



## On The Generalized Inequalities Of The Hermite – Hadamard Type

Juan E. Nápoles Valdés<sup>a</sup>, Bahtiyar Bayraktar<sup>b</sup>

<sup>a</sup>UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina  
UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina

<sup>b</sup>Bursa Uludag University, Faculty of Education, Gorukle Campus, 16059, Bursa, Turkey

**Abstract.** In this paper, we establish new Hermite–Hadamard inequalities for  $h$ -convex functions, with in the framework of a previously defined generalized integral. The results obtained, generalize or complete, several reported in the literature. Some final remarks show the strength and scope of our results.

### 1. Introduction

One of the most fruitful mathematical concepts in recent years, due to its multiplicity of applications and interrelationships with other sciences, is that of convex function. Subsequent extensions of this concept have appeared in recent times, which has turned it into a difficult map to unravel. To get an idea about it, we recommend the reader the work [13], where a fairly complete study is made of different notions of convexity.

In [19], the author defined the  $h$ -convex functions:

**Definition 1.1.** Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval and  $h : I \rightarrow \mathbb{R}$  be a positive function. A function  $\phi : I \rightarrow \mathbb{R}$  is said to be  $h$ -convex, if  $\phi$  is non-negative and

$$\phi(tx + (1-t)y) \leq h(t)\phi(x) + h(1-t)\phi(y) \quad (1)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1) is reversed, then  $\phi$  is said to be  $h$ -concave.

By other hand, it is known that the Fractional Calculus, that is, the Calculus with integral and differential operators of non-integer order, is as old as the classical Calculus itself, in recent times it has had a theoretical development and its applications have increased in such a way that We have many fractional operators, applied in various fields, from comprehensive inequalities to epidemic modeling. In particular, one of the operators that has had the most development has been the Riemann–Liouville Fractional Integral, on which we will focus our work. Throughout the work we use the classical Euler’s gamma function  $\Gamma$  (see [15]) and  $\Gamma_k$  (cf. defined by [5]):

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \text{ and } \Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt, \text{ with } z \in \mathbb{C}, \operatorname{Re}(z) > 0 \text{ and } k > 0$$

2020 Mathematics Subject Classification. Primary 26A33; Secondary 31A10, 26D07, 26D15

Keywords. Generalized integral operator, Hermite–Hadamard inequality, Hölder inequality, power mean inequality,  $h$ -convex function.

Received: 06 December 2020; Accepted: 22 April 2021

Communicated by Miodrag Spalević

Email addresses: [jnapoles@exa.unne.edu.ar](mailto:jnapoles@exa.unne.edu.ar) (Juan E. Nápoles Valdés), [bbayraktar@uludag.edu.tr](mailto:bbayraktar@uludag.edu.tr) (Bahtiyar Bayraktar)

It is clear that if  $k \rightarrow 1$  we have  $\Gamma_k(z) \rightarrow \Gamma(z)$ ,  $\Gamma_k(z) = (k)^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right)$  and  $\Gamma_k(z+k) = z\Gamma_k(z)$ .

To facilitate understanding of the subject, we present several definitions of fractional integrals, some very recent (eith  $0 \leq a < t < b \leq \infty$ ). The first is the classic Riemann–Liouville fractional integrals.

One of the first operators that can be called fractional is that of Riemann–Liouville fractional derivatives of order  $\alpha \in \mathbb{C}$ ,  $Re(\alpha) \geq 0$ , defined by (see [9]).

**Definition 1.2.** Let  $\phi \in L[a_1, a_2]$ ,  $(a_1, a_2) \in \mathbb{R}^2$ ,  $a_1 < a_2$ . The right and life side Riemann–Liouville fractional integrals of order  $\alpha > 0$  are defined by

$${}^{RL}J_{a_1^+}^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^t (t-s)^{\alpha-1} \phi(s) ds, t > a_1 \quad \text{and} \quad {}^{RL}J_{a_2^-}^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_t^{a_2} (s-t)^{\alpha-1} \phi(s) ds, t < a_2.$$

Other definitions of fractional operators are as follows.

The left–sided and right–sided Riemann–Liouville  $k$ –fractional integrals are given in [10].

**Definition 1.3.** Let  $\phi \in L[a_1, a_2]$ . Then the Riemann–Liouville  $k$ –fractional integrals of order  $\alpha \in \mathbb{C}$ ,  $Re(\alpha) > 0$  and  $k > 0$  are given by the expressions:

$${}^\alpha I_{a_1^+}^k \phi(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^x (x-t)^{\frac{\alpha}{k}-1} \phi(t) dt, \quad x > a_1, \quad \text{and} \quad {}^\alpha I_{a_2^-}^k \phi(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{a_2} (t-x)^{\frac{\alpha}{k}-1} \phi(t) dt, \quad x < a_2.$$

In [12] a generalized derivative was defined in the following way (see also [11] and [20]).

**Definition 1.4.** Given a function  $\phi : [0, +\infty) \rightarrow \mathbb{R}$ . Then the  $N$ -derivative of  $\phi$  of order  $\alpha$  is defined by

$$N_T^\alpha \phi(t) = \lim_{\varepsilon \rightarrow 0} \frac{\phi(t + \varepsilon T(t, \alpha)) - \phi(t)}{\varepsilon},$$

for all  $t > 0$ ,  $\alpha \in (0, 1)$  being  $T(\alpha, t)$  is some function. In some cases of  $T$  is defined using  $E_{a,b}(\cdot)$  the classic definition of Mittag–Leffler function with  $Re(a), Re(b) > 0$ . Also we consider  $E_{a,b}(\cdot)_k$  is the  $k$ -nth term of  $E_{a,b}(\cdot)$ .

If  $\phi$  is  $\alpha$ -differentiable in some  $(0, \alpha)$ , and  $\lim_{t \rightarrow 0^+} N_T^\alpha \phi(t)$  exists, then define  $N_T^\alpha \phi(0) = \lim_{t \rightarrow 0^+} N_T^\alpha \phi(t)$ , note that if  $\phi$  is differentiable, then  $N_T^\alpha \phi(t) = T(t, \alpha)\phi'(t)$ , where  $\phi'(t)$  is the ordinary derivative.

Now, we give the definition of a general fractional integral right and left sided (see [8]). Throughout the work we will consider that the integral operator kernel  $T$  defined below is an absolutely continuous function.

**Definition 1.5.** Let  $I$  be an interval  $I \subseteq \mathbb{R}$ ,  $a, t \in I$  and  $\alpha \in \mathbb{R}$ . The integral operator  $J_{T,a_1^+}^\alpha$ , right and left, is defined for every locally integrable function  $\phi$  on  $I$  as

$$J_{T,a_1^+}^\alpha(\phi)(t) = \int_{a_1}^t \frac{\phi(s)}{T(t-s, \alpha)} ds, t > a_1 \quad \text{and} \quad J_{T,b^-}^\alpha(\phi)(t) = \int_t^{a_2} \frac{\phi(s)}{T(s-t, \alpha)} ds, a_2 > t.$$

**Remark 1.6.** It is easy to see that the case of the  $J_T^\alpha$  operator defined above contains, as particular cases, the integral operators obtained from conformable and nonconformable local derivatives. However, we will see that it contains many of the known fractional integral operators.

**Remark 1.7.** We will also use the “central” integral operator defined by (see [8] and [20]):

$$J_{T,a_1}^\alpha(\phi)(a_2) = \int_{a_1}^{a_2} \frac{\phi(t)}{T(t, \alpha)} dt, a_2 > a_1.$$

Additional properties of these operators can be consulted in [8],[14] and [20].

Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function and  $a_1, a_2 \in I^\circ, a_1 < a_2$ . The following double inequality:

$$\phi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \phi(x) dx \leq \frac{\phi(a_1) + \phi(a_2)}{2} \tag{2}$$

it is called Hermite–Hadamard inequality for convex function.

In this work, using the integral operators previously defined in the Definition 1.5 we will obtain new Hermite–Hadamard inequalities for  $h$ -convex functions, throughout the work we present convenient remarks that show the strength and scope of the results obtained.

### 2. Main Results

Our first result is an extension of the Hermite–Hadamard Inequality.

**Theorem 2.1.** *Let  $\phi$  a continuous real function defined on some interval  $I, I \subset \mathbb{R}, a_1, a_2 \in I^\circ, a_1 < a_2$ . Then we have the following inequality:*

$$\frac{\Pi}{h\left(\frac{1}{2}\right)} \phi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha \phi(a_2) + J_{T, a_2-}^\alpha \phi(a_1) \right] \leq (H(t) + H(1 - t)) (\phi(a_1) + \phi(a_2)), \tag{3}$$

with  $\Pi = \int_0^1 \frac{dt}{T((a_2 - a_1)t, \alpha)}, H(t) = \int_0^1 \frac{h(t)dt}{T((a_2 - a_1)t, \alpha)}$  and  $H(1 - t) = \int_0^1 \frac{h(1-t)dt}{T((a_2 - a_1)t, \alpha)}$ .

*Proof.* For  $x = ta_1 + (1 - t)a_2, y = ta_2 + (1 - t)a_1$  and  $t = \frac{1}{2}$  from (1), we obtain

$$\phi\left(\frac{a_1 + a_2}{2}\right) \leq h\left(\frac{1}{2}\right) (\phi(ta_1 + (1 - t)a_2) + \phi(ta_2 + (1 - t)a_1)).$$

Multiplying by  $\frac{1}{T((a_2 - a_1)t, \alpha)}$  and integrating from 0 to 1, and dividing both sides of the inequality by  $h(1/2)$ , we obtain:

$$\phi\left(\frac{a_1 + a_2}{2}\right) \int_0^1 \frac{dt}{T((a_2 - a_1)t, \alpha)} \leq h\left(\frac{1}{2}\right) \left\{ \int_0^1 \frac{\phi(ta_1 + (1 - t)a_2)}{T((a_2 - a_1)t, \alpha)} dt + \int_0^1 \frac{\phi(ta_2 + (1 - t)a_1)}{T((a_2 - a_1)t, \alpha)} dt \right\},$$

or

$$\frac{\Pi}{h\left(\frac{1}{2}\right)} \phi\left(\frac{a_1 + a_2}{2}\right) \leq \int_0^1 \frac{\phi(ta_1 + (1 - t)a_2)}{T((a_2 - a_1)t, \alpha)} dt + \int_0^1 \frac{\phi(ta_2 + (1 - t)a_1)}{T((a_2 - a_1)t, \alpha)} dt.$$

Making the change of variables  $u = ta_1 + (1 - t)a_2$  in the first integral and  $u = ta_2 + (1 - t)a_1$  in the second, we easily obtain the left inequality (3).

To obtain the right inequality (3), from (1) we successively obtain

$$\begin{cases} \phi(ta_1 + (1 - t)a_2) \leq h(t)\phi(a_1) + h(1 - t)\phi(a_2), \\ \phi(ta_2 + (1 - t)a_1) \leq h(t)\phi(a_2) + h(1 - t)\phi(a_1). \end{cases} \tag{4}$$

Adding term to term inequalities (4) and multiplying by  $\frac{1}{T((a_2 - a_1)t, \alpha)}$ , and integrating, we obtain

$$\frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha \phi(a_2) + J_{T, a_2-}^\alpha \phi(a_1) \right] \leq (H(t) + H(1 - t)) (\phi(a_1) + \phi(a_2)),$$

which is the required inequality. This completes the proof.  $\square$

**Remark 2.2.** Let's take a look at some of the results covered by the previous result at  $T \equiv 1$ :

(i) we are dealing with the classical Riemann integral. Since  $\Pi \equiv 1$  if we take  $h(t) = t$ , that is, for the case of a classical convex function, we obtain the Hermite–Hadamard inequality ( 2).

(ii) If we consider the concept of  $s$ -convexity (in the second sense, see [3]), i.e.,  $h(t) = t^s$ , then from the previous theorem we obtain Theorem 2.1 from [7].

(iii) If the function is  $h$ -convex, then this result covers Theorem 6 in [16].

**Remark 2.3.** The right member of the inequality (3) coincides with the left member of Theorem 2.1 in [17], if we consider the kernel  $t^{1-\alpha}$ , that is, if we consider integrals fractional Riemann-Liouville type.

The following definition will be used in our next result (see [1]).

**Definition 2.4.** A function  $h : I \rightarrow \mathbb{R}$  is a superadditive function if

$$h(x + y) \geq h(x) + h(y), \text{ for all } x, y \in I.$$

**Theorem 2.5.** Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable on  $I^\circ$  and  $h$ -convex function such that  $\phi \in L[a_1, a_2]$ ,  $a_1, a_2 \in I^\circ$ . If  $h$  is a superadditive function, then we have the following inequality

$$\frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha \phi(a_2) + J_{T, a_2-}^\alpha \phi(a_1) \right] \leq h(1) (\phi(a_1) + \phi(a_2)) \Pi. \tag{5}$$

*Proof.* From (4) and using the superadditivity of  $h$ , we obtain the inequality:

$$\phi(ta_1 + (1 - t)a_2) + \phi(ta_2 + (1 - t)a_1) \leq h(1)(\phi(a_1) + \phi(a_2)).$$

After multiplying by  $\frac{1}{T((a_2 - a_1)t, \alpha)}$  we have

$$\int_0^1 \frac{\phi(ta_1 + (1 - t)a_2)}{T((a_2 - a_1)t, \alpha)} dt + \int_0^1 \frac{\phi(ta_2 + (1 - t)a_1)}{T((a_2 - a_1)t, \alpha)} dt \leq h(1) (\phi(a_1) + \phi(a_2)) \int_0^1 \frac{dt}{T((a_2 - a_1)t, \alpha)}.$$

From where it is obtained without much difficulty, the necessary inequality.  $\square$

**Remark 2.6.** If we use the kernel  $t^{1-\alpha}$ , instead of  $T((a_2 - a_1)t, \alpha)$ , then from this result we obtain the Theorem 2.4 of [17].

Our next result will be basic from now on. Let us denote  $\Theta(u) = \int_0^u \frac{ds}{T((b-a)s, \alpha)}$ , then true is next lemma:

**Lemma 2.7.** Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable on  $I^\circ$  such that  $\phi' \in L[a_1, a_2]$ ,  $a_1, a_2 \in I^\circ$ . If  $\phi$  is  $h$ -convex, then the following inequality holds:

$$\begin{aligned} -\Theta(1)(\phi(a_1) + \phi(a_2)) + \frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha \phi(a_2) + J_{T, a_2-}^\alpha \phi(a_1) \right] \\ = (a_2 - a_1) \int_0^1 [\Theta(1 - t) - \Theta(t)] \phi'(ta_2 + (1 - t)a_1) dt. \end{aligned} \tag{6}$$

*Proof.* Let's put

$$\begin{aligned} I &= \int_0^1 [\Theta(1 - t) - \Theta(t)] \phi'(ta_2 + (1 - t)a_1) dt \\ &= \int_0^1 \Theta(1 - t) \phi'(ta_2 + (1 - t)a_1) dt - \int_0^1 \Theta(t) \phi'(ta_2 + (1 - t)a_1) dt = I_1 - I_2. \end{aligned}$$

Integrating by parts  $I_1$  and with a change of variables we have

$$I_1 = -\frac{\Theta(1)}{a_2 - a_1} \phi(a_1) + \frac{1}{(a_2 - a_1)^2} \int_{a_1}^{a_2} \frac{\phi(u)}{T(a_2 - u, \alpha)} du.$$

Similarly for  $I_2$  we obtain

$$I_2 = \frac{\Theta(1)}{a_2 - a_1} \phi(a_2) - \frac{1}{(a_2 - a_1)^2} \int_{a_1}^{a_2} \frac{\phi(u)}{T(u - a_1, \alpha)} du.$$

By subtracting the second equality from the first, the required equality is obtained. The proof is completed.  $\square$

**Remark 2.8.** If we take the kernel  $T \equiv 1$ , and  $\Theta(s) = s$ , then this result is equivalent to Lemma 2.1 of [6].

Let's call

$$LHH = -\Theta(1)(\phi(a_1) + \phi(a_2)) + \frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha(\phi)(a_2) + J_{T, a_2-}^\alpha(\phi)(a_1) \right].$$

**Theorem 2.9.** Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable on  $I^\circ$  such that  $\phi' \in L[a_1, a_2]$ ,  $a_1, a_2 \in I^\circ$ . If  $|\phi'|$  is  $h$ -convex then the following inequality

$$|LHH| \leq (a_2 - a_1) \left( A |\phi'(a_2)| + B |\phi'(a_1)| \right) \tag{7}$$

holds, with  $A = \int_0^1 |\Theta(1-t) - \Theta(t)| h(t) dt$  and  $B = \int_0^1 |\Theta(1-t) - \Theta(t)| h(1-t) dt$ .

*Proof.* From Lemma 2.7 and using the  $h$ -convexity of  $|\phi'|$  we have

$$\begin{aligned} |LHH| &\leq (a_2 - a_1) \int_0^1 |\Theta(1-t) - \Theta(t)| |\phi'(ta_2 + (1-t)a_1)| dt \\ &\leq (a_2 - a_1) \int_0^1 |\Theta(1-t) - \Theta(t)| \left[ h(t) |\phi'(a_2)| + h(1-t) |\phi'(a_1)| \right] dt. \end{aligned}$$

Using the definition of A and B, it is very easy to conclude the proof.  $\square$

**Remark 2.10.** Under the conditions of the Remark 2.8, if we take  $h(t) = t$ , then Theorem 2.9 contains as a particular case Theorem 2.2 of [6].

**Theorem 2.11.** Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable on  $I^\circ$  such that  $\phi' \in L[a_1, a_2]$ ,  $a_1, a_2 \in I^\circ$ . If  $|\phi'|^q$  is  $h$ -convex, with  $q \geq 1$ , then we have the following inequality

$$|LHH| \leq (a_2 - a_1) C^{1-\frac{1}{q}} \left( A |\phi'(a_2)|^q + B |\phi'(a_1)|^q \right)^{\frac{1}{q}} \tag{8}$$

with A and B as before and  $C = \int_0^1 |\Theta(1-t) - \Theta(t)| dt$ .

*Proof.* From Lemma 2.7, the well-known power mean inequality and using the  $h$ -convexity of  $|\phi'|^q$  we have

$$\begin{aligned} |LHH| &\leq (a_2 - a_1) \left( \int_0^1 |\Theta(1-t) - \Theta(t)| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 |\Theta(1-t) - \Theta(t)| |\phi'(ta_2 + (1-t)a_1)|^q dt \right)^{\frac{1}{q}} \\ &\leq (a_2 - a_1) C^{1-\frac{1}{q}} \left( \int_0^1 |\Theta(1-t) - \Theta(t)| \left( h(t) |\phi'(a_2)|^q + h(1-t) |\phi'(a_1)|^q \right) dt \right)^{\frac{1}{q}} \\ &= (a_2 - a_1) C^{1-\frac{1}{q}} \left( A |\phi'(a_2)|^q + B |\phi'(a_1)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

We got the required inequality. The proof is complete.  $\square$

**Corollary 2.12.** Under the conditions of the Theorem 2.11, if we use the kernel  $t^{1-\alpha}$ , instead of  $T((a_2 - a_1)t, \alpha)$  and  $h(t) = t^s$ , then from this result we obtain the Theorem 4 of [18]:

$$\begin{aligned} & \left| \frac{\phi(a_1) + \phi(a_2)}{2} - \frac{\alpha\Gamma(\alpha)}{a_2 - a_1} \left[ {}^{RL}J_{a_1+}^\alpha(\phi)(a_2) + {}^{RL}J_{a_2-}^\alpha(\phi)(a_1) \right] \right| \\ & \leq \frac{1}{(a_2 - a_1)^\alpha} \left( \frac{2^\alpha - 1}{(\alpha + 1)2^{\alpha-1}} \right)^{1-\frac{1}{q}} \left[ |\phi'(a_2)|^q + |\phi'(a_1)|^q \right]^{\frac{1}{q}} \\ & \times \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right], \end{aligned} \tag{9}$$

where  $\beta_x(u, v)$  is incomplete Euler Beta function and  $\beta_{\frac{1}{2}}(u, v) = \int_0^x t^{u-1}(1-t)^{v-1}dt, x \in (0, 1)$ .

*Proof.* Since we use the kernel  $t^{1-\alpha}$ , then

$$\begin{aligned} \Theta(t) &= \int_0^t \frac{d\xi}{T((a_2 - a_1)\xi, \alpha)} = \int_0^t \frac{d\xi}{[(a_2 - a_1)\xi]^{1-\alpha}} = \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} t^\alpha \text{ and} \\ \Theta(1-t) &= \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} (1-t)^\alpha. \end{aligned}$$

and it should be noted that

$$|\Theta(1-t) - \Theta(t)| = \begin{cases} \Theta(1-t) - \Theta(t), & t \in [0, \frac{1}{2}] \\ \Theta(t) - \Theta(1-t) & t \in [\frac{1}{2}, 1] \end{cases}.$$

With this in mind:

$$\begin{aligned} C &= \int_0^1 |\Theta(1-t) - \Theta(t)| dt = \int_0^{\frac{1}{2}} [\Theta(1-t) - \Theta(t)] dt + \int_{\frac{1}{2}}^1 [\Theta(1-t) - \Theta(t)] dt \\ &= \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} \left\{ \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] dt \right\} = \frac{2(1 - 2^{-\alpha})}{\alpha(\alpha + 1)(a_2 - a_1)^{1-\alpha}} \end{aligned}$$

and

$$\begin{aligned} A &= \int_0^1 |\Theta(1-t) - \Theta(t)| h(t) dt = \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} \left[ \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] t^s dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] t^s dt \right] \\ &= \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right]. \end{aligned}$$

Similarly

$$\begin{aligned} B &= \int_0^1 |\Theta(1-t) - \Theta(t)| h(1-t) dt \\ &= \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right]. \end{aligned}$$

Thus, for the LHH, we can write:

$$\begin{aligned} |LHH| &= \left| -\Theta(1)(\phi(a_1) + \phi(a_2)) + \frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha(\phi)(a_2) + J_{T, a_2-}^\alpha(\phi)(a_1) \right] \right| \\ &= \left| -\frac{[\phi(a_1) + \phi(a_2)]}{\alpha(a_2 - a_1)^{1-\alpha}} + \frac{\alpha\Gamma(\alpha)}{a_2 - a_1} \left[ {}^{RL}J_{a_1+}^\alpha(\phi)(a_2) + {}^{RL}J_{a_2-}^\alpha(\phi)(a_1) \right] \right|. \end{aligned} \tag{10}$$

Or

$$\begin{aligned}
 |LHH| &\leq (a_2 - a_1)C^{1-\frac{1}{q}} \left[ A|\phi'(a_2)|^q + B|\phi'(a_1)|^q \right]^{\frac{1}{q}} \\
 &\leq (a_2 - a_1) \left[ \frac{2\left(1 - \frac{1}{2^\alpha}\right)}{\alpha(\alpha + 1)(a_2 - a_1)^{1-\alpha}} \right]^{1-\frac{1}{q}} \left[ \frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} \left( |\phi'(a_2)|^q + |\phi'(a_1)|^q \right) \right]^{\frac{1}{q}} \\
 &\times \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right] \\
 &= \frac{a_2 - a_1}{\alpha(a_2 - a_1)^{1-\alpha}} \left[ \frac{2^\alpha - 1}{(\alpha + 1)2^{\alpha-1}} \right]^{1-\frac{1}{q}} \left[ |\phi'(a_2)|^q + |\phi'(a_1)|^q \right]^{\frac{1}{q}} \\
 &\times \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right].
 \end{aligned}
 \tag{11}$$

Taking into account (10) and (11), we have

$$\begin{aligned}
 &\left| -\frac{1}{\alpha(a_2 - a_1)^{1-\alpha}} [\phi(a_1) + \phi(a_2)] + \frac{\alpha\Gamma(\alpha)}{a_2 - a_1} \left[ {}^{RL}J_{a_1+}^\alpha(\phi)(a_2) + {}^{RL}J_{a_2-}^\alpha(\phi)(a_1) \right] \right| \\
 &\leq (a_2 - a_1) \left( \frac{2^\alpha - 1}{(\alpha + 1)2^{\alpha-1}} \right)^{1-\frac{1}{q}} \left[ |\phi'(a_2)|^q + |\phi'(a_1)|^q \right]^{\frac{1}{q}} \\
 &\times \left[ \beta_{\frac{1}{2}}(s + 1, \alpha + 1) - \beta_{\frac{1}{2}}(\alpha + 1, s + 1) + \frac{2^{\alpha+s} - 1}{(\alpha + s + 1)2^{\alpha+s}} \right].
 \end{aligned}$$

or, multiplying the last inequality by the expression  $\frac{\alpha(a_2-a_1)^{1-\alpha}}{2}$  we get (9). Proof is completed.  $\square$

This inequality was proved in [18] (Theorem 4). A variant of the result obtained in Theorem 2.11 is presented below.

**Theorem 2.13.** *Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable on  $I^\circ$  such that  $\phi' \in L[a_1, a_2]$ ,  $a_1, a_2 \in I^\circ$ . If  $|\phi'|^q$  is  $h$ -convex, with  $q > 1$ , then we have the following inequality*

$$|LHH| \leq (a_2 - a_1) \left( D^{\frac{1}{p}} + E^{\frac{1}{q}} \right) \left( |\phi'(a_2)|^q \int_0^1 h(t)dt + |\phi'(a_1)|^q \int_0^1 h(1-t)dt \right)^{\frac{1}{q}}
 \tag{12}$$

with  $D = \int_0^1 \Theta^p(1-t)dt$ ,  $E = \int_0^1 \Theta^p(t)dt$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* From Lemma 2.7, using the Hölder inequality and the  $h$ -convexity of  $|\phi'|^q$  we obtain

$$\begin{aligned}
 |LHH| &\leq (a_2 - a_1) \left( \int_0^1 \Theta(1-t) |\phi'(ta_2 + (1-t)a_1)| dt + \int_0^1 \Theta(t) |\phi'(ta_2 + (1-t)a_1)| dt \right) \\
 &\leq (a_2 - a_1) \left( \int_0^1 \Theta^p(1-t) dt \right)^{\frac{1}{p}} \left( \int_0^1 |\phi'(ta_2 + (1-t)a_1)|^q dt \right)^{\frac{1}{q}} \\
 &\quad + (a_2 - a_1) \left( \int_0^1 \Theta^p(t) dt \right)^{\frac{1}{p}} \left( \int_0^1 |\phi'(ta_2 + (1-t)a_1)|^q dt \right)^{\frac{1}{q}} \\
 &= (a_2 - a_1) \left( D^{\frac{1}{p}} + E^{\frac{1}{q}} \right) \left( |\phi'(a_2)|^q \int_0^1 h(t)dt + |\phi'(a_1)|^q \int_0^1 h(1-t)dt \right)^{\frac{1}{q}}.
 \end{aligned}$$

Required inequality that completes the Theorem’s proof.  $\square$

**Remark 2.14.** *Under the conditions of the Remarks 2.10, the Theorem 2.3 of [6] is contained in this result.*

### 3. Conclusions

In our work, we have obtained different variants of the Hermite–Hadamard Inequality, in the context of the generalized integral operators of the Definition 1.5. The results achieved contain, as a particular case, several known from the literature.

On the other hand, the generalized integral operator used in this paper opens new directions of work, so we can state, by way of illustration, the following results that can be obtained by following the ideas presented. Defining the mapping  $L$  by  $L = \frac{1}{a_2 - a_1} \left[ J_{T, a_1+}^\alpha \phi(a_2) + J_{T, a_2-}^\alpha \phi(a_1) \right]$  we can generalize the results obtained in [4]. On the other hand, if instead of considering the Hermite–Hadamard Inequality, we consider the Hermite–Hadamard–Fejér version, then we can extend the results of [2].

### Acknowledgment

We are grateful to the editors and the anonymous reviewers for their valuable comments and remarks which led to improvements on the paper.

### References

- [1] H. Alzer, A superadditive property of Hadamard’s gamma function, *Abh. Math. Semin. Univ. Hambg.*, 79 (2009) 11–23.
- [2] M. Bombardelli, S. Varosanec, Properties of  $h$ -convex functions related to the Hermite–Hadamard–Fejér inequalities, *Computers and Mathematics with Applications* 58 (2009) 1869–1877
- [3] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen, *Pupl. Inst. Math.*, 23 (1978) 13–20.
- [4] M. R. Delavar, M. De La Sen, A mapping associated to  $h$ -convex version of the Hermite–Hadamard inequality with applications, *J. Math. Inequal.*, 14(2) (2020) 329–335
- [5] R. Díaz, E. Pariguan, On hypergeometric functions and Pochhammer  $k$ -symbol. *Divulgaciones Matemáticas* 15(2) (2007) 179–192
- [6] S. S. Dragomir, P. A. Agarwal, Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, *Appl. Math. Lett.*, 11(5) (1998) 91–95,
- [7] S. S. Dragomir, S. Fitzpatrick, The Hadamard’s inequality for  $s$ -convex functions in the second sense, *Demonstratio Math.*, 32(4) (1999) 687–696.
- [8] P. M. Guzmán, L. M. Lugo, J. E. Nápoles Valdés, M. Vivas, On a New Generalized Integral Operator and Certain Operating Properties, *Axioms* 9(2) (2020), 69; doi:10.3390/axioms9020069
- [9] R. Gorenflo, F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer, Wien (1997) 223–276.
- [10] S. Mubeen, G. M. Habibullah,  $k$ -fractional integrals and applications., *Int. J. Contemp. Math. Sci.* 7,(2012) 89–94
- [11] J. E. Nápoles Valdés, P. M. Guzmán, L. M. Lugo, Some new results on Nonconformable fractional calculus, *Advances in Dynamical Systems and Applications*, 13(2) (2018) 167–175.
- [12] J. E. Nápoles, P. M. Guzmán, L. M. Lugo, A. Kashuri, The local generalized derivative and Mittag Leffler function, *Sigma J Eng & Nat Sci*, 38 (2) (2020) 1007-1017.
- [13] J. E. Nápoles Valdés, F. Rabossi, A. D. Samaniego, Convex functions: Ariadne’s Thread or Charlotte’s Spiderweb?, *Advanced Mathematical Models & Applications*, 5(2) (2020) 176–191
- [14] J. E. Nápoles V., M. Vivas Cortez, On a generalized fractional integral and related methodological remarks. (Submitted)
- [15] E. D. Rainville, *Special Functions*. Macmillan Co., New York 1960.
- [16] M. Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard–type inequalities for  $h$ -convex functions, *J. Math. Inequal.*, 2(3) (2008) 335–341
- [17] M. Tunc, On new inequalities for  $h$ -convex functions via Riemann–Liouville fractional integration, *Filomat*, 27(4) (2013) 559–565
- [18] E. Set, M. Z. Sarikaya, M. E. Özdemir and H. Yildirim, The Hermite–Hadamard’s inequality for some convex functions via fractional integrals and related results, *Journal of Applied Mathematics, Statistics and Informatics*, 10(2) (2014) 69–83
- [19] S. Varosanec, On  $h$ -convexity, *J. Math. Anal. Appl.* 326 (2007) 303–311.
- [20] D. Zhao and M. Luo, General conformable fractional derivative and its physical interpretation, *Calcolo*, 54 (2017) 903–917, DOI 10.1007/s10092-017-0213-8.