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Product Structures and Complex Structures of Hom-Lie-Yamaguti Algebras

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Abstract. In this paper, we mainly discuss linear deformations of a Hom-Lie-Yamaguti algebra and introduce the notion of a Hom-Nijenhuis operator. We introduce the notion of a product structure on a Hom-Lie-Yamaguti algebra, which is a Hom-Nijenhuis operator E satisfying $E^2 = \mathrm{Id}$. There is a product structure on an involutive Hom-Lie-Yamaguti algebra if and only if the Hom-Lie-Yamaguti algebra is the direct sum of two subalgebras (as vector spaces). At the same time, we also introduce the notion of a complex structure on a Hom-Lie-Yamaguti algebra. Finally, we add a compatibility condition between a product structure and a complex structure to introduce the notion of a complex product structure on an involutive Hom-Lie-Yamaguti algebra.

1. Introduction

Hom-type algebras were studied by many researchers. The first examples coming from q-deformations of Witt and Virasora algebras are Hom-Lie algebras, see [9]. The notion of a Lie-Yamaguti algebra was introduced in [12], and can be traced back to [15] and [16]. Roughly speaking, a Lie-Yamaguti algebra owns both a binary operation and a ternary operation that satisfy some compatibility conditions. It is an algebraic structure that generalizes a Lie algebra and a Lie triple system simultaneously. See [6], [17] for more details about recent studies on Lie-Yamaguti algebras. In [8], the authors introduced the concept of Hom-Lie-Yamaguti algebras. It is a Hom-type generalization of a Lie-Yamaguti algebra in [5], a general Lie triple system in [15, 16] and a Lie triple algebra in [11]. The authors studied the formal deformations of Hom-Lie-Yamaguti algebras in [13]. In [18], the authors give general representation and cohomology theory of Hom-Lie-Yamaguti algebras.

A Nijenhuis operator on a Lie algebra can generate its trivial deformation, and plays an important role in the study on integrability of nonlinear evolution equations [7]. Product structures and complex structures on a Lie algebra can be viewed as special Nijenhuis operators. Namely, the Nijenhuis condition on a Lie algebra is exactly the integrability condition of an almost product structure (an almost complex structure) being a product structure (complex structure). See [1]-[4] for more details about product structures, complex structures and complex product structures on Lie algebras. In [14], the authors discussed Nijenhuis operators, product structures and complex structures on Lie-Yamaguti algebras.

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The purpose of this paper is to introduce and study Nijenhuis operators, product structures, complex structures and complex product structures on involutive Hom-Lie-Yamaguti algebras, which are natural generalizations of those on Lie-Yamaguti algebras.

This paper is organized as follows. In section 2, we recall the notions of a Hom-Lie-Yamaguti algebra and its (2,3)-cohomology group.

In section 3, we study linear deformations of a Hom-Lie-Yamaguti algebra and introduce the notion of a Hom-Nijenhuis operator on a Hom-Lie-Yamaguti algebra. We show that a Hom-Nijenhuis operator on a Hom-Lie-Yamaguti algebra can generate a trivial linear deformation.

In section 4, we introduce the notion of a product structure on a Hom-Lie-Yamaguti algebra, and prove that an involutive Hom-Lie-Yamaguti algebra is the direct sum of two subalgebras (as vector spaces) if and only if there is a product structure on it. In particular, we give several special product structures, which are called strict product structures, abelian product structures and perfect product structures, respectively, and they give rise to special decompositions of the original Hom-Lie-Yamaguti algebra, respectively.

In section 5, we introduce the notion of a complex structure on a Hom-Lie-Yamaguti algebra and study it parallel to the case of product structures.

In section 6, by adding a compatibility condition between a product structure and a complex structure we introduce the notion of a complex product structure on an involutive Hom-Lie-Yamaguti algebra.

Throughout this paper, we work on an algebraically closed field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In this section, we first recall some basic definitions of Hom-Lie-Yamaguti algebras.

Definition 2.1. *A Hom-Lie-Yamaguti algebra (or HLYA for short) consists of a vector space L together with a linear map $\alpha : L \to L$, a bilinear map $[\cdot,\cdot] : L \times L \to L$ and a trilinear map $\{\cdot,\cdot,\cdot\} : L \times L \times L \to L$ such that, for all $x_i, y_i \in L (i = 1, 2, 3)$, the following conditions are satisfied:*

- (HLY01) $\alpha([x_1, x_2]) = [\alpha(x_1), \alpha(x_2)]$;
- (HLY02) $\alpha(\{x_1, x_2, x_3\}) = \{\alpha(x_1), \alpha(x_2), \alpha(x_3)\}$;
- (HLY1) $[x_1, x_2] + [x_2, x_1] = 0$;
- (HLY2) $\{x_1, x_2, x_3\} + \{x_2, x_1, x_3\} = 0$;
- (HLY3) $[[x_1, x_2], \alpha(x_3)] + c.p. + \{x_1, x_2, x_3\} + c.p. = 0$;
- (HLY4) $\{[x_1, x_2], \alpha(y_1), \alpha(y_2)\} + \{[x_2, y_1], \alpha(x_1), \alpha(y_2)\} + \{[y_1, x_1], \alpha(x_2), \alpha(y_2)\} = 0$;
- (HLY5) $\{\alpha(x_1), \alpha(x_2), [y_1, y_2]\} = [\{x_1, x_2, y_1\}, \alpha^2(y_2)] + [\alpha^2(y_1), \{x_1, x_2, y_2\}]$;
- (HLY6) $\{\alpha^2(x_1), \alpha^2(x_2), \{y_1, y_2, y_3\}\} = \{\{x_1, x_2, y_1\}, \alpha^2(y_2), \alpha^2(y_3)\} + \{\alpha^2(y_1), \{x_1, x_2, y_2\}, \alpha^2(y_3)\} + \{\alpha^2(y_1), \alpha^2(y_2), \{x_1, x_2, y_3\}\}$,

where c.p. means cyclic permutations with respect to x_1, x_2, x_3. We denote a HLYA by $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ or simply by L.

Definition 2.2. *A homomorphism between two HLYAs $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ and $(L', [\cdot,\cdot]', \{\cdot,\cdot,\cdot\}', \alpha')$ is a linear map $f : L \to L'$ satisfying $f\alpha = \alpha' f$ and*

$$f([x_1, x_2]) = [f(x_1), f(x_2)]', \quad f(\{x_1, x_2, x_3\}) = \{f(x_1), f(x_2), f(x_3)\}',$$

for all $x_i \in L (i = 1, 2, 3)$.

Definition 2.3. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA, $\varphi : L \times L \to L$ a bilinear map and $\omega : L \times L \times L \to L$ a trilinear map. Then a pair (φ, ω) is called a (2,3)-cocycle if for all $x_1, x_2, x_3, y_1, y_2, y_3$ the following conditions hold:*

- $\varphi(\alpha(x_1), \alpha(x_2)) = \alpha(\varphi(x_1, x_2))$;
- $\omega(\alpha(x_1), \alpha(x_2), \alpha(x_3)) = \alpha(\omega(x_1, x_2, x_3))$;
- $\varphi([x_1, x_2], \alpha(x_3)) + c.p. + [\varphi(x_1, x_2), \alpha(x_3)] + c.p. + \omega(x_1, x_2, x_3) + c.p. = 0$;
- $\omega([x_1, x_2], \alpha(y_1), \alpha(y_2)) + c.p. + \{\varphi(x_1, x_2), \alpha(y_1), \alpha(y_2)\} + c.p. = 0$;
- $\omega(\alpha(x_1), \alpha(x_2), [y_1, y_2]) + \{\alpha(x_1), \alpha(x_2), \varphi(y_1, y_2)\} = \varphi(\alpha^2(y_1), \{x_1, x_2, y_2\}) + [\alpha^2(y_1), \omega(x_1, x_2, y_2)]$ $+\varphi(\{x_1, x_2, y_1\}, \alpha^2(y_2)) + [\omega(x_1, x_2, y_1), \alpha^2(y_2)]$;
- $\omega(\alpha^2(x_1), \alpha^2(x_2), \{y_1, y_2, y_3\}) + \{\alpha^2(x_1), \alpha^2(x_2), \omega(y_1, y_2, y_3)\} = \omega(\{x_1, x_2, y_1\}, \alpha^2(y_2), \alpha^2(y_3))$ $+ \{\omega(x_1, x_2, y_1), \alpha^2(y_2), \alpha^2(y_3)\} + \omega(\alpha^2(y_1), \{x_1, x_2, y_2\}, \alpha^2(y_3)) + \{\alpha^2(y_1), \omega(x_1, x_2, y_2), \alpha^2(y_3)\}$ $+ \omega(\alpha^2(y_1), \alpha^2(y_2), \{x_1, x_2, y_3\}) + \{\alpha^2(y_1), \alpha^2(y_2), \omega(x_1, x_2, y_3)\}$.

Now we introduce the definition of the (2, 3)-cohomology group of a HLYA.

Definition 2.4. *A pair (φ, ω) is called a (2,3)-coboundary if there exits a linear map $N \in \mathfrak{gl}(L)$, such that for any $x, y, z \in L$, the following conditions hold:*

- $N\alpha(x) = \alpha N(x)$, $\varphi(\alpha(x), \alpha(y)) = \alpha\varphi(x, y)$, $\omega(\alpha(x), \alpha(y), \alpha(z)) = \alpha\omega(x, y, z)$;
- $\varphi(x, y) = [Nx, y] + [x, Ny] - N[x, y]$;
- $\omega(x, y, z) = \{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\} - N\{x, y, z\}$.

We denote by $\mathcal{Z}^{(2,3)}(L, L)$ and $\mathcal{B}^{(2,3)}(L, L)$ the set of (2,3)-cocycles and the set of (2,3)-coboundaries, respectively. Then the quotient space $\mathcal{Z}^{(2,3)}(L, L)/\mathcal{B}^{(2,3)}(L, L)$ is called the (2, 3)-cohomology group of a Hom-Lie-Yamaguti algebra, and denoted by $\mathcal{H}^{(2,3)}(L, L)$. In the following, we see that the (2, 3)-cohomology group can characterize equivalent deformations of a Hom-Lie-Yamaguti algebra.

3. Linear deformations of Hom-Lie-Yamaguti algebras and Nijenhuis operators

In this section, we discuss linear deformations of Hom-Lie-Yamaguti algebras, and introduce the definition of a Nijenhuis operator on a Hom-Lie-Yamaguti algebra, which can generate a trivial deformation. Moreover, Nijenhuis operators are the main tool to define product structures and complex structures on a Hom-Lie-Yamaguti algebra in the following sections.

Definition 3.1. *Let $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ be a HLYA, $\varphi : L \times L \to L$ and $\omega, \psi : L \times L \times L \to L$ be a bilinear map and trilinear maps satisfying $\varphi(\alpha(x), \alpha(y)) = \alpha\varphi(x, y)$, $\omega(\alpha(x), \alpha(y), \alpha(z)) = \alpha\omega(x, y, z)$, $\psi(\alpha(x), \alpha(y), \alpha(z)) = \alpha\psi(x, y, z)$. Consider the following linear operators: for all $x, y, z \in L$,*

$$[x, y]_t = [x, y] + t\varphi(x, y), \tag{1}$$
$$\{x, y, z\}_t = \{x, y, z\} + t\omega(x, y, z) + t^2\psi(x, y, z). \tag{2}$$

For all $t \in \mathbb{K}$, if $(L, [\cdot, \cdot]_t, \{\cdot, \cdot, \cdot\}_t, \alpha)$ are HLYAs, then we say that (φ, ω, ψ) generates a linear deformation of the HLYA $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$.

Now we'll give the relation between the deformations and cohomologies of HLYA.

Theorem 3.2. *Let $(L, [\cdot, \cdot], \{\cdot, \cdot, \cdot\}, \alpha)$ be a HLYA, $\varphi : L \times L \to L$ and $\omega, \psi : L \times L \times L \to L$ be a bilinear map and trilinear maps. Then the triple (φ, ω, ψ) generates a linear deformation of L if and only if the following conditions are satisfied:*
(*a*) *(φ, ω) is a (2,3)-cocycle;*
(*b*) *(φ, ψ) defines a HLYA structure on L;*
(*c*) *the following equations hold:*

- *(c1)* $\psi([x_1,x_2],\alpha(y_1),\alpha(y_2))+\omega(\varphi(x_1,x_2),\alpha(y_1),\alpha(y_2))+\psi([x_2,y_1],\alpha(x_1),\alpha(y_2))$ $+\omega(\varphi(x_2,y_1),\alpha(x_1),\alpha(y_2))+\psi([y_1,x_1],\alpha(x_2),\alpha(y_2))+\omega(\varphi(y_1,x_1),\alpha(x_2),\alpha(y_2))=0;$
- *(c2)* $\omega(\alpha(x_1),\alpha(x_2),\varphi(y_1,y_2))+\psi(\alpha(x_1),\alpha(x_2),[y_1,y_2])=\varphi(\alpha^2(y_1),\omega(x_1,x_2,y_2))$ $+[\alpha^2(y_1),\psi(x_1,x_2,y_2)]+\varphi(\omega(x_1,x_2,y_1),\alpha^2(y_2))+[\psi(x_1,x_2,y_1),\alpha^2(y_2)];$
- *(c3)* $\psi(\alpha^2(x_1),\alpha^2(x_2),\{y_1,y_2,y_3\})+\omega(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))+\{\alpha^2(x_1),\alpha^2(x_2),\psi(y_1,y_2,y_3)\}$ $=\psi(\{x_1,x_2,y_1\},\alpha^2(y_2),\alpha^2(y_3))+\omega(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))+\{\psi(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)\}$ $+\psi(\alpha^2(y_1),\{x_1,x_2,y_2\},\alpha^2(y_3))+\omega(\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3))+\{\alpha^2(y_1),\psi(x_1,x_2,y_2),\alpha^2(y_3)\}$ $+\psi(\alpha^2(y_1),\alpha^2(y_2),\{x_1,x_2,y_3\})+\omega(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))+\{\alpha^2(y_1),\alpha^2(y_2),\psi(x_1,x_2,y_3)\};$
- *(c4)* $\psi(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))+\omega(\alpha^2(x_1),\alpha^2(x_2),\psi(y_1,y_2,y_3))$ $=\psi(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))+\omega(\psi(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))+\psi(\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3))$ $+\omega(\alpha^2(y_1),\psi(x_1,x_2,y_2),\alpha^2(y_3))+\psi(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))+\omega(\alpha^2(y_1),\alpha^2(y_2),\psi(x_1,x_2,y_3)),$

for all $x_1,x_2,x_3,y_1,y_2,y_3\in L$.

Proof. If the triple (φ,ω,ψ) generates a linear deformation $([\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t)$ of a HLYA $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$, then by Definition 3.1 for all t, $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\alpha)$ is a HLYA. By $[[x_1,x_2]_t,\alpha(x_3)]_t+c.p.+\{x_1,x_2,x_3\}_t+c.p.=0$, we have

$$\varphi([x_1,x_2],\alpha(x_3))+c.p.+[\varphi(x_1,x_2),\alpha(x_3)]+c.p.+\omega(x_1,x_2,x_3)+c.p.=0; \tag{3}$$
$$\varphi(\varphi(x_1,x_2),\alpha(x_3))+c.p.+\psi(x_1,x_2,x_3)+c.p.=0. \tag{4}$$

By $\{[x_1,x_2]_t,\alpha(y_1),\alpha(y_2)\}_t+\{[x_2,y_1]_t,\alpha(x_1),\alpha(y_2)\}_t+\{[y_1,x_1]_t,\alpha(x_2),\alpha(y_2)\}_t=0$, we have

$$\omega([x_1,x_2],\alpha(y_1),\alpha(y_2))+c.p.+\{\varphi(x_1,x_2),\alpha(y_1),\alpha(y_2)\}+c.p.=0; \tag{5}$$
$$\psi([x_1,x_2],\alpha(y_1),\alpha(y_2))+c.p.+\omega(\varphi(x_1,x_2),\alpha(y_1),\alpha(y_2))+c.p.=0; \tag{6}$$
$$\psi(\varphi(x_1,x_2),\alpha(y_1),\alpha(y_2))+c.p.=0. \tag{7}$$

By $\{\alpha(x_1),\alpha(x_2),[y_1,y_2]_t\}_t=[\{x_1,x_2,y_1\}_t,\alpha^2(y_2)]_t+[\alpha^2(y_1),\{x_1,x_2,y_2\}_t]_t$, we have

$$\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])+\{\alpha(x_1),\alpha(x_2),\varphi(y_1,y_2)\}=\varphi(\{x_1,x_2,y_1\},\alpha^2(y_2))$$
$$+[\omega(x_1,x_2,y_1),\alpha^2(y_2)]+\varphi(\alpha^2(y_1),\{x_1,x_2,y_2\})+[\alpha^2(y_1),\omega(x_1,x_2,y_1)]; \tag{8}$$
$$\omega(\alpha(x_1),\alpha(x_2),\varphi(y_1,y_2))+\psi(\alpha(x_1),\alpha(x_2),[y_1,y_2])=\varphi(\omega(x_1,x_2,y_1),\alpha^2(y_2))$$
$$+[\psi(x_1,x_2,y_1),\alpha^2(y_2)]+\varphi(\alpha^2(y_1),\omega(x_1,x_2,y_2))+[\alpha^2(y_1),\psi(x_1,x_2,y_2)]; \tag{9}$$
$$\psi(\alpha(x_1),\alpha(x_2),\varphi(y_1,y_2))=\varphi(\psi(x_1,x_2,y_1),\alpha^2(y_2))+\varphi(\alpha^2(y_1),\psi(x_1,x_2,y_2)). \tag{10}$$

By $\{\alpha^2(x_1),\alpha^2(x_2),\{y_1,y_2,y_3\}_t\}_t=\{\{x_1,x_2,y_1\}_t,\alpha^2(y_2),\alpha^2(y_3)\}_t+\{\alpha^2(y_1),\{x_1,x_2,y_2\}_t,\alpha^2(y_3)\}_t+\{\alpha^2(y_1),\alpha^2(y_2),\{x_1,x_2,y_3\}_t\}_t$, we have

$$\omega(\alpha^2(x_1),\alpha^2(x_2),\{y_1,y_2,y_3\})+\{\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3)\}$$
$$=\omega(\{x_1,x_2,y_1\},\alpha^2(y_2),\alpha^2(y_3))+\{\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)\}+\omega(\alpha^2(y_1),\{x_1,x_2,y_2\},\alpha^2(y_3))$$
$$+\{\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3)\}+\omega(\alpha^2(y_1),\alpha^2(y_2),\{x_1,x_2,y_3\})+\{\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3)\}; \tag{11}$$

$$\psi(\alpha^2(x_1),\alpha^2(x_2),\{y_1,y_2,y_3\})+\omega(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))+\{\alpha^2(x_1),\alpha^2(x_2),\psi(y_1,y_2,y_3)\}$$
$$=\psi(\alpha^2(x_1),\alpha^2(x_2),\{y_1,y_2,y_3\})+\omega(\omega(\alpha^2(x_1),\alpha^2(x_2),y_1),y_2,y_3)+\omega(\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3))$$
$$+\{\psi(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)\}+\psi(\alpha^2(y_1),\{x_1,x_2,y_2\},\alpha^2(y_3))+\{\alpha^2(y_1),\psi(x_1,x_2,y_2),\alpha^2(y_3)\}$$
$$+\psi(\alpha^2(y_1),\alpha^2(y_2),\{x_1,x_2,y_3\})+\omega(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))+\{\alpha^2(y_1),\alpha^2(y_2),\psi(x_1,x_2,y_3)\}; \tag{12}$$

$$\psi(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))+\omega(\alpha^2(x_1),\alpha^2(x_2),\psi(y_1,y_2,y_3))$$
$$=\psi(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))+\omega(\psi(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3))+\psi(\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3))$$
$$+\omega(\alpha^2(y_1),\psi(x_1,x_2,y_2),\alpha^2(y_3))+\psi(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))+\omega(\alpha^2(y_1),\alpha^2(y_2),\psi(x_1,x_2,y_3)); \tag{13}$$

$$\psi(\alpha^2(x_1), \alpha^2(x_2), \psi(y_1, y_2, y_3)) = \psi(\psi(x_1, x_2, y_1), \alpha^2(y_2), \alpha^2(y_3))$$
$$+\psi(\alpha^2(y_1), \psi(x_1, x_2, y_2), \alpha^2(y_3)) + \psi(\alpha^2(y_1), \alpha^2(y_2), \psi(x_1, x_2, y_3)). \tag{14}$$

By (3), (5), (8) and (11), we can obtain (*a*) is satisfied. By (4), (7), (10) and (14), we can obtain (*b*) is satisfied. By (6), (9), (12) and (13), we can obtain (*c*) is satisfied. The conclusion thus follows.

Definition 3.3. *(i) Let $(L, [\cdot,\cdot]_t, \{\cdot,\cdot,\cdot\}_t, \alpha)$ and $(L, [\cdot,\cdot]'_t, \{\cdot,\cdot,\cdot\}'_t, \alpha)$ be two linear deformations of a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ generated by (φ, ω, ψ) and $(\varphi', \omega', \psi')$, respectively. They are called equivalent if there exists a linear map $N \in \mathfrak{gl}(L)$ such that*

$$T_t = \mathrm{Id} + tN : (L, [\cdot,\cdot]'_t, \{\cdot,\cdot,\cdot\}'_t, \alpha) \to (L, [\cdot,\cdot]_t, \{\cdot,\cdot,\cdot\}_t, \alpha)$$

is a homomorphism.

(ii) A linear deformation $(L, [\cdot,\cdot]_t, \{\cdot,\cdot,\cdot\}_t, \alpha)$ of a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is said to be trivial if it is equivalent to $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$.

More precisely, if two linear deformations are equivalent, we have

$$(\mathrm{Id} + tN)[x, y]'_t = [x + tNx, y + tNy]_t, \tag{15}$$
$$(\mathrm{Id} + tN)\{x, y, z\}'_t = \{x + tNx, y + tNy, z + tNz\}_t. \tag{16}$$

Expanding each side of (15) and comparing the coefficients of t, we get

$$\varphi'(x, y) - \varphi(x, y) = [Nx, y] + [x, Ny] - N[x, y]. \tag{17}$$

Similarly, with respect to (16), we obtain that

$$\omega'(x, y, z) - \omega(x, y, z) = \{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\} - N\{x, y, z\}. \tag{18}$$

By (17) and (18), we obtain that equivalent classes of linear deformations of a HLYA can be characterized by the (2, 3)-cohomology group of a HLYA. That is

Theorem 3.4. *Let $(L, [\cdot,\cdot]'_t, \{\cdot,\cdot,\cdot\}'_t, \alpha)$ and $(L, [\cdot,\cdot]_t, \{\cdot,\cdot,\cdot\}_t, \alpha)$ be two linear deformations of a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ generated by $(\varphi', \omega', \psi')$ and (φ, ω, ψ), respectively. Then (φ', ω') and (φ, ω) are in the same cohomology class in the cohomology group $\mathcal{H}^{(2,3)}(L, L)$.*

In the sequel, we introduce the notion of a Hom-Nijenhuis operator on a HLYA by considering trivial deformations. Let $(L, [\cdot,\cdot]_t, \{\cdot,\cdot,\cdot\}_t, \alpha)$ be a trivial deformation of a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ generated by (φ, ω, ψ). Then there exists a linear map $N \in \mathfrak{gl}(L)$ such that $T_t = \mathrm{Id} + tN$ satisfies:

$$T_t[x, y]_t = [T_t(x), T_t(y)], \tag{19}$$
$$T_t\{x, y, z\}_t = \{T_t(x), T_t(y), T_t(y)\}. \tag{20}$$

By expanding each side of (19) and comparing the coefficients of t^i, we have

$$[Nx, Ny] = N\varphi(x, y); \tag{21}$$
$$\varphi(x, y) = [x, Ny] + [Nx, y] - N[x, y]. \tag{22}$$

Similarly, with respect to (20), we get

$$\omega(x, y, z) = \{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\} - N\{x, y, z\}; \tag{23}$$
$$\psi(x, y, z) = \{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\} - N\omega(x, y, z); \tag{24}$$
$$\{Nx, Ny, Nz\} = N\psi(x, y, z). \tag{25}$$

Definition 3.5. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. A Linear map $N : L \to L$ is called a Hom-Nijenhuis operator on L if for all $x, y, z \in L$, the following conditions are satisfied:*

$$[Nx, Ny] = N([x, Ny] + [Nx, y] - N[x, y]); \tag{26}$$
$$\{Nx, Ny, Nz\} = N(\{Nx, Ny, z\} + \{Nx, y, Nz\} + \{x, Ny, Nz\}) - N^2(\{Nx, y, z\}$$
$$+\{x, Ny, z\} + \{x, y, Nz\}) + N^3\{x, y, z\}; \tag{27}$$
$$N\alpha(x) = \alpha N(x). \tag{28}$$

Remark 3.6. *When a HLYA reduces to a Hom-Lie triple system, namely, the binary bracket* $[\cdot,\cdot]$ *is 0, we obtain the notion of a Nijenhuis operator on a Hom-Lie triple system obviously, which we can refer to [10].*

It is obvious that a trivial deformation of a HLYA gives rise to a Hom-Nijenhuis operator. In the sequel, we show that the converse is also true.

Lemma 3.7. *Let* $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$ *be a HLYA and* (T,β) *a vector space endowed with a binary* $[\cdot,\cdot]'$ *and a ternary bracket* $\{\cdot,\cdot,\cdot\}'$*. If there exists an isomorphism between vector spaces* $f: T\to L$ *such that for any* $x,y,z\in T$,

$$\begin{aligned} f\beta(x) &= \alpha f(x); &(29)\\ f([x,y]') &= [f(x),f(y)]; &(30)\\ f(\{x,y,z\}') &= \{f(x),f(y),f(z)\}, &(31)\end{aligned}$$

then $(T,[\cdot,\cdot]',\{\cdot,\cdot,\cdot\}',\beta)$ *is a HLYA.*

Proof. Since $f: T\to L$ is an isomorphism, by (30) and (31), we can get that

$$[x,y]' = f^{-1}[f(x),f(y)];\quad \{x,y,z\}' = f^{-1}\{f(x),f(y),f(z)\},$$

for any $x,y,z\in T$, then the conclusion follows by a direct computation.

Theorem 3.8. *Let* $N: L\to L$ *be a Hom-Nijenhuis operator on a HLYA* $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$*. Then a deformation can be obtained by letting*

$$\begin{aligned} \varphi(x,y) &= [Nx,y]+[x,Ny]-N[x,y]; &(32)\\ \omega(x,y,z) &= \{Nx,y,z\}+\{x,Ny,z\}+\{x,y,Nz\}-N\{x,y,z\}; &(33)\\ \psi(x,y,z) &= \{Nx,Ny,z\}+\{Nx,y,Nz\}+\{x,Ny,Nz\}-N\omega(x,y,z), &(34)\end{aligned}$$

for any $x,y,z\in L$*. Moreover, this deformation is trivial.*

Proof. By the fact that N is a Hom-Nijenhuis operator, the given maps φ,ω,ψ satisfy $[Nx,Ny]=N\varphi(x,y)$, $\{Nx,Ny,Nz\}=N\psi(x,y,z)$ for all $x,y,z\in L$. Hence, the given maps φ,ω and ψ satisfy Conditions (21)-(25). Therefore, $T_t=\mathrm{Id}+tN$ is a homomorphism from $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\alpha)$ to $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$. For t sufficiently small, T_t is an isomorphism between vector spaces. By Lemma 3.7, we can deduce that $(L,[\cdot,\cdot]_t,\{\cdot,\cdot,\cdot\}_t,\alpha)$ is a HLYA for t sufficiently small. Thus, (φ,ω,ψ) generates a linear deformation. It is obvious that the deformation is trivial.

Corollary 3.9. *Let* $N: L\to L$ *be a Hom-Nijenhuis operator on a HLYA* $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$*. Then* (L,φ,ψ,α) *is a HLYA, and N is homomorphism from* (L,φ,ψ,α) *to* $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$*.*

4. Product structures of Hom-Lie-Yamaguti algebras

In this section, we always suppose that $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$ be an involutive HLYA. We introduce the notion of a product structure on a HLYA using the Hom-Nijenhuis operator that we introduced in the above section. Moreover, we find some special product structures, which give special decompositions of the HLYA.

Definition 4.1. *Let* $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$ *be a HLYA. An almost product structure on L is a linear map* $E: L\to L$ *satisfying* $E^2=\mathrm{Id}$ $(E\neq\pm\mathrm{Id})$ *and* $\alpha E=E\alpha$*. An almost product structure is called a product structure if the following integrability conditions are satisfied: for any* $x,y,z\in L$,

$$\begin{aligned} [(\alpha E)x,(\alpha E)y] &= (\alpha E)[(\alpha E)x,y]+(\alpha E)[x,(\alpha E)y]-[x,y], &(35)\\ \{(\alpha E)x,(\alpha E)y,(\alpha E)z\} &= (\alpha E)\{(\alpha E)x,(\alpha E)y,z\}+(\alpha E)\{x,(\alpha E)y,(\alpha E)z\}+(\alpha E)\{(\alpha E)x,y,(\alpha E)z\}\\ &\quad -\{(\alpha E)x,y,z\}-\{x,(\alpha E)y,z\}-\{x,y,(\alpha E)z\}+(\alpha E)\{x,y,z\}. &(36)\end{aligned}$$

Remark 4.2. *A product structure E on a HLYA is exactly a Hom-Nijenhuis operator satisfying $E^2 = \mathrm{Id}$.*

Remark 4.3. *When a HLYA reduces to a Hom-Lie triple system, namely the binary bracket $[\cdot,\cdot]$ is 0, we obtain the notion of a product structure on a Hom-Lie triple system immediately, which we can refer to [10].*

We consider the vector spaces $L_+ = ker(\alpha E - \mathrm{Id}_L)$ and $L_- = ker(\alpha E + \mathrm{Id}_L)$ as eigenspaces corresponding to the eigenvalues 1 and -1 of αE, respectively.

Theorem 4.4. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there exists a product structure on L if and only if L admits a decomposition:*

$$L = L_+ \oplus L_-,$$

where L_+ and L_- are Hom-Lie-Yamaguti subalgebras of L.

Proof. Let E be a product structure on L. By $E^2 = \mathrm{Id}$, we have that $L = L_+ \oplus L_-$ as vector spaces, where L_+ and L_- are eigenspaces corresponding to the eigenvalues 1 and -1 of αE, respectively. For any $x, y, z \in L_+$, Since $(\alpha E)(x) = x$ and $\alpha E = E\alpha$, we have

$$(\alpha E)(\alpha(x)) = \alpha((E\alpha)x) = \alpha((\alpha E)x) = \alpha(x),$$

which gives $\alpha(x) \in L_+$. Similarly, we can obtain $\alpha(x') \in L_-$, for any $x' \in L_-$. Then we prove $[x, y]$, $\{x, y, z\} \in L_+$. For any $x, y, z \in L_+$,

$$\begin{aligned}(\alpha E)\{x,y,z\} &= \{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\} \\ &\quad -(\alpha E)\{(\alpha E)x, (\alpha E)y, z\} - (\alpha E)\{x, (\alpha E)y, (\alpha E)z\} - (\alpha E)\{(\alpha E)x, y, (\alpha E)z\} \\ &= 4\{x,y,z\} - 3(\alpha E)\{x,y,z\}.\end{aligned}$$

Thus, we have $(\alpha E)\{x, y, z\} = \{x, y, z\}$. Moreover, we have

$$[(\alpha E)x, (\alpha E)y] = 2(\alpha E)[x, y] - [x, y].$$

Combining this formula with $[(\alpha E)x, (\alpha E)y] = [x, y]$, we can get that $(\alpha E)[x, y] = [x, y]$. Hence,

$$[x, y], \ \ \{x, y, z\} \in L_+,$$

which implies that L_+ is a Hom-Lie-Yamaguti subalgebra. Similarly, we can get that L_- is a Hom-Lie-Yamaguti subalgebra.

Conversely, let $L = L_+ \oplus L_-$ be a decomposition of L, where $L = L_+$ and $L = L_-$ are Hom-Lie-Yamaguti subalgebras of L. We define a linear map $E : L \to L$ by

$$E(x + l) = \alpha(x) - \alpha(l), \ \ \forall x \in L_+, \ l \in L_-.$$

It's easy to prove $E^2 = \mathrm{Id}$.

Since L_+ is a Hom-Lie-Yamaguti subalgebra of L , for all $x, y, z \in L_+$, we have

$$(\alpha E)[(\alpha E)x, y] + (\alpha E)[x, (\alpha E)y] - [x, y] = 2(\alpha E)[x, y] - [x, y] = [(\alpha E)x, (\alpha E)y],$$

$$\begin{aligned}&\{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\} \\ &\ -(\alpha E)\{(\alpha E)x, (\alpha E)y, z\} - (\alpha E)\{x, (\alpha E)y, (\alpha E)z\} - (\alpha E)\{(\alpha E)x, y, (\alpha E)z\} \\ &= 4\{x,y,z\} - 3(\alpha E)\{x,y,z\} = (\alpha E)\{x,y,z\},\end{aligned}$$

which implies that (35) and (36) hold for all $x, y, z \in L_+$. Similarly, we can show that (35) and (36) hold for all $x, y, z \in L$. Thus, E is a product structure on L.

In the sequel, we study three important cases in detail, which are called strict, abelian and perfect product structures, respectively.

Proposition 4.5. *Let E be an almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. If E satisfies the following equations*

$$\begin{aligned}(\alpha E)[x, y] &= [(\alpha E)x, y], &(37)\\ (\alpha E)\{x, y, z\} &= \{(\alpha E)x, y, z\}, &(38)\\ (\alpha E)\{x, y, z\} &= \{x, y, (\alpha E)z\}, &(39)\end{aligned}$$

for all $x, y, z \in L$, then E is a product structure on L such that

$$[L_+, L_-] = 0, \ \ \{L_+, L_-, \cdot\} = \{L_+, \cdot, L_-\} = \{L_-, \cdot, L_+\} = 0,$$

i.e. L is the direct sum of two Hom-Lie-Yamaguti subalgebras.

Proof. By (37), (38), (39) and $E^2 = \mathrm{Id}$, we have

$$(\alpha E)[(\alpha E)x, y] + (\alpha E)[x, (\alpha E)y] - [x, y] = [(\alpha E)^2x, y] + [(\alpha E)x, (\alpha E)y] - [x, y] = [(\alpha E)x, (\alpha E)y],$$

$$\begin{aligned}&\{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\ &\ -(\alpha E)\{(\alpha E)x, (\alpha E)y, z\} - (\alpha E)\{(\alpha E)x, y, (\alpha E)z\} - (\alpha E)\{x, (\alpha E)y, (\alpha E)z\}\\ &= \{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + (\alpha E)\{x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\ &\ -\{(\alpha E)x, (\alpha E)y, (\alpha E)z\} - \{(\alpha E)^2x, y, (\alpha E)z\} - \{x, (\alpha E)y, (\alpha E)^2z\}\\ &= (\alpha E)\{x, y, z\},\end{aligned}$$

then E is a product structure on L.

For all $x, y \in L_+$, $l \in L_-$, $e \in L$,

$$(\alpha E)[x, l] = [(\alpha E)x, l] = [x, l],$$

which implies that $[L_+, L_-] \subset L_+$. On the other hand, we have

$$(\alpha E)[x, l] = -[(\alpha E)l, x] = -[x, l],$$

which implies that $[L_+, L_-] \subset L_-$, therefore, we have $[L_+, L_-] = 0$.

Meanwhile $(\alpha E)\{x, l, e\} = \{(\alpha E)x, l, e\} = \{x, l, e\}$, which implies that $\{L_+, L_-, \cdot\} \subset L_+$. On the other hand, we have

$$(\alpha E)\{x, l, e\} = -(\alpha E)\{l, x, e\} = \{l, x, e\} = -\{x, l, e\},$$

which implies that $\{L_+, L_-, \cdot\} \subset L_-$. Thus, we obtain that

$$\{L_+, L_-, \cdot\} = 0.$$

Moreover, we have that $(\alpha E)\{x, e, l\} = \{(\alpha E)x, e, l\} = \{x, e, l\}$. While on the other hand, by (39), we have that $(\alpha E)\{x, e, l\} = \{x, e, (\alpha E)l\} = -\{x, e, l\}$. Thus, we obtain that

$$\{L_+, \cdot, L_-\} = 0.$$

Similarly, we can get that $\{L_-, \cdot, L_+\} = 0$.

Definition 4.6. *An almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is called a strict product structure if (37), (38) and (39) hold.*

Corollary 4.7. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there is a strict product structure on L if and only if L admits a decomposition:*

$$L = L_+ \oplus L_-,$$

where L_+ and L_- are Hom-Lie-Yamaguti subalgebras of L such that

$$[L_+, L_-] = 0, \ \ \{L_+, L_-, \cdot\} = \{L_+, \cdot, L_-\} = \{L_-, \cdot, L_+\} = 0.$$

Proposition 4.8. *Let E be an almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. If E satisfies the following equations*

$$
\begin{aligned}
[(\alpha E)x, (\alpha E)y] &= -[x, y], && (40)\\
\{x, y, z\} &= -\{x, (\alpha E)y, (\alpha E)z\} - \{(\alpha E)x, (\alpha E)y, z\} - \{(\alpha E)x, y, (\alpha E)z\}, && (41)
\end{aligned}
$$

for all $x, y, z \in L$, then E is a product structure on L such that L_+ and L_- are abelian Hom-Lie-Yamaguti subalgebras of L.

Proof. By (40), (41) and $E^2 = \mathrm{Id}$, we have

$$
\begin{aligned}
& (\alpha E)[(\alpha E)x, y] + (\alpha E)[x, (\alpha E)y] - [x, y]\\
= \;& -(\alpha E)[(\alpha E)^2x, (\alpha E)y] + (\alpha E)[x, (\alpha E)y] + [(\alpha E)x, (\alpha E)y] = [(\alpha E)x, (\alpha E)y],
\end{aligned}
$$

$$
\begin{aligned}
& \{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\
& -(\alpha E)\{(\alpha E)x, (\alpha E)y, z\} - (\alpha E)\{(\alpha E)x, y, (\alpha E)z\} - (\alpha E)\{x, (\alpha E)y, (\alpha E)z\}\\
& = \{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + (\alpha E)\{x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\
& -\{(\alpha E)x, (\alpha E)y, (\alpha E)z\} - \{(\alpha E)^2x, y, (\alpha E)z\} - \{x, (\alpha E)y, (\alpha E)^2z\}\\
& = (\alpha E)\{x, y, z\},
\end{aligned}
$$

then E is a product structure on L.

For all $x, y, z \in L_+$, we have $[(\alpha E)x, (\alpha E)y] = [x, y]$, by (40), we obtain $[x, y] = 0$. And by (41), we have $\{x, y, z\} = -\{x, (\alpha E)y, (\alpha E)z\} - \{(\alpha E)x, (\alpha E)y, z\} - \{(\alpha E)x, y, (\alpha E)z\} = -3\{x, y, z\}$, which implies that

$$\{x, y, z\} = 0.$$

Similarly for all $l, m, n \in L_-$, we also have $[l, m] = 0$, $\{l, m, n\} = 0$. Thus, L_+ and L_- are abelian Hom-Lie-Yamaguti subalgebras of L.

Definition 4.9. *An almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is called an abelian product structure if (40) and (41) hold.*

Corollary 4.10. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there is an abelian product structure on L if and only if L admits a decomposition:*

$$L = L_+ \oplus L_-,$$

where L_+ and L_- are abelian Hom-Lie-Yamaguti subalgebras of L.

Proposition 4.11. *Let E be an almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. If E satisfies the following equations*

$$
\begin{aligned}
[(\alpha E)x, (\alpha E)y] &= (\alpha E)[(\alpha E)x, y] + (\alpha E)[x, (\alpha E)y] - [x, y], && (42)\\
(\alpha E)\{x, y, z\} &= \{(\alpha E)x, (\alpha E)y, (\alpha E)z\}, && (43)
\end{aligned}
$$

for all $x, y, z \in L$, then E is a product structure on L such that

$$\{L_-, L_-, L_+\} \subset L_+, \;\; \{L_+, L_-, L_-\} \subset L_+, \;\; \{L_+, L_+, L_-\} \subset L_-, \;\; \{L_+, L_-, L_+\} \subset L_-.$$

Proof. By (43) and $E^2 = \mathrm{Id}$, we have

$$
\begin{aligned}
& \{(\alpha E)x, (\alpha E)y, (\alpha E)z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\
& -(\alpha E)\{(\alpha E)x, (\alpha E)y, z\} - (\alpha E)\{(\alpha E)x, y, (\alpha E)z\} - (\alpha E)\{x, (\alpha E)y, (\alpha E)z\}\\
& = (\alpha E)\{x, y, z\} + \{(\alpha E)x, y, z\} + \{x, (\alpha E)y, z\} + \{x, y, (\alpha E)z\}\\
& -\{(\alpha E)^2x, (\alpha E)^2y, (\alpha E)z\} - \{(\alpha E)^2x, (\alpha E)y, (\alpha E)^2z\} - \{(\alpha E)x, (\alpha E)^2y, (\alpha E)^2z\}\\
& = (\alpha E)\{x, y, z\},
\end{aligned}
$$

so E is a product structure on a HLYA L. For any $x, y \in L_+, l \in L_-$, we have $(\alpha E)\{x, y, l\} = \{(\alpha E)x, (\alpha E)y, (\alpha E)l\} = -\{x, y, l\}$, which implies that

$$\{L_+, L_+, L_-\} \subset L_-.$$

Similarly, we can prove that $\{L_-, L_-, L_+\} \subset L_+,\ \{L_+, L_-, L_-\} \subset L_+,\ \{L_+, L_-, L_+\} \subset L_-$.

Definition 4.12. *An almost product structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is called a perfect product structure if (42) and (43) hold.*

Corollary 4.13. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there is a perfect product structure on L if and only if L admits a decomposition:*

$$L = L_+ \oplus L_-,$$

where L_+ and L_- are abelian Hom-Lie-Yamaguti subalgebras of L such that

$$\{L_-, L_-, L_+\} \subset L_+,\ \ \{L_+, L_-, L_-\} \subset L_+,\ \ \{L_+, L_+, L_-\} \subset L_-,\ \ \{L_+, L_-, L_+\} \subset L_-.$$

Example 4.14. *We consider a 2-dimensional involutive HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ with a basis $\{e_1, e_2\}$, where the nonzero brackets and α are given by*

$$[e_1, e_2] = e_2,\ \ \{e_1, e_2, e_2\} = e_1,$$

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ or\ \alpha_2 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},\ (a^2 = 1).$$

Then

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a perfect product structure on L.

Example 4.15. *We consider a 4-dimensional involutive HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ with a basis $\{e_1, e_2, e_3, e_4\}$, where the nonzero brackets and α are given by*

$$[e_1, e_2] = 2e_4,\ \ \{e_1, e_2, e_1\} = e_4,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is an abelien product structure and a perfect product structure on L, respectively.

5. Complex structures of Hom-Lie-Yamaguti algebras

In this section, we always suppose that $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be an involutive HLYA. We introduce the notion of a complex structure on a real Hom-Lie-Yamaguti algebra. There are also some special complex structures, totally parallelling to the case of product structures.

Definition 5.1. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. An almost complex structure on L is a linear map $J : L \to L$ satisfying $J^2 = -\mathrm{Id}$ and $\alpha J = J\alpha$. An almost complex structure is called a complex structure if the following integrability conditions are satisfied:*

$$\begin{aligned}[(\alpha J)x, (\alpha J)y] &= (\alpha J)[(\alpha J)x, y] + (\alpha J)[x, (\alpha J)y] + [x, y],\\ (\alpha J)\{x, y, z\} &= -(\alpha J)\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + \{(\alpha J)x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\}\\ &\quad +(\alpha J)\{(\alpha J)x, (\alpha J)y, z\} + (\alpha J)\{(\alpha J)x, y, (\alpha J)z\} + (\alpha E)\{x, (\alpha J)y, (\alpha J)z\},\end{aligned}$$

for any $x, y, z \in L$.

Remark 5.2. *A complex structure J on a HLYA is exactly a Hom-Nijenhuis operator satisfying $J^2 = -\mathrm{Id}$.*

Remark 5.3. *When a HLYA reduces to a Hom-Lie triple system, namely the binary bracket $[\cdot,\cdot]$ is 0, we obtain the notion of a complex structure on a Hom-Lie triple system immediately, which we can refer to [10].*

Denote by $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C} = \{x + iy : x, y \in L\}$ the complexification of the real Hom-Lie-Yamaguti algebra L, which is obviously a Hom-Lie-Yamaguti algebra. Let σ be the conjugation map in $L_{\mathbb{C}}$, i.e.

$$\sigma(x + iy) = x - iy, \ \ \forall\, x, y \in L.$$

Linear maps $J, \alpha : L \to L$ can be naturally extended to a linear map $J_{\mathbb{C}}, \alpha_{\mathbb{C}} : L_{\mathbb{C}} \to L_{\mathbb{C}}$ by

$$J_{\mathbb{C}}(x + iy) = Jx + iJy,\ \alpha_{\mathbb{C}}(x + iy) = \alpha(x) + i\alpha(y),\ \forall\, x, y \in L.$$

Theorem 5.4. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there exists a complex structure on L if and only if there is a decomposition of $L_{\mathbb{C}}$:*

$$L_{\mathbb{C}} = L_i \oplus L_{-i},$$

where L_i and L_{-i} are Hom-Lie-Yamaguti subalgebras of $L_{\mathbb{C}}$ suah that $L_{-i} = \sigma(L_i)$.

Proof. Let J be a complex structure on L. We extend the linear map $J : L \to L$ to $J_{\mathbb{C}} : L_{\mathbb{C}} \to L_{\mathbb{C}}$ as above. Then we can obtain $J_{\mathbb{C}}^2 = -\mathrm{Id}$. Furthermore, we can prove that $J_{\mathbb{C}}$ is a complex structure on $J_{\mathbb{C}}$. Let i and $-i$ be the eigenvalues of $\alpha J_{\mathbb{C}}$, L_i and L_{-i} the corresponding eigenspaces, i.e.

$$L_i = \{x - i(\alpha J)x : x \in L\},\ L_{-i} = \{x + i(\alpha J)x : x \in L\}.$$

Then $L_{\mathbb{C}} = L_i \oplus L_{-i}$ as vector spaces. Obviously, $L_{-i} = \sigma(L_i)$. For all $X, Y, Z \in L_i$, we have $(\alpha_{\mathbb{C}}J_{\mathbb{C}})[(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y] + (\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y] + [X, Y] = 2i(\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, Y] + [X, Y]$. Combining this formula with $[(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y] = -[X, Y]$, we have

$$(\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, Y] = i[X, Y].$$

Moreover, we have

$$\begin{aligned}&(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, Y, Z\}\\ =\ &-\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} + \{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y, Z\} + \{X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, Z\} + \{X, Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\}\\ &+(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, Z\} + (\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\}\\ &+(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} = 4i\{X, Y, Z\} - 3(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, Y, Z\}.\end{aligned}$$

Thus, we have

$$(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, Y, Z\} = i\{X, Y, Z\},$$

which implies that L_i is a Hom−Lie−Yamaguti subalgebra. Similarly, we can get that L_{-i} is a Hom-Lie-Yamaguti subalgebra.

Conversely, if $L_{\mathbb{C}}$ admits a decomposition $L_{\mathbb{C}} = L_i \oplus L_{-i}$, where $L_{-i} = \sigma(L_i)$ and L_i are Hom-Lie-Yamaguti subalgebras of L. Then we define a map $J_{\mathbb{C}} : L_{\mathbb{C}} \to L_{\mathbb{C}}$ by

$$J_{\mathbb{C}}(X + \sigma Y) = i\alpha_{\mathbb{C}}(X) - i\sigma(\alpha_{\mathbb{C}}(Y)),\ \forall\, X, Y \in L_i.$$

Then we have $J_{\mathbb{C}}^2(X+\sigma Y) = J_{\mathbb{C}}(i\alpha_{\mathbb{C}}(X) + \sigma(\alpha_{\mathbb{C}}(-iY))) = -\mathrm{Id}(X+\sigma Y)$.

For any $X, Y, Z \in L_i$, we get

$$\begin{aligned} & (\alpha_{\mathbb{C}}J_{\mathbb{C}})[(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y] + (\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y] + [X, Y] \\ = \ & 2i(\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, Y] + [X, Y] = [(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y], \end{aligned}$$

and

$$\begin{aligned} & -\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} + \{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y, Z\} + \{X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, Z\} + \{X, Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} \\ & +(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, Z\} + (\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} \\ & +(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, Y, (\alpha_{\mathbb{C}}J_{\mathbb{C}})Z\} = 4i\{X, Y, Z\} - 3(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, Y, Z\} \\ & = (\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X, Y, Z\}. \end{aligned}$$

Thus, $J_{\mathbb{C}}$ is a complex structure on L_i. Similarly, we can get that $J_{\mathbb{C}}$ is a complex structure on $L_{\mathbb{C}}$. Since $(J_{\mathbb{C}}\sigma)(X+\sigma(Y)) = iY - i\sigma(X) = (\sigma J_{\mathbb{C}})(X+\sigma(Y))$, thus, we have $J_{\mathbb{C}}\sigma = \sigma J_{\mathbb{C}}$. Hence, there is a well-defined linear map $J \in \mathfrak{gl}(L)$ given by $J := J_{\mathbb{C}}|_L$, which is a complex structure on L.

Parallel to the case of product structures, there are also some special complex structures on a HLYA. In the sequel, we study three important cases in detail, which are called the strict, abelian and perfect complex structures.

Proposition 5.5. *Let J be an almost complex structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. If J satisfies the following equations*

$$(\alpha J)[x, y] = [(\alpha J)x, y], \tag{44}$$

$$(\alpha J)\{x, y, z\} = \{(\alpha J)x, y, z\}, \tag{45}$$

$$(\alpha J)\{x, y, z\} = \{x, y, (\alpha J)z\}, \tag{46}$$

for all $x, y, z \in L$, then J is a complex structure on L such that

$$[L_i, L_{-i}] = 0, \quad \{L_i, L_{-i}, \cdot\} = \{L_i, \cdot, L_{-i}\} = \{L_{-i}, \cdot, L_i\} = 0,$$

where L_i and L_{-i} are two Hom-Lie-Yamaguti subalgebras of $L_{\mathbb{C}}$ such that $L_{-i} = \sigma(L_i)$.

Proof. By (44), (45), (46) and $J^2 = -\mathrm{Id}$, we have

$$(\alpha J)[(\alpha J)x, y] + (\alpha J)[x, (\alpha J)y] + [x, y] = [(\alpha J)^2x, y] + [(\alpha J)x, (\alpha J)y] + [x, y] = [(\alpha J)x, (\alpha J)y],$$

$$\begin{aligned} & -\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + \{(\alpha J)x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\} \\ & +(\alpha J)\{(\alpha J)x, (\alpha J)y, z\} + (\alpha J)\{(\alpha J)x, y, (\alpha J)z\} + (\alpha J)\{x, (\alpha J)y, (\alpha J)z\} \\ & = -\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + (\alpha J)\{x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\} \\ & +\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + \{(\alpha J)^2x, y, (\alpha J)z\} + \{x, (\alpha J)y, (\alpha J)^2z\} \\ & = (\alpha J)\{x, y, z\}. \end{aligned}$$

Then J is a complex structure on L.

For all $X, Y, Z \in L_i$, $\sigma(Z) \in L_{-i}$, $M \in L_{\mathbb{C}}$, by (44), we have

$$(\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, \sigma(Z)] = [(\alpha_{\mathbb{C}}J_{\mathbb{C}})X, \sigma(Z)] = i[X, \sigma(Z)],$$

which implies that $[L_i, L_{-i}] \subset L_i$. On the other hand, we have

$$(\alpha_{\mathbb{C}}J_{\mathbb{C}})[X, \sigma(Z)] = -(\alpha_{\mathbb{C}}J_{\mathbb{C}})[\sigma(Z), X] = -i[X, \sigma(Z)],$$

which implies that $[L_i, L_{-i}] \subset L_{-i}$, therefore, we have $[L_i, L_{-i}] = 0$.

Meanwhile $(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X,\sigma(Z),M\}=\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X,\sigma(Z),M\}=i\{X,\sigma(Z),M\}$, which implies that $\{L_i,L_{-i},\cdot\}\subset L_i$. On the other hand, we have

$$(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X,\sigma(Z),M\}=-\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})\sigma(Z),X,M\}=-i\{X,\sigma(Z),M\},$$

which implies that $\{L_i,L_{-i},\cdot\}\subset L_{-i}$. Thus, we obtain that

$$\{L_i,L_{-i},\cdot\}=0.$$

Moreover, we have that $(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X,M,\sigma(Z)\}=\{(\alpha_{\mathbb{C}}J_{\mathbb{C}})X,M,\sigma(Z)\}=i\{X,M,\sigma(Z)\}$. While on the other hand, by (46), we have that $(\alpha_{\mathbb{C}}J_{\mathbb{C}})\{X,M,\sigma(Z)\}=\{X,M,(\alpha_{\mathbb{C}}J_{\mathbb{C}})\sigma(Z)\}=-i\{X,M,\sigma(Z)\}$. Thus, we obtain that

$$\{L_i,\cdot,L_{-i}\}=0.$$

Similarly, we can get that $\{L_{-i},\cdot,L_i\}=0$.

Definition 5.6. *An almost complex structure on a HLYA $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$ is called a strict complex structure if (44), (45) and (46) hold.*

Corollary 5.7. *Let $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$ be a HLYA. Then there is a strict complex structure on L if and only if $L_{\mathbb{C}}$ admits a decomposition:*

$$L_{\mathbb{C}}=L_i\oplus L_{-i},$$

where L_i and L_{-i} are Hom-Lie-Yamaguti subalgebras of $L_{\mathbb{C}}$ such that $L_{-i}=\sigma(L_i)$.

Proposition 5.8. *Let J be an almost complex structure on a HLYA $(L,[\cdot,\cdot],\{\cdot,\cdot,\cdot\},\alpha)$. If J satisfies the following equations*

$$[(\alpha J)x,(\alpha J)y] = [x,y], \tag{47}$$
$$\{x,y,z\} = \{x,(\alpha J)y,(\alpha J)z\}+\{(\alpha J)x,(\alpha J)y,z\}+\{(\alpha J)x,y,(\alpha J)z\}, \tag{48}$$

for all $x,y,z\in L$, then J is a complex structure on L such that L_i and $L_{-i}=\sigma(L_i)$ are abelian Hom-Lie-Yamaguti subalgebras of L.

Proof. By (47), (48) and $J^2=-\mathrm{Id}$, we have

$$(\alpha J)[(\alpha J)x,y]+(\alpha J)[x,(\alpha J)y]+[x,y]=(\alpha J)[(\alpha J)^2x,(\alpha J)y]+(\alpha J)[x,(\alpha J)y]+[(\alpha J)x,(\alpha J)y]=[(\alpha J)x,(\alpha J)y],$$

$$\begin{aligned}
&-\{(\alpha J)x,(\alpha J)y,(\alpha J)z\}+\{(\alpha J)x,y,z\}+\{x,(\alpha J)y,z\}+\{x,y,(\alpha J)z\}\\
&\ +(\alpha J)\{(\alpha J)x,(\alpha J)y,z\}+(\alpha J)\{(\alpha J)x,y,(\alpha J)z\}+(\alpha J)\{x,(\alpha J)y,(\alpha J)z\}\\
&=-\{(\alpha J)x,(\alpha J)^2y,(\alpha J)^2z\}+\{(\alpha J)^2x,(\alpha J)^2y,(\alpha J)z\}+\{(\alpha J)^2x,(\alpha J)y,(\alpha J)^2z\}\\
&\ +\{(\alpha J)x,y,z\}+\{x,(\alpha J)y,z\}+\{x,y,(\alpha J)z\}+(\alpha J)\{x,y,z\}\\
&=(\alpha J)\{x,y,z\},
\end{aligned}$$

then J is a complex structure on L.

For all $X,Y,Z\in L_i$, we have $[(\alpha_{\mathbb{C}}J_{\mathbb{C}})X,(\alpha_{\mathbb{C}}J_{\mathbb{C}})Y]=-[X,Y]$, by (47), we obtain $[X,Y]=0$. So, we obtain

$$[L_i,L_i]=0.$$

And by (48), we have $\{X,Y,Z\}=\{X,(\alpha J)Y,(\alpha J)Z\}+\{(\alpha J)X,(\alpha J)Y,Z\}+\{(\alpha J)X,Y,(\alpha J)Z\}=-3\{X,Y,Z\}$, which implies that

$$\{L_i,L_i,L_i\}=0.$$

Thus, L_i is an abelian Hom-Lie-Yamaguti subalgebra of L. Similarly L_{-i} is also an abelian Hom-Lie-Yamaguti subalgebra of L.

Definition 5.9. *An almost complex structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is called an abelian complex structure if (47) and (48) hold.*

Corollary 5.10. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there is an abelian complex structure on L if and only if L admits a decomposition:*

$$L = L_i \oplus L_{-i},$$

where L_i and $L_{-i} = \sigma(L_i)$ are abelian Hom-Lie-Yamaguti subalgebras of L.

Proposition 5.11. *Let J be an almost complex structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. If J satisfies the following equations*

$$[(\alpha J)x, (\alpha J)y] = (\alpha J)[(\alpha J)x, y] + (\alpha J)[x, (\alpha J)y] + [x, y], \tag{49}$$

$$(\alpha J)\{x, y, z\} = -\{(\alpha J)x, (\alpha J)y, (\alpha J)z\}, \tag{50}$$

for all $x, y, z \in L$, then J is a complex structure on L such that

$$\{L_{-i}, L_{-i}, L_i\} \subset L_i,\ \ \{L_i, L_{-i}, L_{-i}\} \subset L_i,\ \ \{L_i, L_i, L_{-i}\} \subset L_{-i},\ \ \{L_i, L_{-i}, L_i\} \subset L_{-i},$$

where L_i and $L_{-i} = \sigma(L_i)$ are Hom-Lie-Yamaguti subalgebras of L.

Proof. By (50) and $J^2 = -\mathrm{Id}$, we have

$$\begin{aligned}
&-\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + \{(\alpha J)x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\}\\
&\ +(\alpha J)\{(\alpha J)x, (\alpha J)y, z\} + (\alpha J)\{(\alpha J)x, y, (\alpha J)z\} + (\alpha J)\{x, (\alpha J)y, (\alpha J)z\}\\
&= (\alpha J)\{x, y, z\} + \{(\alpha J)x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\}\\
&\ -\{(\alpha J)^2x, (\alpha J^2)y, (\alpha J)z\} - \{(\alpha J)^2x, (\alpha J)y, (\alpha J^2)z\} - \{(\alpha J)x, (\alpha J)^2y, (\alpha J)^2z\}\\
&= (\alpha J)\{x, y, z\},
\end{aligned}$$

so J is a complex structure on L.

For any $X, Y \in L_i$, $\sigma(z) \in L_{-i}$, we have

$$(\alpha_{\mathbb{C}} J_{\mathbb{C}})\{X, Y, \sigma(z)\} = -\{(\alpha_{\mathbb{C}} J_{\mathbb{C}})X, (\alpha_{\mathbb{C}} J_{\mathbb{C}})Y, (\alpha_{\mathbb{C}} J_{\mathbb{C}})\sigma(Z)\} = -i\{X, Y, \sigma(z)\},$$

which implies that

$$\{L_i, L_i, L_{-i}\} \subset L_{-i}.$$

Similarly, we can prove that $\{L_{-i}, L_{-i}, L_i\} \subset L_i$, $\{L_i, L_{-i}, L_{-i}\} \subset L_i$, $\{L_i, L_{-i}, L_i\} \subset L_{-i}$.

Definition 5.12. *An almost complex structure on a HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ is called a perfect complex structure if (49) and (50) hold.*

Corollary 5.13. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a HLYA. Then there is a perfect complex structure on L if and only if $L_{\mathbb{C}}$ admits a decomposition:*

$$L_{\mathbb{C}} = L_i \oplus L_{-i},$$

where L_i and $L_{-i} = \sigma(L_i)$ are Hom-Lie-Yamaguti subalgebras of L such that

$$\{L_{-i}, L_{-i}, L_i\} \subset L_i,\ \ \{L_i, L_{-i}, L_{-i}\} \subset L_i,\ \ \{L_i, L_i, L_{-i}\} \subset L_{-i},\ \ \{L_i, L_{-i}, L_i\} \subset L_{-i}.$$

Example 5.14. *Consider the 2-dimensional involutive HLYA $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ given in Example 4.14, $\alpha = \alpha_2$ and $a = 1$. Considering its complex structure,*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is an abelien complex structure on L.

6. Complex product structures of Hom-Lie-Yamaguti algebras

In this section, by adding a compatibility condition between a product structure and a complex structure on an involutive HLYA, we introduce the notion of a complex product structure.

Proposition 6.1. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a complex HLYA. Then the following statements are equivalent:*
(i) E is a product structure on L;
(ii) $J = iE$ is a complex structure on L.

Proof. Let E be a product structure on L. Then we have $J^2 = i^2E^2 = -\mathrm{Id}$. Moreover, we can get

$$\begin{aligned}
& (\alpha J)[(\alpha J)x, y] + (\alpha J)[x, (\alpha J)y] + [x, y] \\
= \;& i(\alpha E)[i(\alpha E)x, y] + i(\alpha E)[x, i(\alpha E)y] + [x, y] = -(\alpha E)[(\alpha E)x, y] - (\alpha E)[x, (\alpha E)y] + [x, y] \\
= \;& -[(\alpha E)x, (\alpha E)y] = [(\alpha J)x, (\alpha J)y],
\end{aligned}$$

and

$$\begin{aligned}
& -\{(\alpha J)x, (\alpha J)y, (\alpha J)z\} + \{(\alpha J)x, y, z\} + \{x, (\alpha J)y, z\} + \{x, y, (\alpha J)z\} \\
& +(\alpha J)\{(\alpha J)x, (\alpha J)y, z\} + (\alpha J)\{(\alpha J)x, y, (\alpha J)z\} + (\alpha J)\{x, (\alpha J)y, (\alpha J)z\} \\
= & -\{i(\alpha E)x, i(\alpha E)y, i(\alpha E)z\} + \{i(\alpha E)x, y, z\} + \{x, i(\alpha E)y, z\} + \{x, y, i(\alpha E)z\} \\
& +i(\alpha E)\{i(\alpha E)x, i(\alpha E)y, z\} + i(\alpha E)\{i(\alpha E)x, y, i(\alpha E)z\} + i(\alpha E)\{x, i(\alpha E)y, i(\alpha E)z\} \\
= & \; i(\alpha E)\{x, y, z\} = (\alpha J)\{x, y, z\}.
\end{aligned}$$

Thus, J is a complex structure on L. The converse part can be proved similarly.

In the sequel, we give the definition of a complex product structure on a real Hom-Lie-Yamaguti algebra.

Definition 6.2. *Let $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ be a real HLYA. A complex product structure on L is a pair (J, E) consisting of a complex structure J and a product structure E such that*

$$JE = -EJ.$$

In the following, we see that a complex product structure on a real Hom-Lie-Yamaguti algebra can be obtained from a given complex structure and a product structure under some certain conditions.

Theorem 6.3. *Let J be a complex structure and E a product structure on a real Hom-Lie-Yamaguti algebra $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$. Then (J, E) is a complex product structure on L if and only if*

$$L_- = (J\alpha)L_+,$$

where $L_\pm$ are the eigenspaces corresponding to the eigenvalues ± 1 of αE.

Proof. Let (J, E) be a complex product structure. Then by Theorem 4.4, L has a decomposition $L = L_+ \oplus L_-$ such that both L_+ and L_- are Hom-Lie-Yamaguti subalgebras of L. On one hand,

$$-(JE)(L_+) = -(JE\alpha)(\alpha(L_+)) = -(J\alpha)((E\alpha)(L_+)) = -(J\alpha)(L_+),$$

and

$$(EJ)(L_+) = (EJ\alpha)(\alpha(L_+)) = (E\alpha)((J\alpha)(L_+)),$$

so we can obtain $(J\alpha)(L_+) \subseteq L_-$. On the other hand,

$$-(JE)(L_-) = -(JE\alpha)(\alpha(L_-)) = (J\alpha)(L_-),$$

and

$$(EJ)(L_-) = (EJ\alpha)(\alpha(L_-)) = (E\alpha)((J\alpha)(L_-)),$$

so we can obtain $(J\alpha)(L_-) \subseteq L_+$, i.e. $(J\alpha)^2(L_-) \subseteq (J\alpha)(L_+)$, which implies $L_- \subseteq (J\alpha)(L_+)$. Thus $L_- = (J\alpha)(L_+)$.

Conversely, for any $x \in L_+$, $y \in L_-$, we have $(\alpha E)(\alpha J)(x + y) = (\alpha E)((\alpha J)y + (\alpha J)x) = (\alpha J)y - (\alpha J)x = -(\alpha J)(\alpha E)(x + y)$, which implies that $EJ = -JE$. Thus, (J, E) is a complex product structure on L.

Example 6.4. *Consider the* 2*-dimensional involutive HLYA* $(L, [\cdot,\cdot], \{\cdot,\cdot,\cdot\}, \alpha)$ *given in Example 4.14 and Example 5.14. Then* (J, E) *is a complex product structure, where*

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

References

[1] A. Andrada, Complex product structures on 6-dimensional nilpotent Lie algebras, Forum Math. 20(2) (2008), 285–315.
[2] A. Andrada, M. L. Barberis, and I. Dotti, Classification of abelian complex structures on 6-dimensional Lie algebras, J. Lond. Math. Soc. 83(1) (2011), 232–255.
[3] A. Andrada, M. L. Barberis, I. Dotti, and G. P. Ovando, Product structures on four dimensional solvable Lie algebras, Homology Homotopy Appl. 7(1) (2005), 9–37.
[4] A. Andrada, and S. Salamon, Complex product structures on Lie algebras, Forum Math. 17(2) (2005), 261–295.
[5] P. Benito, A. Elduque, and F. Martín-Herce, Irreducible Lie-Yamaguti algebras, J. Pure Appl. Algebra 213 (2009), 795–808.
[6] P. Benito, A. Elduque, and F. Marťin-Herce, Irreducible Lie-Yamaguti algebras of generic type, J. Pure Appl. Algebra 215(2) (2011), 108–130.
[7] I. Dorfman, Dirac Structures and Integrability of Nonlinear Evolution Equations (Wiley Chichester, 1993).
[8] D. Gaparayi, and A. Nourou Issa, A twisted generalization of Lie-Yamaguti algebras, Int. J. Algebra 6 (2012), 339–352.
[9] J. T. Hartwig, D. Larsson, and S. D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra 295 (2006), 314–361.
[10] Y. Hou, Y. Mao, and L. Y. Chen, Product and complex structures on Hom-Lie triple systems, https://www. researchgate. net/publication/341360022.
[11] M. Kikkawa, Geometry of homogeneous Lie loops, Hiroshima Math. J. 5(2) (1975), 141–179.
[12] M. K. Kinyon, and A. Weinstein, Leibniz algebras, Courant algebriods and multiplications on reductive homogeneous spaces, Amer. J. Math. 123 (2001), 525–550.
[13] Y. Ma, L.Y. Chen, and J. Lin, One-parameter formal deformations of Hom-Lie-Yamaguti algebras, J. Math. Phys. 56 (2015), 011701.
[14] Y. H. Sheng, J. Zhou, and Y. Q. Zhou, Nijenhuis operators, product structures and complex structures on Lie-Yamaguti algebras, J. Algebra Appl. (2021), 2150146 (22 pages).
[15] K. Yamaguti, On the Lie triple system and its generalization, J. Sci. Hiroshima. Univ. Ser. A 21 (1957/1958), 155–160.
[16] K. Yamaguti, On cohomology groups of general Lie triple systems, Kumamoto J. Sci. A 8 (1967/1969), 135–146.
[17] T. Zhang, and J. Li, Deformations and extension of Lie-Yamaguti algebras, Linear Mulitilinear Algebra 63(11) (2015), 2212–2231.
[18] T. Zhang, and J. Li, Representations and cohomologies of Hom-Lie-Yamaguti algebras with applications. Colloq. Math. 148(1) (2017), 131–155.