Filomat 35:14 (2021), 4675–4690
https://doi.org/10.2298/FIL2114675D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Disjoint Sums of M-Fuzzifying Convex Spaces

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Abstract. In this paper, we first extend the concept of the arity in crisp convex spaces to the case of fuzzification and give some related properties. From the view of arity and hull operator, we study the relations between the disjoint sum of M-fuzzifying convex spaces and its factor spaces. We also examine the additivity of the degree of separability $(S_0, S_1, S_2, S_3, S_4)$. Finally, we show that every factor space is M-fuzzifying JHC iff the corresponding disjoint sum space is JHC.

1. Introduction

Convexity, has been an indispensable tool in studying of extremum problems of many fields. The notion of convexity derives from solving some elementary geometric problems in Euclidean spaces [1]. In fact, many branches of mathematics are closely related to convex theory, such as algebra [12], graphs [3–5], topology [8, 21]. Many mathematical concepts have been generalized to fuzzy case since the notion of fuzzy sets was introduced by Zadeh [35] such as fuzzy algebras [6], fuzzy topology [28], fuzzy convergence [14, 27] and so on. Considering the axiomatic approach, fuzzy convex spaces was introduced by Rosa [17] as a natural extension of the concept of abstract convex structures [22]. Subsequently, Maruyama [13] further proposed the notion of L-fuzzy convex spaces under the framework of a completely distributive lattice L. In both cases of fuzzy convex spaces and L-fuzzy convex spaces, every convex set is fuzzy, but the convex space formed by these fuzzy convex sets is thought to be crisp. Recently, L-convex structures are studied by many researchers in [2, 7, 9, 15, 16, 18, 29].

To provide a new approach to the fuzzification of convex spaces, the notion of M-fuzzifying convex spaces was proposed by Shi and Xiu [20] under the frame of a completely distributive lattice M.

In fact, an M-fuzzifying convexity $\mathscr{C}$ on X is a mapping from 2^X to M and $\mathscr{C}$ satisfying three axiomatic conditions. In this sense, for any subset A of X, $\mathscr{C}(A)$ can be seen as the degree to which A is a convex set. Subsequently, the notion of restricted hull operators in classical convex spaces was extended to the M-fuzzifying case [19], it was shown that M-fuzzifying restricted hull operators and M-fuzzifying convex spaces can be induced by each other, which means that there is a one-to-one correspondence between them. M-fuzzifying JHC property was studied in detail by Wu and Shi [24]. Recently, Liang etl., [10, 11] introduced S_0, S_1, S_2, S_3, S_4 separation axioms in M-fuzzifying convex spaces, it means that every M-fuzzifying convex

2020 *Mathematics Subject Classification*. Primary 03E72; Secondary 52A01, 54A40
Keywords. M-fuzzifying convex space, disjoint sum of M-fuzzifying convex spaces, arity, separability, additivity
Received: 07 November 2020; Revised: 10 August 2021; Accepted: 13 August 2021
Communicated by Ljubiša D.R. Kočinac
This project is supported by the National Natural Science Foundation of China (11871097, 12071033), Beijing Institute of Technology Science and Technology Innovation Plan Cultivation Project (No. 2020CX01030).
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space can be seen as S_0, S_1, S_2, S_3, S_4 separated in a certain degree. There are many other studies related to M-fuzzifying convex spaces [23, 25, 30–33]

As we all know, it is a common method to construct a new space by using given spaces and the new space is closely connected with its initial spaces. There are many studies on subspaces, product spaces and quotient spaces. For example, Zhou and Shi [36] discussed the hereditary properties and productive properties of separability in L-convex spaces. In 2014, the concept of disjoint sums of M-fuzzifying convex spaces was proposed by Shi and Xiu [20], but beyond that disjoint sums of M-fuzzifying convex spaces have not been studied in detail. So it is necessary to continue to study some properties of the disjoint sum of M-fuzzifying convex spaces and establish the relations between the sum space and its factor spaces.

In Section 2, we will review some necessary notations and definitions in M-fuzzifying convex spaces. In Section 3, we will introduce the notion of the arity of an M-fuzzifying convex space. Furthermore, we will investigate the relations between the arity of a disjoint sum of M-fuzzifying convex spaces and the arity of its factor spaces. In Section 4, we will study the additivity of some properties such as separability (S_0, S_1, S_2, S_3, S_4) in M-fuzzifying convex spaces. And we also prove that every factor space is M-fuzzifying JHC iff the corresponding disjoint sum space is M-fuzzifying JHC.

2. Preliminaries

Throughout this paper, 2^X stands for the power set of a nonempty X and 2^X_{fin} represents the collection of all finite subsets of X. In this paper, M is a completely distributive lattice with an order-reversing involution $'$. We denote the minimal element and the maximal element of M by $\perp$ and $\top$, respectively. The symbol M^X represents the family of all M-fuzzy sets of X. For $A \in 2^X$, we use $\bigvee A$ and $\bigwedge A$ to denote the supremum and infimum of A. Let $m, n \in M$, the symbol $m \prec n$ (m is wedge below n) means that for every $E \subseteq M$, $n \leq \bigvee E$ implies the existence of $e \in E$ such that $m \leq e$. The right adjoint $\rightarrow$ of the meet operation $\wedge$ is a mapping from $M \times M$ to M defined as $m \rightarrow n = \bigvee\{q \in M | m \wedge q \leq n\}$. Hence

$$m \wedge q \leq n \Leftrightarrow q \leq m \rightarrow n.$$

The mapping $\psi^{\rightarrow} : M^X \longrightarrow M^Y$ is induced by $\psi : X \longrightarrow Y$ as follows:

$$\forall \lambda \in M^X, \forall y \in Y,\ \ \psi^{\rightarrow}(\lambda)(y) = \bigvee_{\psi(x)=y} \lambda(x).$$

And $\psi^{\leftarrow} : M^Y \longrightarrow M^X$ is induced by ψ as follows:

$$\forall \mu \in M^Y, \forall x \in X\ \ \psi^{\leftarrow}(\mu)(x) = \mu(\psi(x)).$$

Definition 2.1. ([20]) An M-fuzzifying convexity on a set X is a mapping $\mathscr{C} : 2^X \longrightarrow M$ satisfying the following conditions:

(MYC1) $\mathscr{C}(\phi) = \mathscr{C}(X) = \top$;

(MYC2) $\mathscr{C}(\bigcap_{i\in T} G_i) \geq \bigwedge_{i\in T} \mathscr{C}(G_i)$, where $\{G_i\}_{i\in T} \subseteq 2^X \backslash \emptyset$;

(MYC3) $\mathscr{C}(\bigcup_{i\in T} G_i) \geq \bigwedge_{i\in T} \mathscr{C}(G_i)$, where $\{G_i\}_{i\in T} \subseteq 2^X \backslash \emptyset$ is totally ordered by inclusion.

In this case, We say the pair $(X, \mathscr{C})$ is an M-fuzzifying convex space.

Definition 2.2. ([20]) Assume that $(X, \mathscr{C})$ is an M-fuzzifying convex space and $\emptyset \neq G \subseteq X$. Then the mapping $\mathscr{C}|_G : 2^G \longrightarrow M$ given by

$$\forall B \in 2^G, \mathscr{C}|_G(B) = \bigvee_{D\in 2^X, D\cap G = B} \mathscr{C}(D)$$

is an M-fuzzifying convexity on G. Furthermore, $(G, \mathscr{C}|_G)$ is called an M-fuzzifying subspace of $(X, \mathscr{C})$.

Theorem 2.3. ([20]) *Assume that* $(X,\mathscr{C})$ *is an M-fuzzifying convex space. Then the mapping* $co_{\mathscr{C}}: 2^X \longrightarrow M^X$ *(in symbols, co) given by :*

$$\forall G \in 2^X, \forall x \in X,\ co(G)(x) = \bigwedge_{x \notin D \supseteq G} \mathscr{C}(D)'$$

is a hull operator of $\mathscr{C}$ *such that the following conditions hold.*

(MCO1) *for each* $x \in X$, $co(\emptyset)(x) = \perp$;
(MCO2) *for each* $x \in G$, $co(G)(x) = \top$;
(MCO3) $co(G)(x) = \bigwedge_{x \notin D \supseteq G} \bigvee_{y \notin D} co(D)(y)$;

(MFD) $co(G)(x) = \bigvee_{F \in 2^G_{fin}} co(F)(x)$.

On the contrary, an operator $co : 2^X \longrightarrow M^X$ *satisfying* $(MCO1) - (MCO3)$ *and* (MFD) *can be used to induce an M-fuzzifying convexity* $\mathscr{C}_{co}$ *on X as follows:*

$$\forall G \in 2^X,\ \mathscr{C}_{co}(G) = \bigwedge_{x \notin G} [co(G)(x)]'. \qquad (1)$$

Furthermore, co is the hull operator of $\mathscr{C}_{co}$. *That is to say* $co_{\mathscr{C}_{co}} = co$.

Definition 2.4. ([20]) Assume that $\psi : (X,\mathscr{C}) \longrightarrow (Y,\mathscr{D})$ is a function between two M-fuzzifying convex spaces. Then

(i) ψ is called an M-fuzzifying convexity preserving function (in symbols, M-CP) provided that

$$\forall D \in 2^Y,\ \mathscr{C}(\psi^{-1}(D)) \geq \mathscr{D}(D).$$

(ii) ψ is called an M-fuzzifying convex-to-convex function (in symbols, M-CC) provided that

$$\forall B \in M^X,\ \ \mathscr{D}(\psi(B)) \geq \mathscr{C}(B).$$

(iii) ψ is called an M-fuzzifying isomorphism provided that ψ is bijective, M-CP and M-CC.

Theorem 2.5. ([26]) *A function* $\psi : (X,\mathscr{C}) \longrightarrow (Y,\mathscr{D})$ *between two M-fuzzifying convex spaces is M-CP iff*

$$\forall F \in 2^X_{fin},\ \psi^{\rightarrow}(co_X(F)) \leq co_Y(\psi(F)).$$

A function $\psi : (X,\mathscr{C}) \longrightarrow (Y,\mathscr{D})$ *between two M-fuzzifying convex spaces is M-CC if and only if*

$$\forall F \in 2^X_{fin},\ \psi^{\rightarrow}(co_X(F)) \geq co_Y(\psi(F)).$$

Definition 2.6. ([20]) Assume that that $\{(X_i,\mathscr{C}_i)\}_{i\in T}$ is a family of M-fuzzifying convex space and for all $i_1 \neq i_2 \in T$ such that $X_{i_1} \cap X_{i_2} = \emptyset$ (i.e., pairwise disjoint). Put $X = \bigcup_{i\in T} X_i$ and consider the usual inclusion mapping $j_i : X_i \longrightarrow X$ for all $i \in T$ (i.e.,$\forall z \in X_i,\ j_i(z) = z$). Then the mapping $\mathscr{C} : 2^X \longrightarrow M$ given by:

$$\forall B \in 2^X, \mathscr{C}(B) = \bigwedge_{i\in T} \mathscr{C}_i(j_i^{-1}(B)) = \bigwedge_{i\in T} \mathscr{C}_i(B \cap X_i)$$

is an M-fuzzifying convexity on X, which is called the disjoint sum of M-fuzzifying convexity $\{\mathscr{C}_i\}_{i\in T}$ and $\mathscr{C}$ is written as $\sum_{i\in T} \mathscr{C}_i$. And we say the pair $(X, \sum_{i\in T} \mathscr{C}_i)$ is the disjoint sum of M-fuzzifying convex spaces $\{(X_i,\mathscr{C}_i)\}_{i\in T}$.

Definition 2.7. ([10]) Assume that $(X,\mathscr{C})$ is an M-fuzzifying convex space and $B \in 2^X$. We say $\mathcal{H}_{\mathscr{C}}(B)$ given by

$$\mathcal{H}_{\mathscr{C}}(B) = \mathscr{C}(B) \wedge \mathscr{C}(X \backslash B)$$

is the degree that B is a biconvex set.

Definition 2.8. ([10, 11]) Assume that $(X, \mathscr{C})$ is an M-fuzzifying convex space. Then we have the following definitions.

(S0) The degree $S_0(X, \mathscr{C})$ that $(X, \mathscr{C})$ is S_0 separated is defined by:

$$S_0(X, \mathscr{C}) = \bigwedge_{x \neq z} \left(\bigvee_{x \notin B, z \in B} \mathscr{C}(B) \vee \bigvee_{z \notin D, x \in D} \mathscr{C}(D) \right).$$

(S1) The degree $S_1(X, \mathscr{C})$ that $(X, \mathscr{C})$ is S_1 separated is defined by: $S_1(X, \mathscr{C}) = \bigwedge_{z \in X} \mathscr{C}(\{z\})$.

(S2) The degree $S_2(X, \mathscr{C})$ that $(X, \mathscr{C})$ is S_2 separated is defined by: $S_2(X, \mathscr{C}) = \bigwedge_{x \neq z} \bigvee_{x \in B, z \notin B} \mathcal{H}_{\mathscr{C}}(B)$.

(S3) The degree $S_3(X, \mathscr{C})$ that $(X, \mathscr{C})$ is S_3 separated is defined by:

$$S_3(X, \mathscr{C}) = \bigwedge_{B \subseteq X} \bigwedge_{z \notin B} \left[\mathscr{C}(B) \to \left(\bigvee_{B \subseteq D, z \notin D} \mathcal{H}_{\mathscr{C}}(D) \right) \right].$$

(S4) The degree $S_4(X, \mathscr{C})$ that $(X, \mathscr{C})$ is S_4 separated is defined by:

$$S_4(X, \mathscr{C}) = \bigwedge_{B \cap D = \emptyset} \left[\mathscr{C}(B) \wedge \mathscr{C}(D) \to \left(\bigvee_{D \subseteq X \backslash H, B \subseteq H} \mathcal{H}_{\mathscr{C}}(H) \right) \right].$$

Theorem 2.9. ([10, 11]) *Assume that $(G, \mathscr{C}|_G)$ is the subspace of an M-fuzzifying convex space $(X, \mathscr{C})$. Then*

(i) $S_0(X, \mathscr{C}) \leq S_0(G, \mathscr{C}|_G)$,

(ii) $S_1(X, \mathscr{C}) \leq S_1(G, \mathscr{C}|_G)$,

(iii) $S_2(X, \mathscr{C}) \leq S_2(G, \mathscr{C}|_G)$,

(iv) $S_3(X, \mathscr{C}) \leq S_3(G, \mathscr{C}|_G)$,

(v) $S_4(X, \mathscr{C}) \wedge \mathscr{C}(G) \leq S_4(G, \mathscr{C}|_G)$.

Definition 2.10. ([24]) Assume that $(X, \mathscr{C})$ is an M-fuzzifying convex space. We say $\mathscr{C}$ is an M-fuzzifying JHC convexity if for arbitrary $y, c \in X$ and $B \in 2^X \backslash \emptyset$,

$$co(\{c\} \cup B)(y) = \bigvee_{x \in X} \Big(co(\{c, x\})(y) \wedge co(B)(x) \Big).$$

3. The Arity of the Disjoint Sum of Convex Spaces

The arity plays an important role in classical convex spaces because it indicates the ability of finite subsets generating the entire space by hull operators. Yao and Chen [34] gave a formal and strict definition of the arity of classical convex space. Next, we will first generalize this concept to M-fuzzifying convex spaces and give some properties. Based on this, we will further study the relations between the arity of a disjoint sum of M-fuzzifying convex spaces and its factor spaces.

Definition 3.1. The arity of an M-fuzzifying convex space $(X, \mathscr{C})$ is the least natural number n such that:

$$\forall B \in 2^X, \mathscr{C}(B) = \bigwedge_{z \notin B} \bigwedge_{|F| \leq n, F \subseteq B} [co(F)(z)]'. \qquad (2)$$

Let us denote the arity of $(X, \mathscr{C})$ by $ary(\mathscr{C})$.

Notation. According to the above definition, it is evident that $ary(\mathscr{C}) \leq n$ iff it satisfies the equality (2), this happens to be the definition of $arity \leq n$ given in [26].

Remark 3.2. From Theorem 2.3 (MDF) and equality (1), we can see

$$\mathscr{C}(B) = \bigwedge_{z \notin B} [co(B)(z)]' = \bigwedge_{z \notin B} \bigwedge_{F \in 2^B_{fin}} [co(F)(z)]' \leq \bigwedge_{z \notin B} \bigwedge_{|F| \leq n, F \subseteq B} [co(F)(z)]'.$$

So in order to prove the equality (2) holds, it shall be proved that the following inequality holds:

$$\bigwedge_{z \notin B} \bigwedge_{|F| \leq n, F \subseteq B} [co(F)(z)]' \leq \mathscr{C}(B).$$

Proposition 3.3. *Assume that $(X, \mathscr{C})$ is an M-fuzzifying convex space. Then $ary(\mathscr{C}) = n$ implies*

$$\forall m \geq n, \bigwedge_{z \notin B} \bigwedge_{|F| \leq m, F \subseteq B} [co(F)(z)]' = \mathscr{C}(B).$$

Proof. By Remark 3.2, we only need to prove $\bigwedge_{z \notin B} \bigwedge_{|F| \leq m, F \subseteq B} [co(F)(z)]' \leq \mathscr{C}(B)$. Since $ary(\mathscr{C}) = n$ and $m \geq n$, we have

$$\bigwedge_{z \notin B} \bigwedge_{|F| \leq m, F \subseteq B} [co(F)(z)]' \leq \bigwedge_{z \notin B} \bigwedge_{|F| \leq n, F \subseteq B} [co(F)(z)]' = \mathscr{C}(B),$$

which completes the proof. □

Proposition 3.4. *Assume that $\psi : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is an injection between two M-fuzzifying convex spaces. If ψ is M-CP and M-CC, then $ary(\mathscr{C}_X) \leq ary(\mathscr{C}_Y)$.*

Proof. Suppose $ary(\mathscr{C}_Y) = n$, so we have

$$\bigwedge_{y \notin D} \bigwedge_{|G| \leq n, G \subseteq D} [co(G)(y)]' = \mathscr{C}_Y(D).$$

It is sufficient to show that $ary(\mathscr{C}_X) \leq n$. Since ψ is an injection, an M-CP function, and an M-CC function, then by Theorem 2.5 we have

$$co_Y(\psi(F))(\psi(z)) = \psi^{\rightarrow}(co_X(F))(\psi(z)) = \bigvee_{\psi(c) = \psi(z)} co_X(F)(c) = co_X(F)(z).$$

For $B \in 2^X$ and $z \in X$, we can see that

$$\begin{aligned}
\bigwedge_{z\notin B} \bigwedge_{\substack{F\subseteq B\\ |F|\leq n}} [co_X(F)(z)]' &= \bigwedge_{z\notin B} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_X(F)(z)]' \\
&= \bigwedge_{z\notin B} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(\psi(z))]' \\
&= \bigwedge_{\psi(z)\notin\psi(B)} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(\psi(z))]' \\
&= \bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)\neq\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]' \\
&= \left(\bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)\neq\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]' \right) \wedge \top.
\end{aligned}$$

Next, we want to replace $\top$ with $\bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)=\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]'$. To do this, we must prove

$$\bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)=\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]' = \top.$$

Take $y \notin \psi(B)$ such that $\psi^{-1}(y) = \emptyset$ and $F \in 2^B_{fin}$, then by Theorem 2.5

$$co_Y(\psi(F))(y) = \psi^{\rightarrow}(co_X(F))(y) = \bigvee_{\psi(c)=y} co_X(F)(c) = \bot.$$

This implies our statement holds.
Therefore,

$$\begin{aligned}
\bigwedge_{z\notin B} \bigwedge_{\substack{F\subseteq B\\ |F|\leq n}} [co_X(F)(z)]' &= \left(\bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)\neq\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]' \right) \wedge \left(\bigwedge_{\substack{y\notin\psi(B)\\ \psi^{-1}(y)=\emptyset}} \bigwedge_{\substack{\psi(F)\subseteq\psi(B)\\ |\psi(F)|\leq n}} [co_Y(\psi(F))(y)]' \right) \\
&= \bigwedge_{y\notin\psi(B)} \bigwedge_{|U|\leq n, U\subseteq\psi(B)} [co_Y(U)(y)]' \\
&= \mathscr{C}_Y(\psi(B)) \qquad \text{(by } ary(\mathscr{C}_Y) = n\text{)} \\
&= \mathscr{C}_X(\psi^{-1}(\psi(B))) = \mathscr{C}_X(B). \qquad \text{(since } \psi \text{ is injective } M\text{-}CP, M\text{-}CC.\text{)}
\end{aligned}$$

We thus get

$$\bigwedge_{z\notin B} \bigwedge_{|F|\leq n, F\subseteq B} [co_X(F)(z)]' \leq \mathscr{C}_X(B).$$

Therefore, $ary(\mathscr{C}_X) \leq n$. □

Corollary 3.5. *Let* $(X, \mathscr{C}) = (X, \sum_{i\in T} \mathscr{C}_i)$. *Then* $ary(\mathscr{C}_i) = n_i$ *(* $\forall i \in T$*) implies* $ary\left(\sum_{i\in T} \mathscr{C}_i\right) \geq \bigvee_{i\in T} n_i$.

Proof. Consider the usual inclusion mapping $j_i : X_i \longrightarrow X$ ($\forall i \in T$). Obviously, j_i is an injection, an M-CP function, and an M-CC function. It follows from Proposition 3.4 that $ary\left(\sum_{i\in T}\mathscr{C}_i\right) \geq n_i$ ($\forall i \in T$). So we have $ary\left(\sum_{i\in T}\mathscr{C}_i\right) \geq \bigvee_{i\in T} n_i$. □

To further study the arity of a disjoint sum of M-fuzzifying convex spaces, the following lemmas are necessary.

Lemma 3.6. *Assume that $(G, \mathscr{C}|_G)$ is the M-fuzzifying subspace of an M-fuzzifying space $(X, \mathscr{C})$. Then $co_{\mathscr{C}}(B) \geq co_{\mathscr{C}|_G}(B \cap Y)$ for all $B \in 2^X$.*

Proof. By Theorem 2.3 we get $\forall B \in 2^X$ and $\forall z \in X$, $co_{\mathscr{C}}(B)(z) = \bigwedge_{x\notin D\supseteq B}(\mathscr{C}(D))'$. Now we claim that $co_{\mathscr{C}}(B) \geq co_{\mathscr{C}|_G}(B \cap G)$, we consider two cases below:

Case 1: $z \in G$. Then for each $D \in 2^X$,

$$\mathscr{C}|_G(D \cap G) = \bigvee_{E\cap G=D\cap G}\mathscr{C}(E) \geq \mathscr{C}(D).$$

It implies that $\mathscr{C}(D)' \geq (\mathscr{C}|_G(D \cap G))'$. Further, we have

$$\begin{aligned} co_{\mathscr{C}}(B)(z) = \bigwedge_{z\notin D\supseteq B}(\mathscr{C}(D))' &\geq \bigwedge_{z\notin D\cap G\supseteq B\cap G}(\mathscr{C}(D))' \\ &\geq \bigwedge_{z\notin D\cap G\supseteq B\cap G}(\mathscr{C}|_G(D \cap G))' \\ &= \bigwedge_{z\notin U\supseteq B\cap G}(\mathscr{C}|_G(U))' \quad (\textit{where } U \subseteq G) \\ &= co_{\mathscr{C}|_G}(B \cap G)(z). \end{aligned}$$

Case 2: $z \notin G$. Since

$$co_{\mathscr{C}|_G}(B \cap G)(z) = \bigwedge_{z\notin U\supseteq B\cap G}(\mathscr{C}|_G(U))' \leq (\mathscr{C}|_G(G))' = \bot,$$

we have $co_{\mathscr{C}}(B)(z) \geq co_{\mathscr{C}|_G}(B \cap G)(z)$. Therefore, $co_{\mathscr{C}}(B) \geq co_{\mathscr{C}|_G}(B \cap G)$. □

Lemma 3.7. *Assume that $(X, \mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$. If $U \in 2^{X_{i_0}}$ $(i_0 \in T)$, then $\mathscr{C}(U) = \mathscr{C}_{i_0}(U)$.*

Proof. Since $U \in 2^{X_{i_0}}$, we have $\forall i \neq i_0$, $U \cap X_i = \emptyset$. It implies $\mathscr{C}_i(U \cap X_i) = \mathscr{C}_i(\emptyset) = \top$ when $i \neq i_0$. Therefore,

$$\mathscr{C}(U) = \bigwedge_{i\in T}\mathscr{C}_i(U \cap X_i) = \mathscr{C}_{i_0}(U).$$

□

Lemma 3.8. *Assume that $(X, \mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$. Then $\mathscr{C}|_{X_i} = \mathscr{C}_i$.*

Proof. For any $i \in T$ and $V \in 2^{X_i}$, then $V \cap X_i = V$. It follows from Lemma 3.7 that

$$\mathscr{C}|_{X_i}(V) = \bigvee_{A\cap X_i=V}\mathscr{C}(A) \geq \mathscr{C}(V) = \mathscr{C}_i(V).$$

Conversely,

$$\mathscr{C}|_{X_i}(V) = \bigvee_{A\cap X_i=V}\mathscr{C}(A) = \bigvee_{A\cap X_i=V}\ \bigwedge_{j\in T}\mathscr{C}_j(A \cap X_j) \leq \bigvee_{A\cap X_i=V}\mathscr{C}_i(A \cap X_i) = \mathscr{C}_i(V).$$

Therefore, $\mathscr{C}|_{X_i} = \mathscr{C}_i$. □

Lemma 3.9. *Assume that* $(X,\mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$ *and* $x \in X$. *Then there is an unique* $i_0 \in T$ *such that* $x \in X_{i_0}$. *Furthermore,* $\forall A \in 2^X$, $co_i(A \cap X_i)(x) = \bot$ *when* $i \neq i_0$.

Proof. Since $X = \bigcup_{i\in T} X_i$, we know that there exists $i_0 \in T$ such that $x \in X_{i_0}$ and for any $i \neq i_0$, $X_i \cap X_{i_0} = \emptyset$, which gives for any $i \neq i_0$, $x \notin X_i$. Take $A \in 2^X$, it follows from Theorem 2.3 that

$$co_i(A \cap X_i)(x) = \bigwedge_{x\notin U\supseteq A\cap X_i} \mathscr{C}_i(U)' \leq \mathscr{C}_i(X_i)' = \bot.$$

□

In the following, we study the relations between the hull operator of a disjoint sum of M-fuzzifying convex spaces and its factor spaces.

Theorem 3.10. *Assume that* $(X,\mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$. *Then for every* $x \in X$ *and* $A \in 2^X$, $co_{\mathscr{C}}(A)(x) = \bigvee_{i\in T} co_i(A \cap X_i)(x)$.

Proof. By Lemma 3.8, we see that $\forall i \in T$, $\mathscr{C}|_{X_i} = \mathscr{C}_i$. Again by Lemma 3.6 we get that $\forall i \in T, co_{\mathscr{C}}(A) \geq co_i(A \cap X_i)$, which means $co_{\mathscr{C}}(A) \geq \bigvee_{i\in T} co_i(A \cap X_i)$.

Conversely, fixing $x \in X$, so by Lemma 3.9 there exists $i_0 \in T$ such that $x \in X_{i_0}$ and $\forall i \neq i_0$, $co_i(A \cap X_i)(x) = \bot$. It implies that

$$\bigvee_{i\in T} co_i(A \cap X_i)(x) = co_{i_0}(A \cap X_{i_0})(x).$$

Therefore, to show $co_{\mathscr{C}}(A)(x) \leq \bigvee_{i\in T} co_i(A \cap X_i)(x)$, we just need to show that $co_{\mathscr{C}}(A)(x) \leq co_{i_0}(A \cap X_{i_0})(x)$. That is

$$\bigwedge_{x\notin B\supseteq A} \mathscr{C}(B)' \leq \bigwedge_{x\notin V\supseteq A\cap X_{i_0}} \mathscr{C}_{i_0}(V)'.$$

For any $V \in 2^{X_{i_0}}$ with $x \notin V \supseteq A \cap X_{i_0}$, we take $B_* = (\bigcup_{i\neq i_0} X_i) \cup V = (X\backslash X_{i_0}) \cup V$. Since $x \notin X_i$ for all $i \neq i_0$, we have $x \notin B_* \supseteq (A \cap X_{i_0}) \cup \bigcup_{i\neq i_0} X_i \supseteq A$. Thus

$$\begin{aligned}
\bigwedge_{x\notin B\supseteq A} \mathscr{C}(B)' \leq \mathscr{C}(B_*)' &= \Big(\bigwedge_{i\in T} \mathscr{C}_i(B_* \cap X_i)\Big)' \\
&= \bigvee_{i\in T} \mathscr{C}_i\bigg[\bigg(\Big(\bigcup_{i\neq i_0} X_i\Big) \cup V\bigg) \cap X_i\bigg]' \\
&= \mathscr{C}_{i_0}(V)'.
\end{aligned}$$

This gives $\bigwedge_{x\notin B\supseteq A} \mathscr{C}(B)' \leq \bigwedge_{x\notin V\supseteq A\cap X_{i_0}} \mathscr{C}_{i_0}(V)'$ since V is arbitrary. □

With the help of some properties of the arity, the relations between the arity of a disjoint sum of M-fuzzifying convex spaces and the arity of its factor spaces are investigeted as follows.

Theorem 3.11. *Assume that* $(X,\mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$. *If* $ary(\mathscr{C}_i) = n_i$ $(\forall i \in T)$, *then* $ary(\mathscr{C}) = \bigvee_{i\in T} n_i$.

Proof. We write $\bigvee_{i\in T} n_i = n$. Then we can see that $ary(\mathscr{C}) \geq n$ from Corollary 3.5. In order to prove $ary(\mathscr{C}) = n$, by Definition 3.1 and Remark 3.2, we only need to prove

$$\forall A \subseteq X, \bigwedge_{x\notin A} \bigwedge_{|F|\leq n, F\subseteq A} [co_{\mathscr{C}}(F)(x)]' \leq \mathscr{C}(A).$$

Since for any $i \in T$, $ary(\mathscr{C}_i) = n_i$, by Definition 3.1 and Proposition 3.3, we have

$$\begin{aligned}\mathscr{C}(A) = \bigwedge_{i\in T}\mathscr{C}_i(A\cap X_i) &= \bigwedge_{i\in T}\bigwedge_{\substack{y\notin A\cap X_i\\ y\in X_i}}\bigwedge_{\substack{G\subseteq A\cap X_i\\ |G|\le n_i}}[co_i(G)(y)]' \\ &= \bigwedge_{i\in T}\bigwedge_{\substack{y\notin A\cap X_i\\ y\in X_i}}\bigwedge_{\substack{G\subseteq A\cap X_i\\ |G|\le n}}[co_i(G)(y)]'. \quad (3)\end{aligned}$$

Again by Theorem 3.10, we have

$$\begin{aligned}\bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}[co_{\mathscr{C}}(F)(x)]' &= \bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}\left[\bigvee_{i\in T}co_i(F\cap X_i)(x)\right]' \\ &= \bigwedge_{i\in T}\bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}[co_i(F\cap X_i)(x)]'. \qquad (4)\end{aligned}$$

Fixing $i \in T$ in (3). For any $y \in X_i$ with $y \notin A \cap X_i$, so $y \notin A$. Take any $G \subseteq A \cap X_i$ and $|G| \le n$, so $G \subseteq A$. It implies

$$\bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}[co_i(F\cap X_i)(x)]' \le [co_i(G\cap X_i)(y)]' = [co_i(G)(y)]'.$$

By the arbitrariness of y and G, we further get

$$\bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}[co_i(F\cap X_i)(x)]' \le \bigwedge_{\substack{y\notin A\cap X_i\\ y\in X_i}}\bigwedge_{\substack{G\subseteq X_i\\ |G|\le n}}[co_i(G)(y)]'.$$

This implies (4) ≤ (3). Therefore, $\bigwedge_{x\notin A}\bigwedge_{|F|\le n, F\subseteq A}[co_{\mathscr{C}}(F)(x)]' \le \mathscr{C}(A)$. □

4. The Additivity of Separability

In this part, we will verify separability $(S_0, S_1, S_2, S_3, S_4)$ is additive in the sense of the following definition. Moreover, we will show that a disjoint sum of M-fuzzifying convex spaces is JHC iff its every factor space is JHC.

Definition 4.1. Assume that $\{(X_i, \mathscr{C}_i)\}_{i\in T}$ is a family of pairwise disjoint M-fuzzifying convex space. We say that the property P of an M-fuzzifying convex space is additive, provided that the infimum of the degrees that every factor space $(X_i, \mathscr{C}_i)$ possesses the property P, is equal to the degree that the disjoint sum space $(X, \sum_{i\in T}\mathscr{C}_i)$ possesses property P.

Theorem 4.2. *Assume that* $(X, \mathscr{C}) = (X, \sum_{i\in T}\mathscr{C}_i)$. *Then* $S_0(X, \mathscr{C}) = \bigwedge_{i\in T} S_0(X_i, \mathscr{C}_i)$.

Proof. From Theorem 2.9 (i), wa can see that $S_0(X, \mathscr{C}) \le \bigwedge_{i\in T} S_0(X_i, \mathscr{C}_i)$.

Conversely, consider $a \prec \bigwedge_{i\in T} S_0(X_i, \mathscr{C}_i)$. Then for each $i \in T$ and $x, y \in X_i$,

$$a \prec \bigwedge_{x\ne y}\left(\bigvee_{x\notin U, y\in U}\mathscr{C}_i(U) \vee \bigvee_{y\notin V, x\in V}\mathscr{C}_i(V)\right).$$

Further, we aim to verify for $x, y \in X$,

$$a \le S_0(X, \mathscr{C}) = \bigwedge_{x\ne y}\left(\bigvee_{x\notin A, y\in A}\mathscr{C}(A) \vee \bigvee_{y\notin B, x\in B}\mathscr{C}(B)\right).$$

For this purpose, we must show that for $x, y \in X$ with $x \neq y$, $a \leq \bigvee_{x \notin A, y \in A} \mathscr{C}(A) \vee \bigvee_{y \notin B, x \in B} \mathscr{C}(B)$.

Take $x, y \in X$ with $x \neq y$ and consider two cases below:

Case1: $x, y \in X_{i_0}$ for some $i_0 \in T$. Since

$$a \prec \bigwedge_{x \neq y} \Big(\bigvee_{x \notin U, y \in U} \mathscr{C}_{i_0}(U) \vee \bigvee_{y \notin V, x \in V} \mathscr{C}_{i_0}(V) \Big),$$

there exists $U \in 2^{X_{i_0}}$ such that $x \notin U, y \in U$, $a \leq \mathscr{C}_{i_0}(U)$ or $V \in 2^{X_{i_0}}$ such that $x \in V, y \notin V$, $a \leq \mathscr{C}_{i_0}(V)$. By Lemma 3.7 we have

$$\begin{aligned} \bigvee_{x \notin A, y \in A} \mathscr{C}(A) \vee \bigvee_{y \notin B, x \in B} \mathscr{C}(B) &\geq \mathscr{C}(U) \vee \mathscr{C}(V) \\ &= \Big(\bigwedge_{i \in T} \mathscr{C}_i(U \cap X_i) \Big) \vee \Big(\bigwedge_{i \in T} \mathscr{C}_i(V \cap X_i) \Big) \\ &= \mathscr{C}_{i_0}(U) \vee \mathscr{C}_{i_0}(V) \geq a. \end{aligned}$$

Case2: $x \in X_r, y \in X_s$ with $r \neq s$. It implies $x \in X_r, y \notin X_r$ and $y \in X_s, x \notin X_s$. So

$$\bigvee_{x \notin A, y \in A} \mathscr{C}(A) \vee \bigvee_{y \notin B, x \in B} \mathscr{C}(B) \geq \mathscr{C}(X_r) \vee \mathscr{C}(X_s) = \top \geq a.$$

According to the arbitrariness of x and y, we concluded that $a \leq \bigwedge_{x \neq y} \Big(\bigvee_{x \notin A, y \in A} \mathscr{C}(A) \vee \bigvee_{y \notin B, x \in B} \mathscr{C}(B) \Big)$. This implies $S_0(X, \mathscr{C}) \geq \bigwedge_{i \in T} S_0(X_i, \mathscr{C}_i)$. □

Theorem 4.3. *Assume that $(X, \mathscr{C}) = (X, \sum_{i \in T} \mathscr{C}_i)$. Then $S_1(X, \mathscr{C}) = \bigwedge_{i \in T} S_1(X_i, \mathscr{C}_i)$.*

Proof. By Definition 2.8 (S_1), we have

$$\begin{aligned} S_1(X, \mathscr{C}) = \bigwedge_{z \in X} \bigwedge_{i \in T} \mathscr{C}_i(\{z\} \cap X_i) &= \bigwedge_{z \in \bigcup_{j \in T} X_j} \bigwedge_{i \in T} \mathscr{C}_i(\{z\} \cap X_i) \\ &= \bigwedge_{j \in T} \bigwedge_{z \in X_j} \bigwedge_{i \in T} \mathscr{C}_i(\{z\} \cap X_i) \\ &= \bigwedge_{j \in T} \bigwedge_{z \in X_j} \mathscr{C}_j(\{z\}) \\ &= \bigwedge_{j \in T} S_1(X_j, \mathscr{C}_j) = \bigwedge_{i \in T} S_1(X_i, \mathscr{C}_i). \end{aligned}$$

□

Theorem 4.4. *Assume that $(X, \mathscr{C}) = (X, \sum_{i \in T} \mathscr{C}_i)$. Then $S_2(X, \mathscr{C}) = \bigwedge_{i \in T} S_2(X_i, \mathscr{C}_i)$.*

Proof. From Theorem 2.9 (*iii*) it is easy to see $S_2(X, \mathscr{C}) \leq \bigwedge_{i \in T} S_2(X, \mathscr{C}_i)$. The converse inequality can be proved in the following.

Take $a \prec \bigwedge_{i \in T} S_2(X_i, \mathscr{C}_i)$. We thus get for each $i \in T$ and $x, y \in X_i$,

$$a \prec S_2(X_i, \mathscr{C}_i) = \bigwedge_{x \neq y} \bigvee_{x \in U, y \notin U} \mathcal{H}_t(U).$$

In fact, it suffices to see that for $x, y \in X$ with $x \neq y$, $a \leq \bigvee_{x\in A, y\notin A} \mathcal{H}_{\mathscr{C}}(A)$.

Now let $x, y \in X$ with $x \neq y$ and consider two cases below:

Case1: $x, y \in X_{i_0}$ for some $i_0 \in T$. Since

$$a \prec S_2(X_{i_0}, \mathscr{C}_{i_0}) = \bigwedge_{x\neq y} \bigvee_{x\in U, y\notin U} \mathcal{H}_{i_0}(U),$$

we know that there exists $U \in 2^{X_{i_0}}$ with $x \in U, y \notin U$ such that $a \leq \mathcal{H}_{i_0}(U)$. So for $A \in 2^X$,

$$\begin{aligned}
\bigvee_{x\in A, y\notin A} \mathcal{H}_{\mathscr{C}}(A) &\geq \mathscr{C}(U) \wedge \mathscr{C}(X\backslash U) \\
&= \left(\bigwedge_{i\in T} \mathscr{C}_i(U\cap X_i)\right) \wedge \left(\bigwedge_{i\in T} \mathscr{C}_i((X\backslash U)\cap X_i)\right) \\
&= \left(\bigwedge_{i\in T} \mathscr{C}_i(U\cap X_i)\right) \wedge \left(\bigwedge_{i\in T} \mathscr{C}_i(X_i\backslash (U\cap X_i))\right) \\
&= \mathscr{C}_{i_0}(U) \wedge \mathscr{C}_{i_0}(X_{i_0}\backslash U) \qquad \text{(by } U \subseteq X_{i_0}) \\
&= \mathcal{H}_{i_0}(U) \geq a.
\end{aligned}$$

Case2: $x \in X_r, y \in X_s$ with $r \neq s$. It follows that $x \in X_r, y \notin X_r$. So we have

$$\bigvee_{x\in A, y\notin A} \mathcal{H}_{\mathscr{C}}(A) \geq \mathcal{H}_{\mathscr{C}}(X_r) = \mathscr{C}(X_r) \wedge \mathscr{C}(X\backslash X_r) = \top \geq a.$$

Since x and y are arbitrary, we have $a \leq \bigwedge_{x\neq y} \bigvee_{x\in A, y\notin A} \mathcal{H}_{\mathscr{C}}(A)$. This implies $S_2(X, \mathscr{C}) \geq \bigwedge_{i\in T} S_2(X_i, \mathscr{C}_i)$. □

Theorem 4.5. *Assume that* $(X, \mathscr{C}) = (X, \sum_{i\in T} \mathscr{C}_i)$. *Then* $S_3(X, \mathscr{C}) = \bigwedge_{i\in T} S_3(X_i, \mathscr{C}_i)$.

Proof. By Theorem 2.9 (v), it is immediately clear that $S_3(X, \mathscr{C}) \leq \bigwedge_{i\in T} S_3(X_i, \mathscr{C}_i)$. The converse inequality can be proved in the following.

Take $a \prec \bigwedge_{i\in T} S_3(X_i, \mathscr{C}_i)$. We thus get $\forall i \in T$ and $x \in X_i$,

$$a \prec S_3(X_i, \mathscr{C}_i) = \bigwedge_{U\subseteq X_i} \bigwedge_{x\notin U} \left[\mathscr{C}_i(U) \to \left(\bigvee_{U\subseteq V, x\notin V} \mathcal{H}_t(V)\right)\right].$$

We aim to show $a \leq S_3(X, \mathscr{C})$, that is

$$a \leq \bigwedge_{A\subseteq X} \bigwedge_{x\notin A} \left[\mathscr{C}(A) \to \left(\bigvee_{A\subseteq B, x\notin B} \mathcal{H}_{\mathscr{C}}(B)\right)\right].$$

Since for each $i \in T, a \prec S_3(X_i, \mathscr{C}_i)$, we know that for any $U \subseteq X_i$ with $x \notin U$,

$$a \wedge \mathscr{C}_i(U) \leq \bigvee_{U\subseteq V, x\notin V} \mathcal{H}_i(V).$$

Let $A \subseteq X$ with $x \notin A$. Then there is a i_0 such that $x \in X_{i_0}$, so $x \notin A \cap X_{i_0}$. Since $A \cap X_{i_0} \subseteq X_{i_0}$, we have

$$a \wedge \mathscr{C}_{i_0}(A \cap X_{i_0}) \leq \bigvee_{A\cap X_{i_0}\subseteq V, x\notin V} \mathcal{H}_{i_0}(V).$$

Take $V \subseteq X_{i_0}$ with $A \cap X_{i_0} \subseteq V, x \notin V$, and let $B^* = V \cup \bigcup_{i \neq i_0} X_i$. Since $x \in X_{i_0}$, it is clear that $A \subseteq B^*$ and $x \notin B^*$. Further, we have

$$\begin{aligned}
\mathcal{H}_{\mathscr{C}}(B^*) &= \mathscr{C}(B^*) \wedge \mathscr{C}(X \backslash B^*) \\
&= \bigwedge_{i \in T} \mathscr{C}_i(B^* \cap X_i) \wedge \bigwedge_{i \in T} \mathscr{C}_i(X_i \backslash (B^* \cap X_i)) \\
&= \bigwedge_{i \in T} \mathscr{C}_i\Big[\Big(V \cup \bigcup_{i \neq i_0} X_i\Big) \cap X_i\Big] \ \wedge \bigwedge_{i \in T} \mathscr{C}_i\Big[X_i \backslash \Big(\Big(V \cup \bigcup_{i \neq i_0} X_i\Big) \cap X_i\Big)\Big] \\
&= \mathscr{C}_{i_0}(V) \wedge \mathscr{C}_{i_0}(X_{i_0} \backslash V) \qquad \text{(by } V \subseteq X_{i_0}) \\
&= \mathcal{H}_{i_0}(V).
\end{aligned}$$

This implies $\bigvee_{A \subseteq B, x \notin B} \mathcal{H}_{\mathscr{C}}(B) \geq \bigvee_{A \cap X_{i_0} \subseteq V, x \notin V} \mathcal{H}_{i_0}(V)$. Therefore,

$$\begin{aligned}
a \wedge \mathscr{C}(A) = a \wedge \bigwedge_{i \in T} \mathscr{C}_i(A \cap X_i) &\leq a \wedge \mathscr{C}_{i_0}(A \cap X_{i_0}) \\
&\leq \bigvee_{A \cap X_{i_0} \subseteq V, x \notin V} \mathcal{H}_{i_0}(V) \\
&\leq \bigvee_{A \subseteq B, x \notin B} \mathcal{H}_{\mathscr{C}}(B).
\end{aligned}$$

Hence $a \leq \mathscr{C}(A) \rightarrow \bigvee_{A \subseteq B, x \notin B} \mathcal{H}_{\mathscr{C}}(B)$. By the arbitrariness of x and A, we thus get

$$a \leq \bigwedge_{A \subseteq X} \bigwedge_{x \notin A} \Big[\mathscr{C}(A) \rightarrow \Big(\bigvee_{A \subseteq B, x \notin B} \mathcal{H}_{\mathscr{C}}(B)\Big)\Big].$$

Therefore, $S_3(X, \mathscr{C}) \geq \bigwedge_{i \in T} S_3(X_i, \mathscr{C}_i)$. $\square$

Theorem 4.6. *Assume that* $(X, \mathscr{C}) = (X, \sum_{i \in T} \mathscr{C}_i)$. *Then* $S_4(X, \mathscr{C}) = \bigwedge_{i \in T} S_4(X_i, \mathscr{C}_i)$.

Proof. By Theorem 2.9 and Lemma 3.8, it is immediately clear that $S_4(X, \mathscr{C}) \wedge \mathscr{C}(X_i) \leq S_4(X_i, \mathscr{C}_i)$. Again by Lemma 3.8 we can see $\mathscr{C}(X_i) = \top$. This gives for each $i \in T$, $S_4(X, \mathscr{C}) \leq S_4(X_i, \mathscr{C}_i)$. So $S_4(X, \mathscr{C}) \leq \bigwedge_{i \in T} S_4(X_i, \mathscr{C}_i)$.

The converse inequality can be proved in the following.

Take $a \leq \bigwedge_{i \in T} S_4(X_i, \mathscr{C}_i)$, so we have $a \leq S_4(X_i, \mathscr{C}_i)$ for all $i \in T$. From Definition 2.8, we know that for every $U, V \subseteq 2^{X_i}$ with $U \cap V = \emptyset$,

$$a \leq \mathscr{C}_i(U) \wedge \mathscr{C}_i(V) \rightarrow \Big(\bigvee_{U \subseteq Q, V \subseteq X_i \backslash Q} \mathcal{H}_i(Q)\Big).$$

This implies $a \wedge \mathscr{C}_i(U) \wedge \mathscr{C}_i(V) \leq \bigvee_{U \subseteq Q, V \subseteq X_i \backslash Q} \mathcal{H}_i(Q)$. Next, we aim to prove $a \leq S_4(X, \mathscr{C})$, that is

$$a \leq \bigwedge_{A \cap B = \emptyset} \Big[\mathscr{C}(A) \wedge \mathscr{C}(B) \rightarrow \Big(\bigvee_{B \subseteq X \backslash C, A \subseteq C} \mathcal{H}_{\mathscr{C}}(C)\Big)\Big].$$

In fact, for any $A, B \in 2^X$ with $A \cap B = \emptyset$, we consider $U_i = A \cap X_i$ and $V_i = B \cap X_i$ for all $i \in T$, which means $U_i, V_i \subseteq X_i$ and $U_i \cap V_i = \emptyset$. So we have $a \wedge \mathscr{C}_i(U_i) \wedge \mathscr{C}_i(V_i) \leq \bigvee_{U_i \subseteq Q, V_i \subseteq X_i \setminus Q} \mathcal{H}_i(Q)$ Therefore,

$$\begin{aligned} a \wedge \mathscr{C}(A) \wedge \mathscr{C}(B) &= a \wedge \bigwedge_{i\in T} \mathscr{C}_i(A \cap X_i) \wedge \bigwedge_{i\in T} \mathscr{C}_i(B \cap X_i) \\ &= a \wedge \bigwedge_{i\in T} \mathscr{C}_i(U_i) \wedge \bigwedge_{i\in T} \mathscr{C}_i(V_i) \\ &\leq \bigwedge_{i\in T} \bigvee_{U_i \subseteq Q, V_i \subseteq X_i \setminus Q} \mathcal{H}_i(Q) \\ &= \bigvee_{f \in \prod_{i\in T} J_i} \bigwedge_{i\in T} \mathcal{H}_i(f(i)), \end{aligned}$$

where $J_i = \{Q \in X_i : U_i \subseteq Q, V_i \subseteq X_i \setminus Q\}$, $f(i) \in J_i$. Since for each $f \in \prod_{i\in T} J_i$ and $i \in T$, there exists $Q_i \in J_i$ such that $f(i) = Q_i$. Thus $U_i \subseteq Q_i, V_i \subseteq X_i \setminus Q_i$, which implies

$$A = \bigcup_{i\in T}(A \cap X_i) = \bigcup_{i\in T} U_i \subseteq \bigcup_{i\in T} Q_i.$$

In a similar way, we can get $B \subseteq X \setminus (\bigcup_{i\in T} Q_i)$. We take $C^* = \bigcup_{i\in T} Q_i = \bigcup_{j\in T} Q_j$, and so $A \subseteq C^*$, $B \subseteq X \setminus C^*$. We thus get

$$\begin{aligned} \mathcal{H}(C^*) &= \mathscr{C}(C^*) \wedge \mathscr{C}(X \setminus C^*) \\ &= \bigwedge_{i\in T} \mathscr{C}_i(C^* \cap X_i) \wedge \mathscr{C}_i((X \setminus C^*) \cap X_i) \\ &= \bigwedge_{i\in T} \mathscr{C}_i(C^* \cap X_i) \wedge \mathscr{C}_i(X_i \setminus (C^* \cap X_i)) \\ &= \bigwedge_{i\in T} \mathscr{C}_i\left[\left(\bigcup_{j\in T} Q_j\right) \cap X_i\right] \wedge \mathscr{C}_i\left[X_i \setminus \left(\left(\bigcup_{j\in T} Q_j\right) \cap X_i\right)\right] \\ &= \bigwedge_{i\in T} \mathscr{C}_i(Q_i) \wedge \mathscr{C}_i(X_i \setminus Q_i) \qquad (\text{since for } j \neq i, \ Q_j \cap X_i \subseteq X_j \cap X_i = \emptyset) \\ &= \bigwedge_{i\in T} \mathcal{H}_i(Q_i) = \bigwedge_{i\in T} \mathcal{H}_i(f(i)). \end{aligned}$$

Since f is arbitrary, we have

$$\bigvee_{f \in \prod_{i\in T} J_i} \bigwedge_{i\in T} \mathcal{H}_i(f(i)) \leq \bigvee_{B \subseteq X \setminus C, A \subseteq C} \mathcal{H}_{\mathscr{C}}(C).$$

Thus

$$a \wedge \mathscr{C}(A) \wedge \mathscr{C}(B) \leq \bigvee_{B \subseteq X \setminus C, A \subseteq C} \mathcal{H}_{\mathscr{C}}(C).$$

So

$$a \leq \mathscr{C}(A) \wedge \mathscr{C}(B) \to \Big(\bigvee_{B \subseteq X \setminus C, A \subseteq C} \mathcal{H}_{\mathscr{C}}(C) \Big).$$

Since A and B are arbitrary, which gives $a \leq S_4(X, \mathscr{C})$. Therefore, $S_4(X, \mathscr{C}) \geq \bigwedge_{i\in T} S_4(X_i, \mathscr{C}_i)$. □

Theorem 4.7. *Assume that $(X, \mathscr{C}) = (X, \sum_{i\in T} \mathscr{C}_i)$. Then for each $i \in T$, $(X_i, \mathscr{C}_i)$ is an M-fuzzifying JHC convexity if and only if $(X, \mathscr{C})$ is an M-fuzzifying JHC convexity.*

Proof. **Sufficiency.** For our purpose, we must show that for a fixed $i \in T$, $\forall b, y \in X_i, \forall U \subseteq X_i$,

$$co_i(\{b\} \cup U)(y) = \bigvee_{c \in X_i} \Big[co_i(\{b, c\})(y) \wedge co_i(U)(c)\Big].$$

By the fact that $b, y \in X_i$ and $U \subseteq X_i$, it follows from Lemma 3.9 and Theorem 3.10 that

$$\begin{aligned} co_{\mathscr{C}}(\{b\} \cup U)(y) &= \bigvee_{i \in T}\Big[co_i((\{b\} \cup U) \cap X_i)(y)\Big] \\ &= co_i(\{b\} \cup U)(y). \end{aligned}$$

Since $(X, \mathscr{C})$ is M-fuzzifying JHC, we have

$$\begin{aligned} co_{\mathscr{C}}(\{b\} \cup U)(y) &= \bigvee_{c \in X} \Big[co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)\Big] \\ &= \Big(\bigvee_{c \in X_i} \Big[co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)\Big]\Big) \vee \Big(\bigvee_{c \notin X_i} \Big[co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)\Big]\Big) \\ &= \Big(\bigvee_{c \in X_i} \Big[co_i(\{b, c\})(y) \wedge co_i(U)(c)\Big]\Big) \vee \Big(\bigvee_{c \notin X_i} \Big[co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)\Big]\Big). \end{aligned}$$

The last equality holds because by Lemma 3.9, $b, c, y \in X_i$ and $U \subseteq X_i$ implies

$$\bigvee_{c \in X_i} \Big[co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)\Big] = \bigvee_{c \in X_i} \Big[co_i(\{b, c\})(y) \wedge co_i(U)(c)\Big].$$

Now, we note that $\bigvee_{c \notin X_i} [co_{\mathscr{C}}(\{b, c\})(y) \wedge co_{\mathscr{C}}(U)(c)] = \perp$. Since $U \subseteq X_i$, then for every $c \notin X_i$, $co_{\mathscr{C}}(U)(c) = \perp$ by Lemma 3.9, this implies our statement holds. Thus,

$$co_i(\{b\} \cup U)(y) = \bigvee_{c \in X_i} \Big[co_i(\{b, c\})(y) \wedge co_i(U)(c)\Big].$$

Necessity. Since for each $i \in T$, $(X_i, \mathscr{C}_i)$ is an M-fuzzifying JHC convexity, we know that $\forall b, y \in X_i, \forall U \subseteq X_i$,

$$co_i(\{b\} \cup U)(y) = \bigvee_{x \in X_i} \Big[co_i(\{b, x\})(y) \wedge co_i(U)(x)\Big].$$

Next, we prove $\forall a, z \in X, \forall A \subseteq X$,

$$co_{\mathscr{C}}(\{a\} \cup A)(z) = \bigvee_{x \in X} \Big[co_{\mathscr{C}}(\{a, x\})(z) \wedge co_{\mathscr{C}}(A)(x)\Big].$$

We first note that for any $x \in X$,

$$co_{\mathscr{C}}(\{a\} \cup A)(z) \geq co_{\mathscr{C}}(\{a, x\})(z) \wedge co_{\mathscr{C}}(A)(x).$$

To do this, by Theorem 2.3 we need to prove

$$\bigwedge_{z \notin D \supseteq \{a\} \cup A} \mathscr{C}(D)' \geq \bigwedge_{z \notin B \supseteq \{a, x\}} \mathscr{C}(B)' \wedge \bigwedge_{x \notin C \supseteq A} \mathscr{C}(C)'.$$

Now let $D \subseteq X$ such that $z \notin D \supseteq \{a\} \cup A$ and consider two cases below:

Case1: $x \notin D$. So $x \notin D \supseteq A$, which implies that

$$\bigwedge_{x \notin C \supseteq A} \mathscr{C}(C)' \leq \mathscr{C}(D)'.$$

Case2: $x \in D$. So $z \notin D \supseteq \{a, x\}$, which implies

$$\bigwedge_{z \notin B \supseteq \{a,x\}} \mathscr{C}(B)' \leq \mathscr{C}(D)'.$$

Hence we obtain that

$$\bigwedge_{z \notin D \supseteq \{a\} \cup A} \mathscr{C}(D)' \geq \bigwedge_{z \notin B \supseteq \{a,x\}} \mathscr{C}(B)' \wedge \bigwedge_{x \notin C \supseteq A} \mathscr{C}(C)'.$$

The converse inequality can be proved in the following.

Since $a, z \in X$, there exist $r, s \in T$ such that $a \in X_r$ and $z \in X_s$. By Lemma 3.9 and Theorem 3.10, we have

$$co_{\mathscr{C}}(\{a\} \cup A)(z) = \bigvee_{i \in T} co_i((\{a\} \cup A) \cap X_i)(z) = co_s((\{a\} \cup A) \cap X_s)(z) \qquad (*)$$

and

$$\begin{aligned} \bigvee_{x \in X} \Big[co_{\mathscr{C}}(\{a, x\})(z) \wedge co_{\mathscr{C}}(A)(x)\Big] &= \bigvee_{x \in X} \left(\bigvee_{i \in T} co_i(\{a, x\} \cap X_i)(z) \wedge \bigvee_{i \in T} co_i(A \cap X_i)(x) \right) \\ &= \bigvee_{x \in X} \left(co_s(\{a, x\} \cap X_s)(z) \wedge \bigvee_{i \in T} co_i(A \cap X_i)(x) \right) \\ &\geq \bigvee_{x \in X_s} \left(co_s(\{a, x\} \cap X_s)(z) \wedge \bigvee_{i \in T} co_i(A \cap X_i)(x) \right) \\ &= \bigvee_{x \in X_s} \Big[co_s(\{a, x\})(z) \wedge co_s(A \cap X_s)(x)\Big]. \qquad (\star) \end{aligned}$$

The last equality holds because by Lemma 3.9, $x \in X_s$ implies $\bigvee_{i \in T} co_i(A \cap X_i)(x) = co_s(A \cap X_s)(x)$.

Next, we consider two cases below:

Case 1: $r \neq s$. i.e., $a \notin X_s$. Then $* = co_s(A \cap X_s)(z)$. It follows from $z \in X_s$ that

$$\star \geq co_s(\{z\})(z) \wedge co_s(A \cap X_s)(z) = co_s(A \cap X_s)(z) = *.$$

Case 2: $r = s$. i.e., $a, z \in X_s$. Since $\mathscr{C}_s$ is M-fuzzifying JHC, we have

$$* = co_s(\{a\} \cup (A \cap X_s))(z) = \bigvee_{x \in X_s} \Big[co_s(\{a, x\})(z) \wedge co_s(A \cap X_s)(x)\Big].$$

Further, we have

$$\star = \bigvee_{x \in X_s} \Big[co_s(\{a, x\})(z) \wedge co_s(A \cap X_s)(x)\Big] = *.$$

Hence

$$\bigvee_{x \in X} \Big[co_{\mathscr{C}}(\{a, x\})(z) \wedge co_{\mathscr{C}}(A)(x)\Big] \geq co_{\mathscr{C}}(\{a\} \cup A)(z).$$

□

5. Conclusions

Based on the definition of the disjoint sum of M-fuzzifying convex spaces mentioned in [20], some related properties are studied in detail. the notion of the arity of an M-fuzzifying convex space is introduced. With the help of arity, the connections between the disjoint sum of M-fuzzifying convex spaces and its factor spaces are established. It is proved that the arity of the disjoint sum of M-fuzzifying convex spacesis is equal to the supremum of the family of arity of every factor space. Furthermore, we show that some properties of M-fuzzifying convex spaces are additive in the sense of Definition 4.1 such as separability. It is shown that a disjoint sum of M-fuzzifying convex spaces is JHC iff its every factor space is JHC. Of course, there are many other properties of M-fuzzifying convex spaces that can be verified to be additive in a similar way.

Acknowledgement

The authors want to express their heartfelt thanks to the reviewers for their careful reading and constructive comments.

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