, d\left(\int_0^z K(z,t)f(t)dt \right)^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$\leq B_1^+ \left(\int_0^\infty \left(\int_z^\infty u^q(x)dx \right)^{\frac{p'}{q'}} d\left(\int_0^z K(z,t)g(t)dt \right)^{p'(q-1)} \right)^{\frac{1}{p'}}$$

$$B_1^+ \left[\left(\int_0^\infty \left(\int_z^\infty d\left(-\int_y^\infty u^q(x)dx \right) \right)^{\frac{p'}{q'}} d\left(\int_0^z K(z,t)f(t)dt \right)^{p'(q-1)} \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$\le B_1^+\left[\int_0^\infty d\left(-\int_y^\infty u^q(x)dx\right)\left(\int_0^y d\left(\int_0^z K(z,t)f(t)dt\right)^{p'(q-1)}\right)^{\frac{q'}{p'}}\right]^{\frac{1}{q'}}$$

$$= B_1^+\left(\int_0^\infty u^q(y)\left(\int_0^y K(y,t)f(t)dt\right)^q dy\right)^{\frac{1}{q'}} = B_1^+(E^+)^{\frac{1}{q'}}. \tag{14}$$

From (13) and (14) we get

$$E_2^+ \ll B_1^+(E^+)^{\frac{1}{q'}}\|vf\|_p. \tag{15}$$

From (6), (12) and (15) we have

$$\left(\int_0^\infty u^q(x)\left(\int_0^x K(x,s)f(s)ds\right)^q dx\right)^{\frac{1}{q}} \ll \max\{B_1^+, B_2^+\}\|vf\|_p,$$

i.e., inequality (1) holds with the estimate $C \ll \max\{B_1^+, B_2^+\}$ for the best constant C in (1), which, together with (8) and (9), gives

$$C \approx \max\{B_1^+, B_2^+\}.$$

The proof of Theorem 2.1 is complete. □

In the proof of Theorem 2.1 the validity of inequality (1) is found on the basis of the validity of inequality (7) dual to (1). Thus, rewriting inequality (7) as

$$\|\mathcal{K}^* f\|_{q,u} \le C\|f\|_{p,v} \tag{16}$$

with the integral operator

$$\mathcal{K}^* f(x) = \int_s^\infty K(x,s)f(x)dx, \quad s > 0, \tag{17}$$

by Theorem 2.1 we have the following theorem.

Theorem 2.3. *Let $1 < p \le q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (17) belong to the class O_1^+. Then inequality (16) holds if and only if*

$$(B_1^+)^* = \sup_{z>0}\left(\int_z^\infty u^q(s)\left(\int_s^\infty K(t,s)v^{-p'}(s)dt\right)^q ds\right)^{\frac{1}{q}}\left(\int_z^\infty v^{-p'}(t)dt\right)^{-\frac{1}{p}} < \infty,$$

$$(B_2^+)^* = \sup_{z>0}\left(\int_z^\infty u^q(s)\left(\int_s^\infty K(t,s)G^{p'-1}(t,s)v^{-p'}(t)dt\right)^q ds\right)^{\frac{1}{q}}\left(\int_z^\infty G^{p'}(t,z)v^{-p'}(t)dt\right)^{-\frac{1}{p}} < \infty,$$

in addition, $C \approx \max\{(B_1^+)^*, (B_2^+)^*\}$, where C is the best constant in (16).

3. Main results for class O_1^-

In this Section we study inequality (16) for integral operator (17) with kernel from the class O_1^-. The validity of inequality (1) follows from the validity of inequality (16) as a corollary. Let us again present the statement proved in [11].

Theorem B. *Let $1 < p \le q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (17) belong to the class O_1^-. Then inequality (16) holds if and only if*

$$(A_1^-)^* = \sup_{z>0} \left(\int_0^z u^q(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty K^{p'}(s,z)v^{-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

$$(A_2^-)^* = \sup_{z>0} \left(\int_0^z Q^q(z,t)u^q(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty \omega^{p'}(s)v^{-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

in addition, $\max\{(A_1^-)^*, (A_2^-)^*\} \approx C$, *where C is the best constant in (16).*

Now, we state and prove alternative characterizations for (16).

Theorem 3.1. *Let $1 < p \le q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (17) belong to the class O_1^-. Then inequality (16) holds if and only if*

$$(B_1^-)^* = \sup_{z>0} \left(\int_0^z v^{-p'}(t) \left(\int_0^t K(t,s)u^q(s)ds \right)^{p'} dt \right)^{\frac{1}{p'}} \left(\int_0^z u^q(t)dt \right)^{-\frac{1}{q'}} < \infty,$$

$$(B_2^-)^* = \sup_{z>0} \left(\int_0^z v^{-p'}(t) \left(\int_0^t K(t,s)Q^{q-1}(t,s)u^q(s)ds \right)^{p'} dt \right)^{\frac{1}{p'}} \times \left(\int_0^z Q^q(z,t)u^q(t)dt \right)^{-\frac{1}{q'}} < \infty,$$

in addition, $C \approx \max\{(B_1^-)^*, (B_2^-)^*\}$, *where C is the best constant in (16).*

For the proof of Theorem 3.1 we need the following lemma. Assume that

$$E^- = \int_0^\infty u^q(s) \left(\int_s^\infty K(x,s)f(x)dx \right)^q ds,$$

$$E_1^- = \int_0^\infty f(x) \int_0^x K(x,s)u^q(s)ds \left(\int_x^\infty K(t,x)f(t)dt \right)^{q-1} dx,$$

$$E_2^- = \int_0^\infty f(x) \int_0^x K(x,s)Q^{q-1}(x,s)u^q(s)ds \left(\int_x^\infty \omega(t)f(t)dt \right)^{q-1} dx.$$

Lemma 3.2. *Let $1 < q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (17) belong to the class O_1^-. Then*

$$E^- = E_1^- + E_2^-. \tag{18}$$

Proof. First we use condition (5), then we change the order of integration and get

$$E^- = \int_0^\infty u^q(s) \left(\int_s^\infty K(x,s)f(x)dx \right)^q ds = q \int_0^\infty u^q(s) \int_s^\infty K(x,s)f(x) \left(\int_x^\infty K(t,s)f(t)dt \right)^{q-1} dxds$$

$$\approx q\int_0^\infty u^q(s)\int_s^\infty K(x,s)f(x)\left(\int_x^\infty K(t,x)f(t)dt\right)^{q-1}dxds$$
$$+\,q\int_0^\infty u^q(s)\int_s^\infty K(x,s)Q^{q-1}(x,s)f(x)\left(\int_s^\infty \omega(t)f(t)dt\right)^{q-1}dxds$$

$$= q\int_0^\infty f(x)\int_0^x K(x,s)u^q(s)ds\left(\int_x^\infty K(t,x)f(t)dt\right)^{q-1}dx$$
$$+\,q\int_0^\infty f(x)\int_0^x K(x,s)Q^{q-1}(x,s)u^q(s)ds\left(\int_x^\infty \omega(t)f(t)dt\right)^{q-1}dx = q(E_1+E_2).$$

□

Proof of Theorem 3.1. Necessity. Let inequality (16) hold. Then the dual inequality

$$\left(\int_0^\infty v^{-p'}(x)\left(\int_0^x K(x,s)g(s)ds\right)^{p'}dx\right)^{\frac{1}{p'}} \le C\left(\int_0^\infty |u^{-1}(t)g(t)|^{q'}dt\right)^{\frac{1}{q'}},\ g\ge 0, \tag{19}$$

also holds. The validity of inequality (16) gives that $u^q \in L_1(0,z)$ for any $z > 0$. Then $u^q \in L_{q',u^{-1}}(0,z)$. Therefore, assuming $g(s) = \chi_{(0,z)}(s)u^q(s)$ in (19), we have

$$\left(\int_0^z v^{-p'}(x)\left(\int_0^x K(x,s)u^q(s)ds\right)^{p'}dx\right)^{\frac{1}{p'}} \le C\left(\int_0^z u^q(t)dt\right)^{\frac{1}{q'}}$$

for any $z > 0$. The latter gives

$$(B_1^-)^* = \sup_{z>0}\left(\int_0^z v^{-p'}(x)\left(\int_0^x K(x,s)u^q(s)ds\right)^{p'}dx\right)^{\frac{1}{p'}}\left(\int_0^z u^q(t)dt\right)^{-\frac{1}{q'}} \le C. \tag{20}$$

From the validity of (16), using (18), we get

$$\int_0^\infty f(x)\int_0^x K(x,s)Q^{q-1}(x,s)u^q(s)ds\left(\int_x^\infty \omega(t)f(t)dt\right)^{q-1}dx \ll C^q\left(\int_0^\infty |v(t)f(t)|^p dt\right)^{\frac{q}{p}},$$

which, due to $K(x,s) \gg \omega(x)Q(x,s)$, yields

$$\omega(x)\int_0^x Q^q(x,s)u^q(s)ds \ll \int_0^x K(x,s)Q^{q-1}(x,s)u^q(s)ds < \infty$$

for almost all $x > 0$. It is obvious that the function $g(s) = \chi_{(0,z)}(s)Q^{q-1}(z,s)u^q(s)$ belongs to $L_{q',u^{-1}}(0,z)$. Therefore, replacing this function $g(s)$ in (19), we have

$$\left(\int_0^z v^{-p'}(x)\left(\int_0^x K(x,s)Q^{q-1}(z,s)u^q(s)ds\right)^{p'}dx\right)^{\frac{1}{p'}} \le C\left(\int_0^z Q^q(z,s)u^q(s)ds\right)^{\frac{1}{q'}}.$$

The latter, due to $Q(z,s) \geq Q(x,s)$ for $z \geq x$, gives

$$(B_2^-)^* = \sup_{z>0} \left(\int_0^z v^{-p'}(x) \left(\int_0^x K(x,s) Q^{q-1}(x,s) u^q(s) ds \right)^{p'} dx \right)^{\frac{1}{p'}} \left(\int_0^z Q^q(z,s) u^q(s) ds \right)^{-\frac{1}{q'}} \ll C. \tag{21}$$

The proof of necessity is complete.

Sufficiency. Let $\max\{(B_1^-)^*, (B_2^-)^*\} < \infty$. Using Hölder's inequality, we have

$$E_2^- = \int_0^\infty f(x) \int_0^x K(x,s) Q^{q-1}(x,s) u^q(s) ds \left(\int_x^\infty \omega(t) f(t) dt \right)^{q-1} dx \leq \|vf\|_p J_2^-, \tag{22}$$

where

$$J_2^- = \left(\int_0^\infty v^{-p'}(x) \left(\int_0^x K(x,s) Q^{q-1}(x,s) u^q(s) ds \right)^{p'} \left(\int_x^\infty \omega(t) f(t) dt \right)^{p'(q-1)} ds \right)^{\frac{1}{p'}}.$$

Let us estimate J_2^-. Integration by parts gives

$$J_2^- = \left(\int_0^\infty \int_0^z v^{-p'}(x) \left(\int_0^x K(x,s) Q^{q-1}(x,s) u^q(s) ds \right)^{p'} ds d \left(- \left(\int_z^\infty \omega(t) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{1}{p'}}.$$

Then, using the finiteness of $(B_2^-)^*$, Minkowski's inequality and integration by parts, we obtain

$$J_2^- \leq (B_2^-)^* \left(\int_0^\infty \left(\int_0^z Q^q(z,s) u^q(s) ds \right)^{\frac{p'}{q'}} d \left(- \left(\int_z^\infty \omega(t) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{1}{p'}}$$

$$= (B_2^-)^* \left[\left(\int_0^\infty \left(\int_0^z d \left(\int_0^x Q^q(x,s) u^q(s) ds \right) \right)^{\frac{p'}{q'}} d \left(- \left(\int_z^\infty \omega(t) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$\leq (B_2^-)^* \left[\int_0^\infty d \left(\int_0^x Q^q(x,s) u^q(s) ds \right) \left(\int_x^\infty d \left(- \left(\int_z^\infty \omega(t) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$= (B_2^-)^* \left(\int_0^\infty \left(\int_x^\infty \omega(t) f(t) dt \right)^q d \left(\int_0^x Q^q(x,s) u^q(s) ds \right) \right)^{\frac{1}{q'}}$$

$$\approx (B_2^-)^* \left(\int_0^\infty f(x) \left(\int_0^x \omega(t) f(t) dt \right)^{q-1} \int_0^x \omega(x) Q^q(x,s) \omega(y) u^q(s) ds dx \right)^{\frac{1}{q'}}.$$

Taking into account that $K(x,s) \gg \omega(x)Q(x,s)$, we have

$$J_2^- \ll (B_2^-)^* \left(\int_0^\infty f(x) \left(\int_0^x \omega(t) f(t) dt \right)^{q-1} \int_0^x K(x,s) Q^{q-1}(x,s) \omega(y) u^q(s) ds dx \right)^{\frac{1}{q'}} = (B_2^-)^* (E_2^-)^{\frac{1}{q'}}. \tag{23}$$

From (18), (22) and (23) it follows that

$$E_2^- \ll (B_2^-)^* (E^-)^{\frac{1}{q'}} \|vf\|_p. \tag{24}$$

Let us estimate E_1^-. Using Hölder's inequality, we have

$$E_1^- = \int_0^\infty f(x) \int_0^x K(x,s) u^q(s) ds \left(\int_x^\infty K(t,x) f(t) dt \right)^{q-1} dx \le \|vf\|_p J_1^-, \tag{25}$$

where

$$J_1^- = \left(\int_0^\infty v^{-p'}(x) \left(\int_0^x K(x,s) u^q(s) ds \right)^{p'} \left(\int_x^\infty K(t,x) f(t) dt \right)^{p'(q-1)} dx \right)^{\frac{1}{p'}}.$$

Then we estimate J_1^- by the similar steps used to estimate J_2^- and get

$$J_1^- = \left(\int_0^\infty \int_0^z v^{-p'}(x) \left(\int_0^x K(x,s) u^q(s) ds \right)^{p'} ds\, d \left(- \left(\int_z^\infty K(t,z) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{1}{p'}}$$

$$\le (B_1^-)^* \left(\int_0^\infty \left(\int_0^z u^q(t) dt \right)^{\frac{p'}{q'}} d \left(- \left(\int_z^\infty K(t,z) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{1}{p'}}$$

$$(B_1^-)^* \left[\left(\int_0^\infty \left(\int_0^z d \left(\int_0^x u^q(t) dt \right) \right)^{\frac{p'}{q'}} d \left(- \left(\int_z^\infty K(t,z) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$\le (B_1^-)^* \left[\int_0^\infty d \left(\int_0^x u^q(t) dt \right) \left(\int_x^\infty d \left(- \left(\int_z^\infty K(t,z) f(t) dt \right)^{p'(q-1)} \right) \right)^{\frac{q'}{p'}} \right]^{\frac{1}{q'}}$$

$$= (B_1^-)^* \left(\int_0^\infty u^q(x) \left(\int_x^\infty K(t,x) f(t) dt \right)^q dx \right)^{\frac{1}{q'}} = (B_1^-)^* (E^-)^{\frac{1}{q'}}. \tag{26}$$

From (25) and (26) it follows that

$$E_1^- \ll (B_1^-)^* (E^-)^{\frac{1}{q'}} \|vf\|_p. \tag{27}$$

From (18), (24) and (27) we have

$$\left(\int_0^\infty u^q(x)\left(\int_x^\infty K(t,x)f(t)dt\right)^q dx\right)^{\frac{1}{q}} \ll \max\{(B_1^-)^*,(B_2^-)^*\}\|vf\|_p,$$

i.e., inequality (16) holds with the estimate $C \ll \max\{(B_1^-)^*,(B_2^-)^*\}$ for the best constant C in (16), which, together with (20) and (21), gives

$$C \approx \max\{(B_1^-)^*,(B_2^-)^*\}.$$

The proof of Theorem 3.1 is complete. □

Theorem 3.1 is proved on the basis of inequality (19) that is dual to inequality (16). Inequality (19) has the form of inequality (1), thus from Theorem 3.1 we get the following theorem.

Theorem 3.3. *Let $1 < p \le q < \infty$ and the kernel $K(\cdot,\cdot)$ of operator (2) belong to the class O_1^-. Then inequality (1) holds if and only if*

$$B_1^- = \sup_{z>0}\left(\int_0^z u^q(s)\left(\int_0^s K(s,x)v^{-p'}(x)dx\right)^q\right)^{\frac{1}{q}}\left(\int_0^z v^{-p'}(t)dt\right)^{-\frac{1}{p}} < \infty,$$

$$B_2^- = \sup_{z>0}\left(\int_0^z u^q(t)\left(\int_0^t K(t,s)Q^{p'-1}(t,s)v^{-p'}(s)dt\right)^q\right)^{\frac{1}{q}}\left(\int_0^z Q^{p'}(z,t)v^{-p'}(t)dt\right)^{-\frac{1}{p}} < \infty,$$

in addition, $C \approx \max\{B_1^-, B_2^-\}$, where C is the best constant in (1).

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