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Symmetries in Yetter-Drinfel'd-Long Categories

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Abstract. Let H be a Hopf algebra and $\mathcal{LR}(H)$ the category of Yetter-Drinfel'd-Long bimodules over H. We first give sufficient and necessary conditions for $\mathcal{LR}(H)$ to be symmetry and pseudosymmetry, respectively. We then introduce the definition of the u-condition in $\mathcal{LR}(H)$ and discuss the relation between the u-condition and the symmetry of $\mathcal{LR}(H)$. Finally, we show that $\mathcal{LR}(H)$ over a triangular (cotriangular, resp.) Hopf algebra contains a rich symmetric subcategory.

1. Introduction

The notion of symmetric category is a classical concept in category theory. Cohen and Westreich [1] tested symmetries and the u-condition in the Yetter-Drinfel'd category ${}^H_H\mathcal{YD}$ over Hopf algebra H. Pareigis [7] found the necessary and sufficient condition for ${}^H_H\mathcal{YD}$ to be symmetric. Later, Panaite et al. [8] proposed the definition of pseudosymmetric braided categories which can be viewed as a kind of weakened symmetric braided categories, and showed that the category ${}_H\mathcal{YD}^H$ is pseudosymmetric if and only if H is commutative and cocommutative. The generalization of those classical structures and results have been introduced and discussed by many authors [5, 12, 13].

It is known that the Radford biproduct has a categorical interpretation (due to Majid): (H, A) is an admissible pair (see [11]) if and only if A is a bialgebra in the Yetter-Drinfel'd category ${}^H_H\mathcal{YD}$. Panaite and Van Oystaeyen [9] described a similar interpretation for L-R-admissible pairs and defined a prebraided category $\mathcal{LR}(H)$ (which is braided if H has a bijective antipode) which contains ${}^H_H\mathcal{YD}$ and $\mathcal{YD}^H_H$ as braided subcategories. They then showed that (H, B) is an L-R-admissible pair with an extra condition

$$b_{(0)} \triangleleft b'_{[-1]} \otimes b_{(1)} \triangleright b'_{[0]} = b \otimes b', \qquad for\ any\ b, b' \in B$$

is equivalent to B is a bialgebra in $\mathcal{LR}(H)$, where the L-R-admissible pair is the sufficient condition for L-R smash biproduct $B \bowtie H$ to be a bialgebra. The Radford biproduct is a particular case. Lu and Zhang in [4] discussed the equivalence on Hom-Hopf algebra.

The aim of the present paper is to discuss the symmetries, the pseudosymmetries and the u-condition in Yetter-Drinfel'd-Long categories.

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This paper is organized as follows: In section 2, we recall some basic definitions and results related to Yetter-Drinfel'd-Long bimodules. Then we give some examples of Yetter-Drinfel'd-Long bimodules. In section 3, we show that the Yetter-Drinfel'd-Long category $\mathcal{LR}(H)$ is symmetric if and only if H is trivial in four different methods, and that $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative. In section 4, we introduce the definition of the u-condition in $\mathcal{LR}(H)$ and give a necessary and sufficient condition for H_i $(i = 1, 2, 3, 4)$ to satisfy the u-condition, where H_i is defined in Example 2.4. Then we study the relation between the u-condition and the symmetry of $\mathcal{LR}(H)$. In section 5, we prove that the subcategory ${}_H\mathcal{M}_H$ of $\mathcal{LR}(H)$ over triangular Hopf algebra H is symmetric. If we consider $M = H\otimes H$, we prove the converse. That is, assume that the braiding $\psi_{H\otimes H,H\otimes H}$ is symmetric forces H to be triangular. In section 6, we give the dual cases of section 5. he total integral introduced by Chen and Wang in T-coalgebras setting.

2. Preliminaries

Throughout this paper, all algebraic systems are over a field $\Bbbk$. For a coalgebra C, the comultiplication will be denoted by Δ. We follow the Sweedler's notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$, in which we often omit the summation symbols for convenience. For any vector spaces M and N, we use $\tau : M \otimes N \to N \otimes M$ for the flip map.

Let A be a algebra, A *right A-module* is a pair $(M, \triangleleft)$, in which M is a vector space and $\triangleleft : M \otimes A \to M$ is a linear map, called the action of A on M, with notation $\triangleleft(m\otimes a) = m \triangleleft a$, such that, for any $a, b \in A$ and $m \in M$:

$$\begin{cases} m \triangleleft ab = (m \triangleleft a) \triangleleft b, \\ m \triangleleft 1 = m. \end{cases}$$

Similarly, we can define the left A-module. A *right A-linear* is a linear map $f : M \to N$ such that $f(m) \triangleleft a = f(m \triangleleft a)$, for any $a \in A$ and $m \in M$.

Let C be a coalgebra, A *right C-comodule* is a pair (M, ρ), in which M is a vector space and $\rho : M \to M \otimes C$ is a linear map, called the coaction of C on M, with notation $\rho(m) = m_{(0)} \otimes m_{(1)}$, such that, for any $m \in M$:

$$\begin{cases} m_{(0)(0)} \otimes m_{(0)(1)} \otimes m_{(1)} = m_{(0)} \otimes m_{(1)1} \otimes m_{(1)2}, \\ m_{(0)}\varepsilon(m_{(1)}) = m. \end{cases}$$

Similarly, we can define the left C-comodule. A *right C-colinear* is a linear map $f : M \to N$ such that $\rho_N \circ f = (f \otimes id) \circ \rho_M$.

Let A be a algebra, and assume that M are both left A-module via $\triangleright : A \otimes M \to M, a \otimes m \mapsto a \triangleright m$ and right A-module via $\triangleleft : M \otimes A \to M, m \otimes b \mapsto m \triangleleft b$, then M is called an *A-bimodule* if

$$(a \triangleright m) \triangleleft b = a \triangleright (m \triangleleft b), \tag{2.1}$$

for any $a, b \in A$ and $m \in M$.

Let C be a coalgebra, and assume that M are both left C-comodule via $\rho^l : M \to C \otimes M, m \mapsto m_{[-1]} \otimes m_{[0]}$ and right C-comodule via $\rho^r : M \to M \otimes C, m \mapsto m_{(0)} \otimes m_{(1)}$, then M is called a *C-bicomodule* if

$$m_{[-1]} \otimes m_{[0](0)} \otimes m_{[0](1)} = m_{(0)[-1]} \otimes m_{(0)[0]} \otimes m_{(1)}, \tag{2.2}$$

for any $m \in M$.

Let H be a Hopf algebra, we can denote those categories by ${}_H\mathcal{M}_H$ and ${}^H\mathcal{M}^H$. Take ${}_H\mathcal{M}_H$ whose objects are all H-bimodules, the morphisms in the category are morphisms of H-bilinear.

Definition 2.1. *([9]) Let H be a Hopf algebra. A Yetter-Drinfel'd-Long bimodule over H is a vector space M endowed with H-bimodule and H-bicomodule structures (denoted by $h \otimes m \mapsto h \triangleright m, m \otimes h \mapsto m \triangleleft h, m \mapsto m_{[-1]} \otimes m_{[0]}, m \mapsto$*

$m_{(0)} \otimes m_{(1)}$, for any $h \in H$ and $m \in M$), such that M is a left-left Yetter-Drinfel'd module, a left-right Long module, a right-right Yetter-Drinfel'd module and a right-left Long module, i.e.

$$(h_1 \triangleright m)_{[-1]}h_2 \otimes (h_1 \triangleright m)_{[0]} = h_1 m_{[-1]} \otimes h_2 \triangleright m_{[0]}, \tag{2.3}$$
$$(h \triangleright m)_{(0)} \otimes (h \triangleright m)_{(1)} = h \triangleright m_{(0)} \otimes m_{(1)}, \tag{2.4}$$
$$(m \triangleleft h_2)_{(0)} \otimes h_1(m \triangleleft h_2)_{(1)} = m_{(0)} \triangleleft h_1 \otimes m_{(1)}h_2, \tag{2.5}$$
$$(m \triangleleft h)_{[-1]} \otimes (m \triangleleft h)_{[0]} = m_{[-1]} \otimes m_{[0]} \triangleleft h. \tag{2.6}$$

We denote by $\mathcal{LR}(H)$ the category whose objects are all Yetter-Drinfel'd-Long bimodules M over H, the morphisms in the category are morphisms of H-bilinear and H-bicolinear.

If H has a bijective antipode S, $\mathcal{LR}(H)$ becomes a strict braided monoidal category with the following structures: for any $M, N \in \mathcal{LR}(H)$, and $h \in H$, $m \in M$ and $n \in N$,

$$h \triangleright (m \otimes n) = h_1 \triangleright m \otimes h_2 \triangleright n, \tag{2.7}$$
$$(m \otimes n) \triangleleft h = m \triangleleft h_1 \otimes n \triangleleft h_2, \tag{2.8}$$
$$(m \otimes n)_{[-1]} \otimes (m \otimes n)_{[0]} = m_{[-1]}n_{[-1]} \otimes m_{[0]} \otimes n_{[0]}, \tag{2.9}$$
$$(m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}, \tag{2.10}$$

the braiding
$$\psi_{M,N} : M \otimes N \to N \otimes M : m \otimes n \mapsto m_{[-1]} \triangleright n_{(0)} \otimes m_{[0]} \triangleleft n_{(1)}$$
and the inverse
$$\psi^{-1}_{N,M} : N \otimes M \to M \otimes N : n \otimes m \mapsto m_{[0]} \triangleleft S^{-1}(n_{(1)}) \otimes S^{-1}(m_{[-1]}) \triangleright n_{(0)}.$$

Definition 2.2. *([6]) A quasitriangular (QT) Hopf algebra is a pair (H, R), where H is a Hopf algebra over $\Bbbk$ and $R = R^1 \otimes R^2 \in H \otimes H$ is invertible, such that the following conditions hold ($r = R$):*

(QT1) $\Delta(R^1) \otimes R^2 = R^1 \otimes r^1 \otimes R^2 r^2$;
(QT2) $R^1 \otimes \Delta(R^2) = R^1 r^1 \otimes r^2 \otimes R^2$;
(QT3) $\Delta^{cop}(h)R = R\Delta(h)$;
(QT4) $\varepsilon(R^1)R^2 = 1 = R^1\varepsilon(R^2)$;
(QT5) *If $R^{-1} = R^2 \otimes R^1$, then (H, R) is called a triangular Hopf algebra.*

Definition 2.3. *([6]) A coquasitriangular (CQT) Hopf algebra is a pair (H, ζ), where H is a Hopf algebra over $\Bbbk$ and $\zeta : H \otimes H \to \Bbbk$ is a $\Bbbk$-bilinear form (braiding) which is convolution invertible in $Hom_{\Bbbk}(H \otimes H, \Bbbk)$ such that the following conditions hold:*

(CQT1) $\zeta(h, gl) = \zeta(h_1, g)\zeta(h_2, l)$;
(CQT2) $\zeta(hg, l) = \zeta(h, l_2)\zeta(g, l_1)$;
(CQT3) $\zeta(h_1, g_1)g_2h_2 = h_1g_1\zeta(h_2, g_2)$;
(CQT4) $\zeta(h, 1) = \varepsilon(h) = \zeta(1, h)$;
(CQT5) *If $\zeta(h_1, g_1)\zeta(g_2, h_2) = \varepsilon(g)\varepsilon(h)$, then (H, ζ) is called a cotriangular Hopf algebra.*

The following are some examples of objects in $\mathcal{LR}(H)$.

Example 2.4. *Let H be a Hopf algebra. Then*

(1) *$H_1 = H \otimes H$ is a Yetter-Drinfel'd-Long bimodule with the following structures, for any $h, k, l \in H$:*

$$h \triangleright (k \otimes l) = hk \otimes l, \qquad \rho^l(k \otimes l) = (k \otimes l)_{[-1]} \otimes (k \otimes l)_{[0]} = k_1S(k_3) \otimes (k_2 \otimes l),$$
$$(k \otimes l) \triangleleft h = k \otimes S(h_1)lh_2, \qquad \rho^r(k \otimes l) = (k \otimes l)_{(0)} \otimes (k \otimes l)_{(1)} = (k \otimes l_1) \otimes l_2.$$

(2) *$H_2 = H \otimes H$ is a Yetter-Drinfel'd-Long bimodule with the following structures, for any $h, k, l \in H$:*

$$h \triangleright (k \otimes l) = h_1kS(h_2) \otimes l, \qquad \rho^l(k \otimes l) = (k \otimes l)_{[-1]} \otimes (k \otimes l)_{[0]} = k_1 \otimes (k_2 \otimes l),$$

$(k \otimes l) \triangleleft h = k \otimes lh,$ $\rho^r(k \otimes l) = (k \otimes l)_{(0)} \otimes (k \otimes l)_{(1)} = (k \otimes l_2) \otimes S(l_1)l_3.$

(3) $H_3 = H \otimes H$ *is a Yetter-Drinfel'd-Long bimodule with the following structures, for any* $h, k, l \in H$:

$$h \triangleright (k \otimes l) = hk \otimes l, \qquad \rho^l(k \otimes l) = (k \otimes l)_{[-1]} \otimes (k \otimes l)_{[0]} = k_1 S(k_3) \otimes (k_2 \otimes l),$$
$$(k \otimes l) \triangleleft h = k \otimes lh, \qquad \rho^r(k \otimes l) = (k \otimes l)_{(0)} \otimes (k \otimes l)_{(1)} = (k \otimes l_2) \otimes S(l_1)l_3.$$

(4) $H_4 = H \otimes H$ *is a Yetter-Drinfel'd-Long bimodule with the following structures, for any* $h, k, l \in H$:

$$h \triangleright (k \otimes l) = h_1 k S(h_2) \otimes l, \qquad \rho^l(k \otimes l) = (k \otimes l)_{[-1]} \otimes (k \otimes l)_{[0]} = k_1 \otimes (k_2 \otimes l),$$
$$(k \otimes l) \triangleleft h = k \otimes S(h_1) l h_2, \qquad \rho^r(k \otimes l) = (k \otimes l)_{(0)} \otimes (k \otimes l)_{(1)} = (k \otimes l_1) \otimes l_2.$$

Note that $H \otimes H$ *is also a Hopf algebra with usual tensor product and usual tensor coproduct.*

3. Symmetric Yetter-Drinfel'd-Long categories

In this section, we give necessary and sufficient conditions for Yetter-Drinfel'd-Long category $\mathcal{LR}(H)$ to be symmetric and pseudosymmetric, respectively.

Let C be a monoidal category and ψ a braiding on C. The braiding ψ is called a symmetry if $\psi_{W,V} \circ \psi_{V,W} = id_{V \otimes W}$ for any $V, W \in C$. In this case, C is called a symmetric braided category (see [2]). The braiding ψ is called a pseudosymmetry if the following condition holds, for any $U, V, W \in C$:

$$(id_W \otimes \psi_{U,V})(\psi^{-1}_{W,U} \otimes id_V)(id_U \otimes \psi_{V,W}) = (\psi_{V,W} \otimes id_U)(id_V \otimes \psi^{-1}_{W,U})(\psi_{U,V} \otimes id_W).$$

In this case, C is called a pseudosymmetric braided category (see [8]).

Note that if ψ is a symmetry, that is, $\psi^{-1}_{W,V} = \psi_{V,W}$, then obviously ψ is a pseudosymmetry.

Theorem 3.1. *Let H be a Hopf algebra such that the canonical braiding of the Yetter-Drinfel'd-Long category $\mathcal{LR}(H)$ is a symmetry if and only if $H = \Bbbk$.*

Proof. By Example 2.4, H_1 and H_2 are two Yetter-Drinfel'd-Long bimodules. If the canonical braiding ψ is a symmetry, that is, $\psi_{H_2,H_1} \circ \psi_{H_1,H_2} = id_{H_1 \otimes H_2}$. Apply $\psi_{H_2,H_1} \circ \psi_{H_1,H_2}$ to the element $1 \otimes k \otimes 1 \otimes 1 \in H_1 \otimes H_2$, we have

$$\begin{aligned}
\psi_{H_2,H_1} \circ \psi_{H_1,H_2}(1 \otimes k \otimes 1 \otimes 1) &= \psi_{H_2,H_1}((1 \otimes k)_{[-1]} \triangleright (1 \otimes 1)_{(0)} \otimes (1 \otimes k)_{[0]} \triangleleft (1 \otimes 1)_{(1)}) \\
&= \psi_{H_2,H_1}(1 \triangleright (1 \otimes 1) \otimes (1 \otimes k) \triangleleft 1) \\
&= \psi_{H_2,H_1}(1 \otimes 1 \otimes 1 \otimes k) \\
&= (1 \otimes 1)_{[-1]} \triangleright (1 \otimes k)_{(0)} \otimes (1 \otimes 1)_{[0]} \triangleleft (1 \otimes k)_{(1)} \\
&= 1 \triangleright (1 \otimes k_1) \otimes (1 \otimes 1) \triangleleft k_2 \\
&= 1 \otimes k_1 \otimes 1 \otimes k_2.
\end{aligned}$$

Thus we have $1 \otimes k \otimes 1 \otimes 1 = 1 \otimes k_1 \otimes 1 \otimes k_2$. Apply $\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes id$ to both sides of the equation, we have $\varepsilon(k)1_H = k$. So $H = \Bbbk$.

The converse is straightforward, This completes the proof. □

Here, we will give three other proofs of Theorem 3.1, and they are different from each other.

- By Example 2.4, H_1 and H_3 are two Yetter-Drinfel'd-Long bimodules. If canonical braiding is a symmetry, that is, $\psi_{H_3,H_1} \circ \psi_{H_1,H_3} = id_{H_1 \otimes H_3}$. For any $1 \otimes k \otimes 1 \otimes 1 \in H_1 \otimes H_3$, we easily get that $\psi_{H_3,H_1} \circ \psi_{H_1,H_3}(1 \otimes k \otimes 1 \otimes 1) = 1 \otimes k_1 \otimes 1 \otimes k_2$.
  Thus we have $1 \otimes k \otimes 1 \otimes 1 = 1 \otimes k_1 \otimes 1 \otimes k_2$. Apply $\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes id$ to both sides of the equation, we have $\varepsilon(k)1_H = k$. So $H = \Bbbk$.

- By Example 2.4, H_2 and H_4 are two Yetter-Drinfel'd-Long bimodules. If canonical braiding is a symmetry, that is, $\psi_{H_2,H_4} \circ \psi_{H_4,H_2} = id_{H_4 \otimes H_2}$. For any $1 \otimes k \otimes 1 \otimes 1 \in H_4 \otimes H_2$, we easily get that $\psi_{H_2,H_4} \circ \psi_{H_4,H_2}(1 \otimes k \otimes 1 \otimes 1) = 1 \otimes k_1 \otimes 1 \otimes k_2$.
Thus we have $1 \otimes k \otimes 1 \otimes 1 = 1 \otimes k_1 \otimes 1 \otimes k_2$. Apply $\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes id$ to both sides of the equation, we have $\varepsilon(k)1_H = k$. So $H = \Bbbk$.
- By Example 2.4, H_3 and H_4 are two Yetter-Drinfel'd-Long bimodules. If canonical braiding is a symmetry, that is, $\psi_{H_3,H_4} \circ \psi_{H_4,H_3} = id_{H_4 \otimes H_3}$. For any $1 \otimes k \otimes 1 \otimes 1 \in H_4 \otimes H_3$, we easily get that $\psi_{H_3,H_4} \circ \psi_{H_4,H_3}(1 \otimes k \otimes 1 \otimes 1) = 1 \otimes k_1 \otimes 1 \otimes k_2$.
Thus we have $1 \otimes k \otimes 1 \otimes 1 = 1 \otimes k_1 \otimes 1 \otimes k_2$. Apply $\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes id$ to both sides of the equation, we have $\varepsilon(k)1_H = k$. So $H = \Bbbk$.

If $H_1 = \Bbbk \otimes H$ and $H_2 = \Bbbk \otimes H$, then H_1 and H_2 are two right-right Yetter-Drinfel'd modules. Hence using Theorem 3.1, we can improve the main result in [7].

Corollary 3.2. *Let H be a Hopf algebra such that the canonical braiding of right-right Yetter-Drinfel'd category $\mathcal{YD}_H^H$ is a symmetry. Then $H = \Bbbk$.*

In the following, we will introduce the pseudosymmetry on $\mathcal{LR}(H)$ over a Hopf algebra H. For this purpose, we need the following Lemma.

Lemma 3.3. *Let H be a cocommutative Hopf algebra. Then the canonical braiding ψ_{H_1,H_2} of the category $\mathcal{LR}(H)$ is the usual flip map.*

Proof. For any $g \otimes h \otimes k \otimes l \in H_1 \otimes H_2$, we have

$$\begin{aligned}
\psi_{H_1,H_2}(g \otimes h \otimes k \otimes l) &= (g \otimes h)_{[-1]} \triangleright (k \otimes l)_{(0)} \otimes (g \otimes h)_{[0]} \triangleleft (k \otimes l)_{(1)} \\
&= g_1 S(g_3) \triangleright (k \otimes l_2) \otimes (g_2 \otimes h) \triangleleft l_1 S(l_3) \\
&= g_1 S(g_2) \triangleright (k \otimes l_3) \otimes (g_3 \otimes h) \triangleleft l_1 S(l_2) \quad \textit{by cocommutative} \\
&= 1 \triangleright (k \otimes l) \otimes (g \otimes h) \triangleleft 1 \\
&= k \otimes l \otimes g \otimes h.
\end{aligned}$$

This completes the proof. □

We now give necessary and sufficient conditions for the canonical braiding of the category $\mathcal{LR}(H)$ to be a pseudosymmetry, we prove the necessary condition by a new method which is different from Proposition 2.5 in [10].

Theorem 3.4. *Let H be a Hopf algebra. Then the canonical braiding of the category $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative.*

Proof. Assume that the canonical braiding ψ of the category $\mathcal{LR}(H)$ is pseudosymmetric. We first check that H is cocommutative. For any $1 \otimes 1 \otimes k \otimes 1 \otimes 1 \otimes 1 \in H_1 \otimes H_2 \otimes H_1$, we have

$$\begin{aligned}
&(id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_1,H_1} \otimes id) \circ (id \otimes \psi_{H_2,H_1})(1 \otimes 1 \otimes k \otimes 1 \otimes 1 \otimes 1) \\
&= (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_1,H_1} \otimes id)(1 \otimes 1 \otimes (k \otimes 1)_{[-1]} \triangleright (1 \otimes 1)_{(0)} \otimes (k \otimes 1)_{[0]} \triangleleft (1 \otimes 1)_{(1)}) \\
&= (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_1,H_1} \otimes id)(1 \otimes 1 \otimes k_1 \triangleright (1 \otimes 1) \otimes (k_2 \otimes 1) \triangleleft 1) \\
&= (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_1,H_1} \otimes id)(1 \otimes 1 \otimes k_1 \otimes 1 \otimes k_2 \otimes 1) \\
&= (id \otimes \psi_{H_1,H_2})((k_1 \otimes 1)_{[0]} \triangleleft S^{-1}((1 \otimes 1)_{(1)}) \otimes S^{-1}((k_1 \otimes 1)_{[-1]}) \triangleright (1 \otimes 1)_{(0)} \otimes k_2 \otimes 1) \\
&= (id \otimes \psi_{H_1,H_2})((k_2 \otimes 1) \triangleleft 1 \otimes S^{-1}(k_1 S(k_3)) \triangleright (1 \otimes 1) \otimes k_4 \otimes 1) \\
&= (id \otimes \psi_{H_1,H_2})(k_2 \otimes 1 \otimes k_3 S^{-1}(k_1) \otimes 1 \otimes k_4 \otimes 1) \\
&= k_2 \otimes 1 \otimes (k_3 S^{-1}(k_1) \otimes 1)_{[-1]} \triangleright (k_4 \otimes 1)_{(0)} \otimes (k_3 S^{-1}(k_1) \otimes 1)_{[0]} \triangleleft (k_4 \otimes 1)_{(1)}
\end{aligned}$$

$$
\begin{aligned}
&= k_2 \otimes 1 \otimes (k_3 S^{-1}(k_1))_1 S((k_3 S^{-1}(k_1))_3) \triangleright (k_4 \otimes 1) \otimes ((k_3 S^{-1}(k_1))_2 \otimes 1) \triangleleft 1 \\
&= k_2 \otimes 1 \otimes [(k_3 S^{-1}(k_1))_1 S((k_3 S^{-1}(k_1))_3)]_1 k_4 S([(k_3 S^{-1}(k_1))_1 S((k_3 S^{-1}(k_1))_3)]_2) \otimes 1 \\
&\quad \otimes (k_3 S^{-1}(k_1))_2 \otimes 1
\end{aligned}
$$

and

$$
\begin{aligned}
&(\psi_{H_2,H_1} \otimes id) \circ (id \otimes \psi^{-1}_{H_1,H_1}) \circ (\psi_{H_1,H_2} \otimes id)(1 \otimes 1 \otimes k \otimes 1 \otimes 1 \otimes 1) \\
&\quad = (\psi_{H_2,H_1} \otimes id) \circ (id \otimes \psi^{-1}_{H_1,H_1}) \\
&\qquad ((1 \otimes 1)_{[-1]} \triangleright (k \otimes 1)_{(0)} \otimes (1 \otimes 1)_{[0]} \triangleleft (k \otimes 1)_{(1)} \otimes 1 \otimes 1) \\
&\quad = (\psi_{H_2,H_1} \otimes id) \circ (id \otimes \psi^{-1}_{H_1,H_1})(1 \triangleright (k \otimes 1) \otimes (1 \otimes 1) \triangleleft 1 \otimes 1 \otimes 1) \\
&\quad = (\psi_{H_2,H_1} \otimes id) \circ (id \otimes \psi^{-1}_{H_1,H_1})(k \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1) \\
&\quad = (\psi_{H_2,H_1} \otimes id)(k \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1) \\
&\quad = (k \otimes 1)_{[-1]} \triangleright (1 \otimes 1)_{(0)} \otimes (k \otimes 1)_{[0]} \triangleleft (1 \otimes 1)_{(1)} \otimes 1 \otimes 1 \\
&\quad = k_1 \triangleright (1 \otimes 1) \otimes (k_2 \otimes 1) \triangleleft 1 \otimes 1 \otimes 1 \\
&\quad = k_1 \otimes 1 \otimes k_2 \otimes 1 \otimes 1 \otimes 1.
\end{aligned}
$$

By assumption, $\mathcal{LR}(H)$ is pseudosymmetric, it follows that

$$
\begin{aligned}
k_1 \otimes 1 \otimes k_2 \otimes 1 \otimes 1 \otimes 1 = k_2 \otimes 1 \otimes [(k_3 S^{-1}(k_1))_1 S((k_3 S^{-1}(k_1))_3)]_1 k_4 \\
\times S([(k_3 S^{-1}(k_1))_1 S((k_3 S^{-1}(k_1))_3)]_2) \otimes 1 \otimes (k_3 S^{-1}(k_1))_2 \otimes 1
\end{aligned}
$$

Apply $id \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes id \otimes \varepsilon$ to both sides of the above equation, we get $k_2 \otimes k_3 S^{-1}(k_1) = k \otimes 1$. Therefore, we have

$$k_2 \otimes k_1 = k_2 \otimes 1 k_1 = k_3 \otimes k_4 S^{-1}(k_2) k_1 = k_1 \otimes k_2.$$

So H is cocommutative.

Next, we verify that H is commutative. For any $1 \otimes 1 \otimes k \otimes 1 \otimes g \otimes 1 \in H_1 \otimes H_2 \otimes H_2$, we have

$$
\begin{aligned}
&(id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_2,H_1} \otimes id) \circ (id \otimes \psi_{H_2,H_2})(1 \otimes 1 \otimes k \otimes 1 \otimes g \otimes 1) \\
&\quad = (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_2,H_1} \otimes id)(1 \otimes 1 \otimes (k \otimes 1)_{[-1]} \triangleright (g \otimes 1)_{(0)} \otimes (k \otimes 1)_{[0]} \triangleleft (g \otimes 1)_{(1)}) \\
&\quad = (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_2,H_1} \otimes id)(1 \otimes 1 \otimes k_1 \triangleright (g \otimes 1) \otimes (k_2 \otimes 1) \triangleleft 1) \\
&\quad = (id \otimes \psi_{H_1,H_2}) \circ (\psi^{-1}_{H_2,H_1} \otimes id)(1 \otimes 1 \otimes k_1 g S(k_2) \otimes 1 \otimes k_3 \otimes 1) \\
&\quad = (id \otimes \psi_{H_1,H_2})((k_1 g S(k_2) \otimes 1)_{[0]} \triangleleft S^{-1}((1 \otimes 1)_{(1)}) \\
&\qquad \otimes S^{-1}((k_1 g S(k_2) \otimes 1)_{[-1]}) \triangleright (1 \otimes 1)_{(0)} \otimes k_3 \otimes 1) \\
&\quad = (id \otimes \psi_{H_1,H_2})((k_2 g_2 S(k_3) \otimes 1) \triangleleft 1 \otimes S^{-1}(k_1 g_1 S(k_4)) \triangleright (1 \otimes 1) \otimes k_5 \otimes 1) \\
&\quad = (id \otimes \psi_{H_1,H_2})(k_2 g_2 S(k_3) \otimes 1 \otimes S^{-1}(k_1 g_1 S(k_4)) \otimes 1 \otimes k_5 \otimes 1) \\
&\quad = k_2 g_2 S(k_3) \otimes 1 \otimes k_5 \otimes 1 \otimes S^{-1}(k_1 g_1 S(k_4)) \otimes 1 \quad \textit{by Lemma } 3.3
\end{aligned}
$$

and

$$
\begin{aligned}
&(\psi_{H_2,H_2} \otimes id) \circ (id \otimes \psi^{-1}_{H_2,H_1}) \circ (\psi_{H_1,H_2} \otimes id)(1 \otimes 1 \otimes k \otimes 1 \otimes g \otimes 1) \\
&\quad = (\psi_{H_2,H_2} \otimes id) \circ (id \otimes \psi^{-1}_{H_2,H_1})(k \otimes 1 \otimes 1 \otimes 1 \otimes g \otimes 1) \quad \textit{by Lemma } 3.3 \\
&\quad = (\psi_{H_2,H_2} \otimes id)(k \otimes 1 \otimes (g \otimes 1)_{[0]} \triangleleft S^{-1}((1 \otimes 1)_{(1)}) \otimes S^{-1}((g \otimes 1)_{[-1]}) \triangleright (1 \otimes 1)_{(0)}) \\
&\quad = (\psi_{H_2,H_2} \otimes id)(k \otimes 1 \otimes (g_2 \otimes 1) \triangleleft 1 \otimes S^{-1}(g_1) \triangleright (1 \otimes 1)) \\
&\quad = (\psi_{H_2,H_2} \otimes id)(k \otimes 1 \otimes g_2 \otimes 1 \otimes S^{-1}(g_1) \otimes 1)
\end{aligned}
$$

$$= (k \otimes 1)_{[-1]} \triangleright (g_2 \otimes 1)_{(0)} \otimes (k \otimes 1)_{[0]} \triangleleft (g_2 \otimes 1)_{(1)} \otimes S^{-1}(g_1) \otimes 1$$
$$= k_1 \triangleright (g_2 \otimes 1) \otimes (k_2 \otimes 1) \triangleleft 1 \otimes S^{-1}(g_1) \otimes 1$$
$$= k_1 g_2 S(k_2) \otimes 1 \otimes k_3 \otimes 1 \otimes S^{-1}(g_1) \otimes 1.$$

Since $\mathcal{LR}(H)$ is pseudosymmetric, we get

$$k_2 g_2 S(k_3) \otimes 1 \otimes k_5 \otimes 1 \otimes S^{-1}(k_1 g_1 S(k_4)) \otimes 1 = k_1 g_2 S(k_2) \otimes 1 \otimes k_3 \otimes 1 \otimes S^{-1}(g_1) \otimes 1.$$

Apply $(\varepsilon \otimes \varepsilon \otimes id \otimes \varepsilon \otimes id \otimes \varepsilon)(id \otimes id \otimes id \otimes id \otimes S \otimes id)$ to both sides of the above equation, we get $k_3 \otimes k_1 g S(k_2) = k \otimes g$. Hence, we have

$$gk = k_1 g S(k_2) k_3 = k_1 g \varepsilon(k_2) = kg.$$

So H is commutative.

The proof of the converse can refer to Proposition 2.5 in [10]. This completes the proof. □

If we consider $H_1 = H \otimes \Bbbk$ and $H_2 = H \otimes \Bbbk$, then H_1 and H_2 are two left-left Yetter-Drinfel'd modules. By the proof of Theorem 3.4, we have the following result:

Corollary 3.5. *The canonical braiding of ${}^H_H\mathcal{YD}$ is pseudosymmetric if and only if H is cocommutative and commutative.*

4. The u-condition in $\mathcal{LR}(H)$

In this section, we introduce the definition of the u-condition in $\mathcal{LR}(H)$ over Hopf algebra H and discuss some properties and results related to the u-condition. It is easy to obtain the u-condition in ${}^H_H\mathcal{YD}$ when the right action and coaction are trivial.

Definition 4.1. *Let H be a Hopf algebra and $M \in \mathcal{LR}(H)$. Then M is said to satisfy the u-condition if*

$$m_{[-1]} \triangleright m_{[0](0)} \triangleleft m_{[0](1)} = m, \tag{4.1}$$

for any $m \in M$.

Note that Eq.(4.1) is equivalent to the following equation:

$$m_{(0)[-1]} \triangleright m_{(0)[0]} \triangleleft m_{(1)} = m, \tag{4.2}$$

for any $m \in M$.

In the following, we will give a necessary and sufficient condition for H_1, H_2, H_3 and H_4 in Example 2.4 to satisfy the u-condition.

Proposition 4.2. *Let H be a Hopf algebra. Then*

(1) *H_1 satisfies the u-condition if and only if $S^2 = id$.*
(2) *H_2 satisfies the u-condition if and only if $S^2 = id$.*
(3) *H_3 satisfies the u-condition if and only if $S^2 = id$.*
(4) *H_4 satisfies the u-condition if and only if $S^2 = id$.*

Proof. It is basic in [3] that $S^2 = id$ if and only if $S(h_2)h_1 = \varepsilon(h)$ or $h_2 S(h_1) = \varepsilon(h)$.

For (1), if $S^2 = id$, we only need to check that Eq.(4.1) holds. For any $k, l \in H$, we have

$$\begin{aligned}(k \otimes l)_{[-1]} \triangleright (k \otimes l)_{[0](0)} \triangleleft (k \otimes l)_{[0](1)} &= k_1 S(k_3) \triangleright (k_2 \otimes l)_{(0)} \triangleleft (k_2 \otimes l)_{(1)} \\ &= k_1 S(k_3) \triangleright (k_2 \otimes l_1) \triangleleft l_2 \\ &= k_1 S(k_3) k_2 \otimes S(l_2) l_1 l_3 \\ &= k_1 \varepsilon(k_2) \otimes \varepsilon(l_1) l_2\end{aligned}$$

$$= k \otimes l.$$

Conversely, assume that H_1 satisfies the u-condition. For any $k \otimes 1 \in H_1$, we have

$$\begin{aligned}(k\otimes 1)_{[-1]} \triangleright (k\otimes 1)_{[0](0)} \triangleleft (k\otimes 1)_{[0](1)} &= k_1S(k_3) \triangleright (k_2\otimes 1)_{(0)} \triangleleft (k_2\otimes 1)_{(1)}\\ &= k_1S(k_3) \triangleright (k_2\otimes 1) \triangleleft 1\\ &= k_1S(k_3)k_2\otimes 1.\end{aligned}$$

By assumption, we have $k_1S(k_3)k_2 \otimes 1 = k \otimes 1$. Apply $id \otimes \varepsilon$ to both sides, we get

$$k_1S(k_3)k_2 = k. \tag{4.3}$$

By computing we have

$$\begin{aligned}S(k_2)k_1 &= \varepsilon(k_1)S(k_3)k_2\\ &= (S(k_1)k_2)S(k_4)k_3\\ &= S(k_1)(k_2S(k_4)k_3)\\ &= S(k_1)k_2 \quad \textit{by (4.3) applied to } k_2\\ &= \varepsilon(k).\end{aligned}$$

Hence $S^2 = id$.

For (2), if $S^2 = id$, for any $k, l \in H$, we have

$$\begin{aligned}(k\otimes l)_{[-1]} \triangleright (k\otimes l)_{[0](0)} \triangleleft (k\otimes l)_{[0](1)} &= k_1 \triangleright (k_2\otimes l)_{(0)} \triangleleft (k_2\otimes l)_{(1)}\\ &= k_1 \triangleright (k_2\otimes l_2) \triangleleft S(l_1)l_3\\ &= k_1k_3S(k_2)\otimes l_2S(l_1)l_3\\ &= k_1\varepsilon(k_2)\otimes \varepsilon(l_1)l_2\\ &= k\otimes l.\end{aligned}$$

Conversely, assume that H_2 satisfies the u-condition. For any $k \otimes 1 \in H_2$, we have

$$\begin{aligned}(k\otimes 1)_{[-1]} \triangleright (k\otimes 1)_{[0](0)} \triangleleft (k\otimes 1)_{[0](1)} &= k_1 \triangleright (k_2\otimes 1)_{(0)} \triangleleft (k_2\otimes 1)_{(1)}\\ &= k_1 \triangleright (k_2\otimes 1) \triangleleft 1\\ &= k_1k_3S(k_2)\otimes 1.\end{aligned}$$

By assumption, we have $k_1k_3S(k_2) \otimes 1 = k \otimes 1$. Apply $id \otimes \varepsilon$ to both sides, we get

$$k_1k_3S(k_2) = k. \tag{4.4}$$

By computing we have

$$\begin{aligned}k_2S(k_1) &= \varepsilon(k_1)k_3S(k_2)\\ &= (S(k_1)k_2)k_4S(k_3)\\ &= S(k_1)(k_2k_4S(k_3))\\ &= S(k_1)k_2 \quad \textit{by (4.4) applied to } k_2\\ &= \varepsilon(k).\end{aligned}$$

Hence $S^2 = id$.

Similarly, we can check that the statements (3) and (4) hold. □

Proposition 4.3. *Let H be a Hopf algebra and $S^2 = id$, and assume that M and N satisfy the u-condition. Then $M \otimes N$ satisfies the u-condition if and only if $\psi_{M,N}$ is a symmetry.*

Proof. For any $m \in M$ and $n \in N$, we have

$$
\begin{aligned}
&(m\otimes n)_{[-1]} \triangleright (m \otimes n)_{[0](0)} \triangleleft (m \otimes n)_{[0](1)} \\
&\quad = (m_{[-1]}n_{[-1]}) \triangleright (m_{[0]} \otimes n_{[0]})_{(0)} \triangleleft (m_{[0]} \otimes n_{[0]})_{(1)} \\
&\quad = (m_{[-1]}n_{[-1]}) \triangleright (m_{[0](0)} \otimes n_{[0](0)}) \triangleleft (m_{[0](1)}n_{[0](1)}) \\
&\quad = m_{[-1]} \triangleright [n_{[-1]} \triangleright (m_{[0](0)} \otimes n_{[0](0)}) \triangleleft m_{[0](1)}] \triangleleft n_{[0](1)} \\
&\quad = m_{[-1]} \triangleright [n_{[-1]1} \triangleright (m_{[0](0)} \triangleleft m_{[0](1)1}) \otimes (n_{[-1]2} \triangleright n_{[0](0)}) \triangleleft m_{[0](1)2}] \triangleleft n_{[0](1)} \\
&\quad = m_{[-1]} \triangleright [n_{(0)[-1]1} \triangleright (m_{[0](0)} \triangleleft m_{[0](1)1}) \otimes (n_{(0)[-1]2} \triangleright n_{(0)[0]}) \triangleleft m_{[0](1)2}] \triangleleft n_{(1)} \quad by\ (2.2) \\
&\quad = m_{[-1]} \triangleright [n_{(0)[-1]1}(n_{(0)[-1]4}S(n_{(0)[-1]3})) \triangleright (m_{[0](0)} \triangleleft m_{[0](1)3}) \\
&\qquad \otimes (n_{(0)[-1]2} \triangleright n_{(0)[0]}) \triangleleft (S(m_{[0](1)2})m_{[0](1)1})m_{[0](1)4}] \triangleleft n_{(1)} \quad by\ S^2 = id \\
&\quad = m_{[-1]} \triangleright [(n_{(0)[-1]11}n_{(0)[-1]2})S(n_{(0)[-1]13}) \triangleright (m_{[0](0)} \triangleleft m_{[0](1)22}) \\
&\qquad \otimes (n_{(0)[-1]12} \triangleright n_{(0)[0]}) \triangleleft S(m_{[0](1)21})(m_{[0](1)1}m_{[0](1)23})] \triangleleft n_{(1)} \\
&\quad = m_{[-1]} \triangleright [(n_{(0)[-1]1}n_{(0)[0][-1]})S(n_{(0)[-1]3}) \triangleright (m_{[0](0)(0)} \triangleleft m_{[0](1)2}) \\
&\qquad \otimes (n_{(0)[-1]2} \triangleright n_{(0)[0][0]}) \triangleleft S(m_{[0](1)1})(m_{[0](0)(1)}m_{[0](1)3})] \triangleleft n_{(1)} \\
&\quad = m_{[-1]} \triangleright [(n_{(0)[-1]1} \triangleright n_{(0)[0]})_{[-1]}n_{(0)[-1]2}S(n_{(0)[-1]3}) \triangleright (m_{[0](0)} \triangleleft m_{[0](1)3})_{(0)} \\
&\qquad \otimes (n_{(0)[-1]1} \triangleright n_{(0)[0]})_{[0]} \triangleleft S(m_{[0](1)1})m_{[0](1)2}(m_{[0](0)} \triangleleft m_{[0](1)3})_{(1)}] \triangleleft n_{(1)}\ by\ (2.3),\ (2.5) \\
&\quad = m_{[-1]} \triangleright [(n_{(0)[-1]} \triangleright n_{(0)[0]})_{[-1]} \triangleright (m_{[0](0)} \triangleleft m_{[0](1)})_{(0)} \\
&\qquad \otimes (n_{(0)[-1]} \triangleright n_{(0)[0]})_{[0]} \triangleleft (m_{[0](0)} \triangleleft m_{[0](1)})_{(1)}] \triangleleft n_{(1)} \\
&\quad = m_{[-1]} \triangleright [\psi_{N,M}(n_{(0)[-1]} \triangleright n_{(0)[0]} \otimes m_{[0](0)} \triangleleft m_{[0](1)})] \triangleleft n_{(1)} \\
&\quad = \psi_{N,M}(m_{[-1]} \triangleright [n_{(0)[-1]} \triangleright n_{(0)[0]} \otimes m_{[0](0)} \triangleleft m_{[0](1)}] \triangleleft n_{(1)}) \\
&\quad = \psi_{N,M}(m_{[-1]1}n_{(0)[-1]} \triangleright n_{(0)[0]} \triangleleft n_{(1)1} \otimes m_{[-1]2} \triangleright m_{[0](0)} \triangleleft m_{[0](1)}n_{(1)2}) \\
&\quad = \psi_{N,M}(m_{[-1]}n_{(0)(0)[-1]} \triangleright n_{(0)(0)[0]} \triangleleft n_{(0)(1)} \otimes m_{[0][-1]} \triangleright m_{[0][0](0)} \triangleleft m_{[0][0](1)}n_{(1)}) \\
&\quad = \psi_{N,M}(m_{[-1]} \triangleright n_{(0)} \otimes m_{[0]} \triangleleft n_{(1)}) \quad by\ (4.1),\ (4.2) \\
&\quad = \psi_{N,M} \circ \psi_{M,N}(m \otimes n).
\end{aligned}
$$

This completes the proof. □

If we consider $M = H_i$ and $N = H_j$, for any $i, j = 1, 2, 3, 4$ (see Example 2.4). By Proposition 4.2 and 4.3, we obtain:

Corollary 4.4. *Let H be a Hopf algebra, and assume that H_i and H_j satisfy the u-condition. Then $H_i \otimes H_j$ satisfies the u-condition if and only if ψ_{H_i,H_j} is a symmetry, for any $i, j = 1, 2, 3, 4$.*

5. Yetter-Drinfel'd-Long categories over quasitriangular Hopf algebras

In this section, we focus on $M \in \mathcal{LR}(H)$ for which $\psi_{M,M}$ is a symmetry. Triangular Hopf algebras give rise to such M.

Theorem 5.1. *Let (H, R) be a quasitriangular Hopf algebra. Then the category ${}_H\mathcal{M}_H$ of H-bimodules is a Yetter-Drinfel'd-Long subcategory of $\mathcal{LR}(H)$ under the coactions $\rho^l(m) = R^2 \otimes R^1 \triangleright m$ and $\rho^r(m) = m \triangleleft R^1 \otimes R^2$, where $\triangleright$ ($\triangleleft$, resp.) is the left (right, resp.) action on M.*

Proof. First, we check that M is a right H-comodule. By the definition of right H-comodule, for any $m \in M$, we have

$$
\begin{aligned}
(id \otimes \Delta)\rho^r(m) &= (id \otimes \Delta)(m \triangleleft R^1 \otimes R^2) \\
&= m \triangleleft R^1 \otimes R^2_1 \otimes R^2_2 \\
&= m \triangleleft R^1 r^1 \otimes r^2 \otimes R^2 \quad by\ (QT2)
\end{aligned}
$$

$$
\begin{aligned}
&= (\rho^r \otimes id)(m \triangleleft R^1 \otimes R^2) \\
&= (\rho^r \otimes id)\rho^r(m),
\end{aligned}
$$

and it is clear that $m_{(0)}\varepsilon(m_{(1)}) = m \triangleleft R^1\varepsilon(R^2) = m \triangleleft 1 = m$. Similarly, we can get that M is a left H-comodule.

Next, we verify the compatible condition of H-bicomodule. For any $m \in M$, we have

$$
\begin{aligned}
(id \otimes \rho^r)\rho^l(m) &= (id \otimes \rho^r)(R^2 \otimes R^1 \triangleright m) \\
&= R^2 \otimes (R^1 \triangleright m) \triangleleft r^1 \otimes r^2 \\
&= R^2 \otimes R^1 \triangleright (m \triangleleft r^1) \otimes r^2 \quad by\ (2.1) \\
&= (\rho^l \otimes id)(m \triangleleft r^1 \otimes r^2) \\
&= (\rho^l \otimes id)\rho^r(m).
\end{aligned}
$$

We now prove that M satisfies the four compatible conditions (2.3) $\sim$ (2.6). Indeed, for any $h \in H$ and $m \in M$, we have

$$
\begin{aligned}
(h \triangleright m)_{(0)} \otimes (h \triangleright m)_{(1)} &= (h \triangleright m) \triangleleft R^1 \otimes R^2 \\
&= h \triangleright (m \triangleleft R^1) \otimes R^2 \\
&= h \triangleright m_{(0)} \otimes m_{(1)}.
\end{aligned}
$$

Thus Eq.(2.4) holds. For Eq.(2.5) , we have

$$
\begin{aligned}
m_{(0)} \triangleleft h_1 \otimes m_{(1)}h_2 &= (m \triangleleft R^1) \triangleleft h_1 \otimes R^2h_2 \\
&= m \triangleleft R^1h_1 \otimes R^2h_2 \\
&= m \triangleleft h_2R^1 \otimes h_1R^2 \quad by\ (QT3) \\
&= (m \triangleleft h_2) \triangleleft R^1 \otimes h_1R^2 \\
&= (m \triangleleft h_2)_{(0)} \otimes h_1(m \triangleleft h_2)_{(1)}.
\end{aligned}
$$

Similarly, we can show that Eq.(2.3) and (2.6) hold.

Finally, we need to show that any morphisms in ${}_H\mathcal{M}_H$ are both left H-colinear and right H-colinear. For this purpose, we take any $M, N \in {}_H\mathcal{M}_H$, and assume that $f : M \to N$ is a morphism in ${}_H\mathcal{M}_H$, we get

$$(f \otimes id) \circ \rho^r_M(m) = f(m \triangleleft R^1) \otimes R^2 = f(m) \triangleleft R^1 \otimes R^2 = \rho^r_N \circ f(m).$$

So f is right H-colinear. Similarly, we can obtain that f described above is left H-colinear.

This completes the proof. $\square$

Proposition 5.2. *Let H be a triangular Hopf algebra. Then the Yetter-Drinfel'd-Long subcategory ${}_H\mathcal{M}_H$ defined above is symmetric.*

Proof. For any $m \in M$ and $n \in N$, we have

$$
\begin{aligned}
\psi_{N,M} \circ \psi_{M,N}(m \otimes n) &= \psi_{N,M}(R^2 \triangleright n \triangleleft r^1 \otimes R^1 \triangleright m \triangleleft r^2) \\
&= Q^2 \triangleright (R^1 \triangleright m \triangleleft r^2) \triangleleft q^1 \otimes Q^1 \triangleright (R^2 \triangleright n \triangleleft r^1) \triangleleft q^2 \\
&= Q^2R^1 \triangleright m \triangleleft r^2q^1 \otimes Q^1R^2 \triangleright n \triangleleft r^1q^2 \quad by\ (QT5) \\
&= 1 \triangleright m \triangleleft 1 \otimes 1 \triangleright n \triangleleft 1 \\
&= m \otimes n.
\end{aligned}
$$

Thus the subcategory ${}_H\mathcal{M}_H$ is symmetric. $\square$

By Theorem 5.1 and Proposition 5.2, we know that If (H,R) be a triangular Hopf algebra then the subcategory ${}_H\mathcal{M}_H$ described above is symmetric. A particular example is $M = H \otimes H$. In the following we prove the converse. That is, assume that the braiding $\psi_{H\otimes H,H\otimes H}$ is a symmetry forces (H,R) to be triangular, where $H \otimes H$ is a Hopf algebra with usual tensor product and tensor coproduct.

Theorem 5.3. *Let H be a Hopf algebra with a bijective antipode, and assume that $(H\otimes H, \triangleright = m\otimes id, \rho^l = \rho_1 \otimes id, \triangleleft = id \otimes m, \rho^r = id \otimes \rho_2) \in \mathcal{LR}(H)$, where m is usual multiplication and ρ_1 (ρ_2, resp.) is a left (right, resp.) coaction on H. Then $\psi_{H\otimes H,H\otimes H}$ is a symmetry if and only if there exists $R \in H \otimes H$ so that (H,R) is triangular. And then ρ^l and ρ^r are induced by R. That is,*

$$\rho^l(k \otimes l) = R^2 \otimes R^1 k \otimes l, \quad \rho^r(k \otimes l) = k \otimes l R^1 \otimes R^2,$$

for any $k,l \in H$, in particular, $R^\tau \otimes 1 = \rho^l(1 \otimes 1)$ and $1 \otimes R = \rho^r(1 \otimes 1)$.

Proof. If $\psi = \psi_{H\otimes H,H\otimes H}$ is a symmetry, for any $k,l,g,h \in H$, we have

$$\begin{aligned}\psi(k\otimes l\otimes g\otimes h) &= (k\otimes l)_{[-1]} \triangleright (g\otimes h)_{(0)} \otimes (k\otimes l)_{[0]} \triangleleft (g\otimes h)_{(1)}\\ &= (g\otimes h)_{[0]} \triangleleft S^{-1}((k\otimes l)_{(1)}) \otimes S^{-1}((g\otimes h)_{[-1]}) \triangleright (k\otimes l)_{(0)}.\end{aligned} \tag{5.1}$$

In particular, let $\rho^l(1\otimes 1) = x_i \otimes y_i \otimes 1$ and $\rho^r(1 \otimes 1) = 1 \otimes s_i \otimes t_i$. Then

$$\begin{aligned}x_i\otimes s_i\otimes y_i\otimes t_i &= x_i \triangleright (1\otimes s_i)\otimes (y_i\otimes 1)\triangleleft t_i\\ &= (1\otimes 1)_{[-1]} \triangleright (1\otimes 1)_{(0)} \otimes (1\otimes 1)_{[0]} \triangleleft (1\otimes 1)_{(1)}\\ &= (1\otimes 1)_{[0]} \triangleleft S^{-1}((1\otimes 1)_{(1)}) \otimes S^{-1}((1\otimes 1)_{[-1]}) \triangleright (1\otimes 1)_{(0)} \quad by\ (5.1)\\ &= (y_i\otimes 1)\triangleleft S^{-1}(t_i)\otimes S^{-1}(x_i)\triangleright (1\otimes s_i)\\ &= y_i\otimes S^{-1}(t_i)\otimes S^{-1}(x_i)\otimes s_i.\end{aligned}$$

Thus

$$x_i\otimes s_i\otimes y_i\otimes t_i = y_i\otimes S^{-1}(t_i)\otimes S^{-1}(x_i)\otimes s_i.$$

Apply $id\otimes\varepsilon\otimes id\otimes\varepsilon$ and $\varepsilon\otimes id\otimes\varepsilon\otimes id$ to both sides, respectively, we have

$$x_i\otimes y_i = y_i\otimes S^{-1}(x_i), \tag{5.2}$$

$$s_i\otimes t_i = S^{-1}(t_i)\otimes s_i. \tag{5.3}$$

Apply $id\otimes S$ to Eq.(5.2) yields

$$x_i\otimes S(y_i) = y_i\otimes x_i. \tag{5.4}$$

Set $R\otimes 1 = y_i\otimes x_i\otimes 1 = (\tau\otimes id)\circ\rho^l(1\otimes 1)$ and $1\otimes R = 1\otimes s_i\otimes t_i = \rho^r(1\otimes 1)$. In the following, we wish to show that (H,R) is triangular and that ρ^l and ρ^r are induced by R. For this purpose, we first need the following equations $\rho^l(k\otimes l) = (id\otimes\varepsilon\otimes id^2)\psi(k\otimes l\otimes 1\otimes 1)$ and $\rho^r(k\otimes l) = (id^2\otimes\varepsilon\otimes id)\psi(1\otimes 1\otimes k\otimes l)$. Indeed, for any $k,l \in H$:

$$\begin{aligned}(id\otimes\varepsilon\otimes id^2)\psi(k\otimes l\otimes 1\otimes 1) &= (id\otimes\varepsilon\otimes id^2)((k\otimes l)_{[-1]} \triangleright (1\otimes 1)_{(0)} \otimes (k\otimes l)_{[0]} \triangleleft (1\otimes 1)_{(1)})\\ &= (id\otimes\varepsilon\otimes id^2)((k\otimes l)_{[-1]} \triangleright (1\otimes s_i) \otimes (k\otimes l)_{[0]} \triangleleft t_i)\\ &= (id\otimes\varepsilon\otimes id^2)((k\otimes l)_{[-1]} \otimes s_i \otimes (k\otimes l)_{[0]} \triangleleft t_i)\\ &= (id\otimes\varepsilon\otimes id^2)((k\otimes l)_{[-1]} \otimes S^{-1}(t_i) \otimes (k\otimes l)_{[0]} \triangleleft s_i) \quad by\ (5.3)\\ &= (k\otimes l)_{[-1]} \otimes (k\otimes l)_{[0]} \triangleleft 1\\ &= (k\otimes l)_{[-1]} \otimes (k\otimes l)_{[0]}\\ &= \rho^l(k\otimes l)\end{aligned}$$

and

$$
\begin{aligned}
(id^2\otimes\varepsilon\otimes id)\psi(1\otimes 1\otimes k\otimes l) &= (id^2\otimes\varepsilon\otimes id)((1\otimes 1)_{[-1]}\triangleright (k\otimes l)_{(0)}\otimes (1\otimes 1)_{[0]}\triangleleft (k\otimes l)_{(1)})\\
&= (id^2\otimes\varepsilon\otimes id)(x_i\triangleright (k\otimes l)_{(0)}\otimes (y_i\otimes 1)\triangleleft (k\otimes l)_{(1)})\\
&= (id^2\otimes\varepsilon\otimes id)(x_i\triangleright (k\otimes l)_{(0)}\otimes y_i\otimes (k\otimes l)_{(1)})\\
&= (id^2\otimes\varepsilon\otimes id)(y_i\triangleright (k\otimes l)_{(0)}\otimes S^{-1}(x_i)\otimes (k\otimes l)_{(1)}) \quad by\ (5.2)\\
&= 1\triangleright (k\otimes l)_{(0)}\otimes (k\otimes l)_{(1)}\\
&= (k\otimes l)_{(0)}\otimes (k\otimes l)_{(1)}\\
&= \rho^r(k\otimes l).
\end{aligned}
$$

We now prove that ρ^l and ρ^r are induced by R. For any $k, l\in H$, we have

$$
\begin{aligned}
\rho^l(k\otimes l) &= (id\otimes\varepsilon\otimes id^2)\psi(k\otimes l\otimes 1\otimes 1)\\
&= (id\otimes\varepsilon\otimes id^2)((1\otimes 1)_{[0]}\triangleleft S^{-1}((k\otimes l)_{(1)})\otimes S^{-1}((1\otimes 1)_{[-1]})\triangleright (k\otimes l)_{(0)}) \quad by\ (5.1)\\
&= (id\otimes\varepsilon\otimes id^2)((y_i\otimes 1)\triangleleft S^{-1}((k\otimes l)_{(1)})\otimes S^{-1}(x_i)\triangleright (k\otimes l)_{(0)})\\
&= (id\otimes\varepsilon\otimes id^2)(y_i\otimes S^{-1}((k\otimes l)_{(1)})\otimes S^{-1}(x_i)\triangleright (k\otimes l)_{(0)})\\
&= y_i\otimes S^{-1}(x_i)\triangleright (k\otimes l)\\
&= y_i\otimes S^{-1}(x_i)k\otimes l\\
&= x_i\otimes y_ik\otimes l \quad by\ (5.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho^r(k\otimes l) &= (id^2\otimes\varepsilon\otimes id)\psi(1\otimes 1\otimes k\otimes l)\\
&= (id^2\otimes\varepsilon\otimes id)((k\otimes l)_{[0]}\triangleleft S^{-1}((1\otimes 1)_{(1)})\otimes S^{-1}((k\otimes l)_{[-1]})\triangleright (1\otimes 1)_{(0)}) \quad by\ (5.1)\\
&= (id^2\otimes\varepsilon\otimes id)((k\otimes l)_{[0]}\triangleleft S^{-1}(t_i)\otimes S^{-1}((k\otimes l)_{[-1]})\triangleright (1\otimes s_i))\\
&= (id^2\otimes\varepsilon\otimes id)((k\otimes l)_{[0]}\triangleleft S^{-1}(t_i)\otimes S^{-1}((k\otimes l)_{[-1]})\otimes s_i)\\
&= (k\otimes l)\triangleleft S^{-1}(t_i)\otimes s_i\\
&= k\otimes lS^{-1}(t_i)\otimes s_i\\
&= k\otimes ls_i\otimes t_i. \quad by\ (5.3)
\end{aligned}
$$

Thus

$$\rho^l(k\otimes l) = x_i\otimes y_ik\otimes l, \tag{5.5}$$
$$\rho^r(k\otimes l) = k\otimes ls_i\otimes t_i. \tag{5.6}$$

Finally, we verify that (H, R) is triangular. By definition, we need to prove the five equations (QT1) $\sim$ (QT5). For (QT1), we only have to check that $\Delta(y_i)\otimes x_i = y_i\otimes y_j\otimes x_ix_j$.

$$
\begin{aligned}
\Delta(y_i)\otimes x_i &= (id^3\otimes\varepsilon)(\Delta(y_i)\otimes x_i\otimes 1)\\
&= (id^3\otimes\varepsilon)(\Delta(x_i)\otimes S(y_i)\otimes 1) \quad by\ (5.4)\\
&= (id^2\otimes S\otimes\varepsilon)(\Delta\otimes id^2)(x_i\otimes y_i\otimes 1)\\
&= (id^2\otimes S\otimes\varepsilon)(\Delta\otimes id^2)\rho^l(1\otimes 1)\\
&= (id^2\otimes S\otimes\varepsilon)(id\otimes\rho^l)\rho^l(1\otimes 1)\\
&= (id^2\otimes S\otimes\varepsilon)(x_i\otimes\rho^l(y_i\otimes 1))\\
&= (id^2\otimes S\otimes\varepsilon)(x_i\otimes x_j\otimes y_jy_i\otimes 1)) \quad by\ (5.5)
\end{aligned}
$$

$$= (id^2 \otimes S \otimes \varepsilon)(y_i \otimes y_j \otimes S^{-1}(x_j)S^{-1}(x_i) \otimes 1)) \quad by\ (5.2)$$
$$= y_i \otimes y_j \otimes x_i x_j.$$

Similarly, we can check that (QT2) holds. For (QT3), we only need to show that $h_2 y_i \otimes h_1 x_i = y_i h_1 \otimes x_i h_2$. Since both ψ and ε are H-module maps, we have

$$\begin{aligned}
h_1 x_i \otimes h_2 y_i &= (id \otimes \varepsilon \otimes id \otimes \varepsilon)(h_1 x_i \otimes 1 \otimes h_2 y_i \otimes 1) \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)(h_1 \triangleright (x_i \otimes 1) \otimes h_2 \triangleright (y_i \otimes 1)) \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[h \triangleright (x_i \otimes 1 \otimes y_i \otimes 1)] \\
&= h \triangleright [(id \otimes \varepsilon \otimes id \otimes \varepsilon)(x_i \otimes 1 \otimes y_i \otimes 1)] \\
&= h \triangleright [(id \otimes id \otimes \varepsilon) \circ \rho^l(1 \otimes 1)] \\
&= h \triangleright [(id \otimes \varepsilon \otimes id \otimes \varepsilon)\psi(1 \otimes 1 \otimes 1 \otimes 1)] \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[h \triangleright \psi(1 \otimes 1 \otimes 1 \otimes 1)] \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[\psi(h \triangleright (1 \otimes 1 \otimes 1 \otimes 1))] \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[\psi(h_1 \otimes 1 \otimes h_2 \otimes 1)] \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[(h_1 \otimes 1)_{[-1]} \triangleright (h_2 \otimes 1)_{(0)} \otimes (h_1 \otimes 1)_{[0]} \triangleleft (h_2 \otimes 1)_{(1)}] \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[x_i \triangleright (h_2 \otimes s_i) \otimes (y_i h_1 \otimes 1) \triangleleft t_i] \quad by\ (5.5),\ (5.6) \\
&= (id \otimes \varepsilon \otimes id \otimes \varepsilon)[x_i h_2 \otimes s_i \otimes y_i h_1 \otimes t_i] \\
&= x_i h_2 \otimes y_i h_1.
\end{aligned}$$

For (QT4), we have

$$\begin{aligned}
\varepsilon(R^1)R^2 &= (\varepsilon \otimes id \otimes \varepsilon)(R^1 \otimes R^2 \otimes 1) \\
&= (\varepsilon \otimes id \otimes \varepsilon)(y_i \otimes x_i \otimes 1) \\
&= (\varepsilon \otimes id \otimes \varepsilon)(S^{-1}(x_i) \otimes y_i \otimes 1) \quad by\ (5.2) \\
&= (\varepsilon \otimes id \otimes \varepsilon)(x_i \otimes y_i \otimes 1) \\
&= (\varepsilon \otimes id \otimes \varepsilon)\rho^l(1 \otimes 1) \\
&= 1.
\end{aligned}$$

Similarly, we can check that $\varepsilon(R^2)R^1 = 1$. For (QT5), we have

$$\begin{aligned}
1 \otimes 1 \otimes 1 \otimes 1 &= \psi^2(1 \otimes 1 \otimes 1 \otimes 1) \\
&= \psi((1 \otimes 1)_{[-1]} \triangleright (1 \otimes 1)_{(0)} \otimes (1 \otimes 1)_{[0]} \triangleleft (1 \otimes 1)_{(1)}) \\
&= \psi(x_i \triangleright (1 \otimes s_i) \otimes (y_i \otimes 1) \triangleleft t_i) \\
&= \psi(x_i \otimes s_i \otimes y_i \otimes t_i) \\
&= (x_i \otimes s_i)_{[-1]} \triangleright (y_i \otimes t_i)_{(0)} \otimes (x_i \otimes s_i)_{[0]} \triangleleft (y_i \otimes t_i)_{(1)} \\
&= x_j \triangleright (y_i \otimes t_i s_j) \otimes (y_j x_i \otimes s_i) \triangleleft t_j \\
&= x_j y_i \otimes t_i s_j \otimes y_j x_i \otimes s_i t_j.
\end{aligned}$$

Thus, R is invertible and $R^{-1} = x_i \otimes y_i = t_i \otimes s_i$.

The converse is Theorem 5.1 and Proposition 5.2. This completes the proof. □

As a corollary we have:

Corollary 5.4. *Let H be a Hopf algebra with a bijective antipode. Then, for $H_3 \in \mathcal{LR}(H)$, the braiding ψ_{H_3,H_3} is a symmetry if and only if H is cocommutative.*

Proof. If the braiding satisfies $\psi^2_{H_3,H_3} = id$, then by Theorem 5.3 (H,R) is triangular with $\rho^l(1\otimes 1) = R^\tau \otimes 1$. Since $\rho^l(k\otimes l) = k_1S(k_3)\otimes k_2\otimes l$ for any $k,l \in H$, we have $\rho^l(1\otimes 1) = 1\otimes 1\otimes 1$, so $R = 1\otimes 1$. Thus (QT3) implies that H is cocommutative.

Conversely, assume that H is cocommutative, for any $k,l,g,h \in H$, we have

$$\begin{aligned}
\psi_{H_3,H_3}(k\otimes l\otimes g\otimes h) &= (k\otimes l)_{[-1]} \triangleright (g\otimes h)_{(0)} \otimes (k\otimes l)_{[0]} \triangleleft (g\otimes h)_{(1)}\\
&= k_1S(k_3) \triangleright (g\otimes h_2) \otimes (k_2\otimes l) \triangleleft h_1S(h_3)\\
&= k_1S(k_2) \triangleright (g\otimes h_3) \otimes (k_3\otimes l) \triangleleft h_1S(h_2) \quad \textit{by } H \textit{ is cocommutative}\\
&= 1 \triangleright (g\otimes h) \otimes (k\otimes l) \triangleleft 1\\
&= g\otimes h\otimes k\otimes l.
\end{aligned}$$

It is clear that the braiding ψ_{H_3,H_3} is a symmetry. □

If we consider $H \otimes \Bbbk$, by Theorem 5.3, we generalize the important result in [1].

Corollary 5.5. *Let H be a Hopf algebra with a bijective antipode, and assume that $(H,m,\rho) \in {}^H_H\mathcal{YD}$, where m is usual multiplication. Then $\psi_{H,H}$ is a symmetry if and only if there exists $R \in H\otimes H$ so that (H,R) is triangular. And then ρ is induced by R. That is,*

$$\rho(k) = R^2 \otimes R^1 k,$$

for any $k \in H$, in particular, $R^\tau = \rho(1)$.

6. Yetter-Drinfel'd-Long categories over coquasitriangular Hopf algebras

In this section, we discuss the dual cases of section 5.

Theorem 6.1. *Let (H,ζ) be a coquasitriangular Hopf algebra. Then the category ${}^H\mathcal{M}^H$ of H-bicomodules is a Yetter-Drinfel'd-Long subcategory of $\mathcal{LR}(H)$ under the actions $h \triangleright m = \zeta(h, m_{[-1]})m_{[0]}$ and $m \triangleleft h = m_{(0)}\zeta(h, m_{(1)})$, for any $h \in H$ and $m \in M \in {}^H\mathcal{M}^H$.*

Proof. First, we prove that $(M,\triangleleft)$ is a right H-module. For any $h,g \in H$ and $m \in M$, we have

$$\begin{aligned}
(m\triangleleft g)\triangleleft h &= m_{(0)} \triangleleft h\zeta(g, m_{(1)})\\
&= m_{(0)(0)}\zeta(h, m_{(0)(1)})\zeta(g, m_{(1)})\\
&= m_{(0)}\zeta(h, m_{(1)1})\zeta(g, m_{(1)2})\\
&= m_{(0)}\zeta(gh, m_{(1)}) \quad \textit{by } (CQT2)\\
&= m \triangleleft gh,
\end{aligned}$$

and it is clear that $m \triangleleft 1 = m_{(0)}\zeta(1, m_{(1)}) = m_{(0)}\varepsilon(m_{(1)}) = m$. Similarly, we can obtain that $(M,\triangleright)$ is a left H-module.

Next, we check the compatible condition of H-bimodule. For any $h,g \in H$ and $m \in M$, we have

$$\begin{aligned}
(h\triangleright m)\triangleleft g &= \zeta(h, m_{[-1]})m_{[0]} \triangleleft g\\
&= \zeta(h, m_{[-1]})m_{[0](0)}\zeta(g, m_{[0](1)})\\
&= \zeta(h, m_{(0)[-1]})m_{(0)[0]}\zeta(g, m_{(1)}) \quad \textit{by } (2.2)\\
&= h \triangleright m_{(0)}\zeta(g, m_{(1)})\\
&= h \triangleright (m\triangleleft g).
\end{aligned}$$

We now check that the four compatible conditions (2.3) ∼ (2.6). For any $h \in H$ and $m \in M$, we have

$$\begin{aligned}
(h\triangleright m)_{(0)} \otimes (h\triangleright m)_{(1)} &= \zeta(h, m_{[-1]})m_{[0](0)} \otimes (h\triangleright m)_{[0](1)}\\
&= \zeta(h, m_{(0)[-1]})m_{(0)[0]} \otimes m_{(1)} \quad \textit{by } (2.2)
\end{aligned}$$

$$= h \triangleright m_{(0)} \otimes m_{(1)}.$$

Thus Eq.(2.4) holds. For Eq.(2.5), we have

$$\begin{aligned} m_{(0)} \triangleleft h_1 \otimes m_{(1)}h_2 &= m_{(0)(0)}\zeta(h_1, m_{(0)(1)}) \otimes m_{(1)}h_2 \\ &= m_{(0)} \otimes \zeta(h_1, m_{(1)1})m_{(1)2}h_2 \\ &= m_{(0)} \otimes h_1 m_{(1)1}\zeta(h_2, m_{(1)2}) \quad by\ (CQT3) \\ &= m_{(0)(0)}\zeta(h_2, m_{(1)}) \otimes h_1 m_{(0)(1)} \\ &= (m \triangleleft h_2)_{(0)} \otimes h_1(m \triangleleft h_2)_{(1)}. \end{aligned}$$

Similarly, we can verify that Eq.(2.3) and (2.6) hold.

Finally, we have to prove that any morphisms in ${}^H\mathcal{M}^H$ are both left H-linear and right H-linear. For this purpose, we take any $M, N \in {}^H\mathcal{M}^H$, and assume that $f : M \to N$ is a morphism in ${}^H\mathcal{M}^H$, we have

$$f(m \triangleleft h) = f(m_{(0)})\zeta(h, m_{(1)}) = f(m)_{(0)}\zeta(h, f(m)_{(1)}) = f(m) \triangleleft h.$$

So f is right H-linear. Similarly, we can obtain that f is left H-linear.

This completes the proof. □

Proposition 6.2. *Let H be a cotriangular Hopf algebra. Then the Yetter-Drinfel'd-Long subcategory ${}^H\mathcal{M}^H$ defined above is symmetric.*

Proof. For any $m \in M$ and $n \in N$, we have

$$\begin{aligned} \psi_{N,M} \circ \psi_{M,N}(m \otimes n) &= \psi_{N,M}(m_{[-1]} \triangleright n_{(0)} \otimes m_{[0]} \triangleleft n_{(1)}) \\ &= \psi_{N,M}(\zeta(m_{[-1]}, n_{(0)[-1]})n_{(0)[0]} \otimes m_{[0](0)}\zeta(n_{(1)}, m_{[0](1)})) \\ &= \zeta(m_{[-1]}, n_{(0)[-1]})\zeta(n_{(1)}, m_{[0](1)})n_{(0)[0][-1]} \triangleright m_{[0](0)(0)} \otimes n_{(0)[0][0]} \triangleleft m_{[0](0)(1)} \\ &= \zeta(m_{[-1]}, n_{(0)[-1]1})\zeta(n_{(1)}, m_{[0](1)2})n_{(0)[-1]2} \triangleright m_{[0](0)} \otimes n_{(0)[0]} \triangleleft m_{[0](1)1} \\ &= \zeta(m_{(0)[-1]}, n_{[-1]1})\zeta(n_{[0](1)}, m_{(1)2})n_{[-1]2} \triangleright m_{(0)[0]} \otimes n_{[0](0)} \triangleleft m_{(1)1} \quad by\ (2.2) \\ &= \zeta(m_{(0)[-1]}, n_{[-1]1})\zeta(n_{[0](1)}, m_{(1)2}) \\ &\quad \zeta(n_{[-1]2}, m_{(0)[0][-1]})m_{(0)[0][0]} \otimes n_{[0](0)(0)}\zeta(m_{(1)1}, n_{[0](0)(1)}) \\ &= \zeta(m_{(0)[-1]1}, n_{[-1]1})\zeta(n_{[-1]2}, m_{(0)[-1]2}) \\ &\quad \zeta(m_{(1)1}, n_{[0](1)1})\zeta(n_{[0](1)2}, m_{(1)2})m_{(0)[0]} \otimes n_{[0](0)} \quad by\ (CQT5) \\ &= m \otimes n. \end{aligned}$$

So the subcategory ${}^H\mathcal{M}^H$ is symmetric. □

Theorem 6.3. *Let H be a Hopf algebra with a bijective antipode, and assume that $(H \otimes H, \triangleright = \rightharpoonup \otimes id, \rho^l = \Delta \otimes id, \triangleleft = id \otimes \leftharpoonup, \rho^r = id \otimes \Delta) \in \mathcal{LR}(H)$, where Δ is usual comultiplication and $\rightharpoonup$ ($\leftharpoonup$, resp.) is a left (right, resp.) action on H. Then $\psi_{H\otimes H, H\otimes H}$ is a symmetry if and only if there exists a braiding $\zeta : H \otimes H \to \Bbbk$ so that (H, ζ) is cotriangular Hopf algebra. And then $\zeta(k, g)\zeta(h, l) = (\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon)\psi(k \otimes l \otimes g \otimes h)$, for any $k, l, g, h \in H$. That is,*

$$h \triangleright (k \otimes l) = h \rightharpoonup k \otimes l = \zeta(h, k_1)k_2 \otimes l,$$

$$(k \otimes l) \triangleleft h = k \otimes l \leftharpoonup h = k \otimes l_1\zeta(h, l_2).$$

Proof. Assume that $\psi = \psi_{H\otimes H, H\otimes H}$ is a symmetry, then for any $k, l, g, h \in H$,

$$\begin{aligned} \psi(k \otimes l \otimes g \otimes h) &= (k \otimes l)_{[-1]} \triangleright (g \otimes h)_{(0)} \otimes (k \otimes l)_{[0]} \triangleleft (g \otimes h)_{(1)} \\ &= (g \otimes h)_{[0]} \triangleleft S^{-1}((k \otimes l)_{(1)}) \otimes S^{-1}((g \otimes h)_{[-1]}) \triangleright (k \otimes l)_{(0)}, \end{aligned}$$

i.e.

$$\psi(k\otimes l\otimes g\otimes h)=k_1\rightharpoonup g\otimes h_1\otimes k_2\otimes l\leftharpoonup h_2$$
$$=g_2\otimes h\leftharpoonup S^{-1}(l_2)\otimes S^{-1}(g_1)\rightharpoonup k\otimes l_1. \tag{6.1}$$

Define for any $k,l,g,h\in H$, $\zeta(k,g)\zeta(h,l)=(\varepsilon\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon)\psi(k\otimes l\otimes g\otimes h)$. Let $l=h=1$, and apply $\varepsilon\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon$ to Eq.(6.1), we get

$$\zeta(k,g)=\varepsilon(k\rightharpoonup g)=\varepsilon(S^{-1}(g)\rightharpoonup k)=\zeta(S^{-1}(g),k). \tag{6.2}$$

By applying $\zeta(k,g)=\zeta(S^{-1}(g),k)$ to $\zeta(g,S(k))$, we get

$$\zeta(k,g)=\zeta(g,S(k)). \tag{6.3}$$

Similarly, we can get that

$$\zeta(h,l)=\varepsilon(l\leftharpoonup h)=\varepsilon(h\leftharpoonup S^{-1}(l))=\zeta(S^{-1}(l),h)=\zeta(l,S(h)). \tag{6.4}$$

Moreover, let $l=h=1$, and apply $id\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon$ to Eq.(6.1), we get by (6.2), that for any $k,g\in H$,

$$k\rightharpoonup g=\zeta(S^{-1}(g_1),k)g_2=\zeta(k,g_1)g_2. \tag{6.5}$$

Similarly, we can get by (6.4), that for any $l,h\in H$,

$$l\leftharpoonup h=\zeta(S^{-1}(l_2),h)l_1=\zeta(h,l_2)l_1.$$

Thus we have

$$h\triangleright(k\otimes l)=h\rightharpoonup k\otimes l=\zeta(h,k_1)k_2\otimes l,$$
$$(k\otimes l)\triangleleft h=k\otimes l\leftharpoonup h=k\otimes l_1\zeta(h,l_2).$$

By definition of cotriangular, we need to prove the five equations (CQT1) ~ (CQT5). First, we prove (CQT2). For any $h,g,l\in H$, we have

$$\begin{aligned}\zeta(hg,l)&=\varepsilon(hg\rightharpoonup l)\\&=(\varepsilon\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon)(h_1\rightharpoonup(g\rightharpoonup l)\otimes 1\otimes h_2\otimes 1)\\&=(\varepsilon\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon)(h_1\triangleright(g\rightharpoonup l\otimes 1)\otimes(h_2\otimes 1)\triangleleft 1)\\&=(\varepsilon\otimes\varepsilon\otimes\varepsilon\otimes\varepsilon)\psi(h\otimes 1\otimes g\rightharpoonup l\otimes 1)\\&=\zeta(h,g\rightharpoonup l)\zeta(1,1)\\&=\zeta(h,\zeta(g,l_1)l_2)\quad by\ (6.5)\\&=\zeta(h,l_2)\zeta(g,l_1).\end{aligned}$$

Next, we prove (CQT1). For any $h,g,l\in H$, we have

$$\begin{aligned}\zeta(h,gl)&=\zeta(gl,S(h))\quad by\ (6.3)\\&=\zeta(g,S(h)_2)\zeta(l,S(h)_1)\quad by\ (CQT2)\\&=\zeta(g,S(h_1))\zeta(l,S(h_2))\\&=\zeta(h_1,g)\zeta(h_2,l).\quad by\ (6.3)\end{aligned}$$

We prove now (CQT3).

$$\begin{aligned}h_1g_1\zeta(h_2,g_2)&=h_1g_1\varepsilon(h_2\rightharpoonup g_2)\quad by\ (6.2)\\&=(id\otimes\varepsilon\otimes\varepsilon)(h_1g_1\otimes h_2\rightharpoonup g_2\otimes 1)\end{aligned}$$

$$
\begin{aligned}
&= (id \otimes \varepsilon \otimes \varepsilon)(h_1(g \otimes 1)_{[-1]} \otimes h_2 \triangleright (g \otimes 1)_{[0]}) \\
&= (id \otimes \varepsilon \otimes \varepsilon)((h_1 \triangleright (g \otimes 1))_{[-1]} h_2 \otimes (h_1 \triangleright (g \otimes 1))_{[0]}) \quad by\ (2.3) \\
&= (id \otimes \varepsilon \otimes \varepsilon)((h_1 \rightharpoonup g \otimes 1)_{[-1]} h_2 \otimes (h_1 \rightharpoonup g \otimes 1)_{[0]}) \\
&= (id \otimes \varepsilon \otimes \varepsilon)((h_1 \rightharpoonup g)_1 h_2 \otimes (h_1 \rightharpoonup g)_2 \otimes 1) \\
&= (h_1 \rightharpoonup g) h_2 \\
&= \zeta(h_1, g_1) g_2 h_2. \quad by\ (6.5)
\end{aligned}
$$

It is easy to check that (CQT4) and (CQT5) hold.

The converse is Theorem 6.1 and Proposition 6.2. This completes the proof. □

As a corollary we have:

Corollary 6.4. *Let H be a Hopf algebra with a bijective antipode. Then, for $H_4 \in \mathcal{LR}(H)$, the braiding ψ_{H_4,H_4} is a symmetry if and only if H is commutative.*

Proof. If the braiding satisfies $\psi^2_{H_4,H_4} = id$, then by (6.2) $\zeta(k, g) = \varepsilon(k \rightharpoonup g) = (\varepsilon \otimes \varepsilon)(k \triangleright (g \otimes 1)) = (\varepsilon \otimes \varepsilon)(k_1 g S(k_2) \otimes 1) = \varepsilon(g)\varepsilon(k)$ for any $k, g \in H$. Thus by Theorem 6.3 $(H, \varepsilon \otimes \varepsilon)$ is a cotriangular Hopf algebra, which by (CQT3) implies that H is commutative.

Conversely, assume that H is commutative, for any $k, l, g, h \in H$, we have

$$
\begin{aligned}
\psi_{H_4,H_4}(k \otimes l \otimes g \otimes h) &= (k \otimes l)_{[-1]} \triangleright (g \otimes h)_{(0)} \otimes (k \otimes l)_{[0]} \triangleleft (g \otimes h)_{(1)} \\
&= k_1 \triangleright (g \otimes h_1) \otimes (k_2 \otimes l) \triangleleft h_2 \\
&= k_1 g S(k_2) \otimes h_1 \otimes k_3 \otimes h_2 l S(h_3) \\
&= k_1 S(k_2) g \otimes h_1 \otimes k_3 \otimes l h_2 S(h_3) \quad by\ H\ is\ commutative \\
&= g \otimes h \otimes k \otimes l.
\end{aligned}
$$

It is clear that the braiding ψ_{H_4,H_4} is a symmetry. □

If we consider $H \otimes \Bbbk$, by Theorem 6.3, we generalize the another important result in [1].

Corollary 6.5. *Let H be a Hopf algebra with a bijective antipode, and assume that $(H, \rightharpoonup, \Delta) \in {}^H_H\mathcal{YD}$, where Δ is usual comultiplication and $\rightharpoonup$ is a left action on H. Then $\psi_{H,H}$ is a symmetry if and only if there exists a braiding $\zeta : H \otimes H \to \Bbbk$ so that (H, ζ) is cotriangular Hopf algebra. And then $\zeta(k, g) = (\varepsilon \otimes \varepsilon)\psi(k \otimes g)$, for any $k, g \in H$. That is,*

$$k \rightharpoonup g = \zeta(k, g_1) g_2.$$

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