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Szász Type Operators Involving Charlier Polynomials and Approximation Properties

A.A.H. Al-Abied[a], M. Ayman Mursaleen[b,c], M. Mursaleen[d]

[a]*Department of Mathematics, Dhamar University, Dhamar, Yemen*
[b]*School of Information & Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia*
[c]*Department of Mathematics & Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[d]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

Abstract. Our aim is to define modified Szász type operators involving Charlier polynomials and obtain some approximation properties. We prove some results on the order of convergence by using the modulus of smoothness and Peetre's K-functional. We also establish Voronoskaja type theorem for these operators. Moreover, we prove a Korovkin type approximation theorem via q-statistical convergence.

1. Introduction and background

It is immensely acknowledged that the most studied positive linear operators are Bernstein operators which have a lot of generalizations over the time. The Szász–Mirakyan operators were introduced by Otto Szász in 1950 [20] and G. M. Mirakjan in 1941 [17] to overcome the disadvantage of Bernstein polynomials which were defined only for finite intervals. The Szász–Mirakyan operators are generalizations of Bernstein polynomials to infinite intervals which are defined as follows

$$S_n(f;\zeta) = e^{-n\zeta}\sum_{k=0}^{\infty}\frac{(n\zeta)^k}{k!}f\left(\frac{k}{n}\right) \tag{1}$$

for each positive n and $f \in C[0,\infty)$, the space of continuous functions on $[0,\infty)$. Many authors generalize the above operators under different conditions and variations depending upon the nature of functions to be approximated by these polynomials. One of such polynomials are the Charlier polynomials [8] which have the generating function of the form

$$e^t\left(1-\frac{t}{a}\right)^u = \sum_{k=0}^{\infty}\mathfrak{C}_k^{(l)}(u)\frac{t^k}{k!}, \qquad |t| < a \tag{2}$$

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Corresponding author: M. Mursaleen
Email addresses: abeid1979@gmail.com (A.A.H. Al-Abied), mohdaymanm@gmail.com (M. Ayman Mursaleen), mursaleenm@gmail.com (M. Mursaleen)

where $\mathfrak{C}_k^{(l)}(u) = \sum\limits_{r=0}^{k} \binom{k}{r} (-u)_r \left(\frac{1}{l}\right)^r$ and $(u)_0 = 1, (u)_j = u(u+1)...(u+j-1)$, for $j \geq 1$. The Szász type operators involving Charlier polynomials [22] were defined by

$$\mathbb{D}_n(f;\zeta,l) = e^{-1}\left(1-\frac{1}{l}\right)^{(l-1)n\zeta} \sum_{k=0}^{\infty} \frac{\mathfrak{C}_k^{(l)}(-(l-1)n\zeta)}{k!} f\left(\frac{k}{n}\right) \tag{3}$$

where $l > 1$ and $\zeta \geq 0$.

There are several generalizations and variants of the Szász operators given by (1) depending on the situation under consideration for approximating certain functions. For more details about the generalizations and variations of the Szász operators, one can refer to [2, 4, 10–15, 18, 23]. One of the general classes are the following operators which were studied in [9, 19, 21, 24]

$$S_n(f, \alpha_n, \beta_n; \zeta) = e^{-\alpha_n \zeta} \sum_{k=0}^{\infty} \frac{(\alpha_n \zeta)^k}{k!} f\left(\frac{k}{\beta_n}\right) \tag{4}$$

known as the generalized Favard-Szász type operators, where α_n and β_n denote the unbounded and increasing sequences of positive numbers such that

$$\lim_{n\to\infty} \beta_n^{-1} = 0, \ \ \alpha_n \beta_n^{-1} = 1 + O\left(\beta_n^{-1}\right). \tag{5}$$

The case $\alpha_n = \beta_n = n$ gives the operators (1). The approximation properties of (4) have been studied in [9] and [24].

We define here a more general class of operators (3) as follows

$$\mathbb{D}_n^*(f;\zeta,a) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta} \sum_{k=0}^{\infty} \frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} f\left(\frac{k}{\beta_n}\right), \tag{6}$$

for $f \in C_B[0,\infty)$, the space of continuous and bounded functions on $[0,\infty)$. We establish some approximation properties for these operators and also find the rate of convergence. Note that if $\alpha_n = \beta_n = n$ for all $n \in \mathbb{N}$ then the operators (6) are reduced to the operators (3).

2. Moments

We first obtain the moments of the operators (6).

Lemma 2.1. *We have*

(i) $\mathbb{D}_n^*(1;\zeta,a) = 1,$

(ii) $\mathbb{D}_n^*(t;\zeta,a) = \frac{\alpha_n}{\beta_n}\zeta + \frac{1}{\beta_n},$

(iii) $\mathbb{D}_n^*(t^2;\zeta,a) = \frac{\alpha_n^2}{\beta_n^2}\zeta^2 + \frac{\alpha_n}{\beta_n^2}\left(3+\frac{1}{a-1}\right)\zeta + \frac{2}{\beta_n^2},$

(iv) $\mathbb{D}_n^*(t^3;\zeta,a) = \frac{\alpha_n^3}{\beta_n^3}\zeta^3 + \frac{\alpha_n^2}{\beta_n^3}\left(6+\frac{3}{a-1}\right)\zeta^2 + \frac{\alpha_n}{\beta_n^3}\left(10+\frac{6}{a-1}+\frac{2}{(a-1)^2}\right)\zeta + \frac{5}{\beta_n^3},$

(v)
$$\mathbb{D}_n^*(t^4;\zeta,a) = \frac{\alpha_n^4}{\beta_n^4}\zeta^4 + \frac{\alpha_n^3}{\beta_n^4}\left(10+\frac{6}{a-1}\right)\zeta^3 + \frac{\alpha_n^2}{\beta_n^4}\left(32+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right)\zeta^2 + \frac{\alpha_n}{\beta_n^4}\left(37+\frac{32}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)\zeta + \frac{15}{\beta_n^4}.$$

Proof. With the help of (2), we get

(i) $\sum_{k=0}^{\infty} \frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta}$,

(ii) $\sum_{k=0}^{\infty} \frac{k\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta} (1+\alpha_n\zeta)$,

(iii) $\sum_{k=0}^{\infty} \frac{k^2\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta} \left\{\alpha_n^2\zeta^2 + \alpha_n x\left(3+\frac{1}{a-1}\right)+2\right\}$,

(iv) $\sum_{k=0}^{\infty} \frac{k^3\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta}$
$\times\left\{\alpha_n^3\zeta^3 + \alpha_n^2\left(6+\frac{3}{a-1}\right)\zeta^2 + \alpha_n\left(10+\frac{6}{a-1}+\frac{2}{(a-1)^2}\right)\zeta+5\right\}$,

(v) $\sum_{k=0}^{\infty} \frac{k^4\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta}$
$\times\left\{\alpha_n^4\zeta^4 + \alpha_n^3\left(10+\frac{6}{a-1}\right)\zeta^3 + \alpha_n^2\left(32+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right)\zeta^2\right.$
$\left.+\alpha_n\left(37+\frac{32}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)\zeta+15\right\}$.

Using the above relations, we get the required moments. □

Lemma 2.2. *We have*

(i) $\mathbb{D}_n^*(t-\zeta;\zeta,a) = \left(\frac{\alpha_n}{\beta_n}-1\right)\zeta + \frac{1}{\beta_n}$,

(ii) $\mathbb{D}_n^*\left((t-\zeta)^2;\zeta,a\right) = \left(\frac{\alpha_n}{\beta_n}-1\right)^2\zeta^2 + \left\{\frac{\alpha_n}{\beta_n^2}\left(3+\frac{1}{a-1}\right)-\frac{2}{\beta_n}\right\}\zeta + \frac{2}{\beta_n^2}$,

(iii) $\mathbb{D}_n^*\left((t-\zeta)^4;\zeta,a\right) = \left(\frac{\alpha_n}{\beta_n}-1\right)^4\zeta^4$
$+2\left\{\frac{\alpha_n^3}{\beta_n^4}\left(5+\frac{3}{a-1}\right)-\frac{6\alpha_n^2}{\beta_n^3}\left(2+\frac{1}{a-1}\right)+\frac{3\alpha_n}{\beta_n^2}\left(3+\frac{1}{a-1}\right)-\frac{2}{\beta_n}\right\}\zeta^3$
$+\left\{\frac{\alpha_n^2}{\beta_n^4}\left(32+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right)-\frac{4\alpha_n}{\beta_n^3}\left(10+\frac{6}{a-1}+\frac{2}{(a-1)^2}\right)+\frac{12}{\beta_n^2}\right\}\zeta^2$
$+\left\{\frac{\alpha_n}{\beta_n^4}\left(37+\frac{32}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)-\frac{20}{\beta_n^3}\right\}\zeta+\frac{15}{\beta_n^4}$.

3. Approximation in weighted spaces

Let $\mathbf{B}_\rho = \{f :| f(\zeta) |\leq \mathbf{M}_f\rho(\zeta), \mathbf{M}_f > 0\}$ with the norm $\| f \|_\rho = \sup_{\zeta\geq 0}\frac{f(\zeta)}{\rho(\zeta)}$. Let $\mathbf{C}_\rho = \{f \in \mathbf{B}_\rho : f$ is continuous$\}$ and $\mathbf{C}_\rho^k = \left\{f\in\mathbf{C}_\rho : \lim_{|\zeta|\to\infty}\frac{f(\zeta)}{\rho(\zeta)} = \mathbf{k}_f\right\}$, where $\rho(\zeta) = 1+\zeta^2$, $\zeta\in(-\infty,\infty)$ (see [7], [16]).

Theorem 3.1. *([7]) Let $(B_n)_{n\geq 0}$ be the sequence of positive linear operators which acts from $\mathbf{C}_\rho$ to $\mathbf{B}_\rho$ such that*

$$\lim_{n\to\infty} \|B_n(t^i;\zeta)-\zeta^i\|_\rho = 0, \quad i\in\{0,1,2\}.$$

Then

$$\lim_{n\to\infty} \|B_n f - f\|_\rho = 0,$$

for $f \in C_\rho^k$, *and there exists* $f^* \in C_\rho \backslash C_\rho^k$ *such that*

$$\lim_{n\to\infty} \|B_n f^* - f^*\|_\rho \geq 1.$$

Theorem 3.2. *For* $f \in C_\rho^k$, *we have*

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(f;\zeta,a) - f(\zeta)\|_\rho = 0.$$

Proof. From Lemma 2.1(i), it is immediate that

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(1;\zeta,a) - 1\|_\rho = 0. \tag{7}$$

Using Lemma 2.1(ii) and (5), we have

$$\|\mathbb{D}_n^*(t;\zeta,a) - \zeta\|_\rho = \Big(\frac{\alpha_n}{\beta_n} - 1\Big)\sup_{\zeta\geq 0}\frac{\zeta}{1+\zeta^2} + \frac{1}{\beta_n}\sup_{\zeta\geq 0}\frac{1}{1+\zeta^2}.$$

Hence we obtain

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(t;\zeta,a) - \zeta\|_\rho = 0. \tag{8}$$

By means of Lemma 2.1(iii) and (5), we get

$$\begin{aligned}&\|\mathbb{D}_n^*(t^2;\zeta,a) - \zeta^2\|_\rho\\ &= \left(\frac{\alpha_n^2}{\beta_n^2} - 1\right)\sup_{\zeta\geq 0}\frac{\zeta^2}{1+\zeta^2} + \frac{\alpha_n}{\beta_n^2}\left(3 + \frac{1}{a-1}\right)\sup_{\zeta\geq 0}\frac{\zeta}{1+\zeta^2} + \frac{2}{\beta_n^2}\sup_{\zeta\geq 0}\frac{1}{1+\zeta^2},\end{aligned}$$

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(t^2;\zeta,a) - \zeta^2\|_\rho = 0. \tag{9}$$

From (7), (8) and (9), for $i \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(t^i;\zeta,a) - \zeta^i\|_\rho = 0.$$

Using Theorem 3.1, we get the desired result. □

4. Order of convergence

Definition 4.1. *The modulus of continuity of second order is defined by*

$$\varpi_2(f,\delta) = \sup_{0<h\leq\delta}\sup_{\zeta\in[0,\infty)} |f(\zeta+h) - 2f(\zeta) + f(\zeta-h)|,\ f \in C_B[0,\infty).$$

Definition 4.2. *[5] The Peetre's K-functional of* $f \in C_B[0,\infty)$ *is defined by*

$$K(f,\delta) := \inf_{g\in C_B^2[0,\infty)}\Big\{\|f-g\|_\infty + \delta\|g\|_{C_B^2}\Big\},$$

where $C_B^2[0,\infty)$ *is the space of all* $g \in C_B[0,\infty)$ *such that* $g',\ g \in C_B[0,\infty)$ *with* $\|g\|_{C_B^2} := \|g\|_\infty + \|g'\|_\infty + \|g''\|_\infty$.

Note that

$$K(f,\delta) \leq M\Big\{\varpi_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_\infty\Big\},\ \delta > 0, \tag{10}$$

where the constant M is independent of f and δ.

Theorem 4.3. *Let $f \in C[0,\infty)$ and $|f(\zeta)| \le Me^{A\zeta}$, $A \in \mathbb{R}, M \in \mathbb{R}^+$. Then*

$$\lim_{n\to\infty} \mathbb{D}_n^*(f;\zeta,a) = f(\zeta)$$

and the operators (6) *converge uniformly in each compact subset of* $[0,\infty)$.

Let $\tilde{C_B}[0,\infty) = \{f : [0,\infty) \to \mathbb{R}$ such that f is uniformly continuous and bounded$\}$, $\|f\|_{\tilde{C_B}[0,\infty)} = \sup_{\zeta\in[0,\infty)} |f(\zeta)|$.

Theorem 4.4. *For $f \in C_B^2[0,\infty)$, we have*

$$|\mathbb{D}_n^*(f;\zeta,a) - f(\zeta)| \le \frac{1}{\beta_2}\left\{1 + \frac{1}{2}\left(1 + \frac{1}{a-1}\right)\right\} \|f\|_{C_B^2}.$$

Proof. By using the Taylor formula and the linearity of $\mathbb{D}_n^*$, we have

$$\mathbb{D}_n^*(f;\zeta,a) - f(\zeta) = \mathbb{D}_n^*(t-\zeta;\zeta,a)f'(\zeta) + \frac{1}{2}\mathbb{D}_n^*\left((t-\zeta)^2;\zeta,a\right)f''(\xi) \tag{11}$$

where $\zeta < \xi < t$. From Lemma 2.2 (i)-(ii) and (11), we obtain

$$\begin{aligned}
&|\mathbb{D}_n^*(f;\zeta,a) - f(\zeta)| \\
&\quad\le \left[\left(\frac{\alpha_n}{\beta_n} - 1\right)\zeta + \frac{1}{\beta_n}\right]\|f'\|_{C_B} \\
&\qquad + \frac{1}{2}\left[\left(\frac{\alpha_n}{\beta_n} - 1\right)^2\zeta^2 + \left\{\frac{\alpha_n}{\beta_n^2}\left(3 + \frac{1}{a-1}\right) - \frac{2}{\beta_n}\right\}\zeta + \frac{2}{\beta_n^2}\right]\|f''\|_{C_B}.
\end{aligned}$$

For sufficiently large n, we obtain

$$\begin{aligned}
&|\mathbb{D}_n^*(f;\zeta,a) - f(\zeta)| \\
&\quad\le \frac{1}{\beta_n}\|f'\|_{C_B} + \frac{1}{2}\left[\frac{1}{\beta_n}\left(1 + \frac{1}{a-1}\right) + \frac{2}{\beta_n}\right]\|f''\|_{C_B} \\
&\quad\le \frac{1}{\beta_n}\left\{1 + \frac{1}{2}\left(1 + \frac{1}{a-1}\right)\right\}\left(\|f'\|_{C_B} + \|f''\|_{C_B}\right).
\end{aligned}$$

This completes the proof. □

Theorem 4.5. *For $f \in C_B[0,\infty)$, we have*

$$\begin{aligned}
&|\mathbb{D}_n^*(f;\zeta,a) - f(\zeta)| \\
&\quad\le 2M\left[\varpi_2\left(f, \sqrt{\frac{1}{2}\tau(\alpha_n,\beta_n;\zeta)}\right) + \min\left\{\left(1, \frac{1}{2}\tau(\alpha_n,\beta_n;\zeta)\right)\right\}\|f\|_\infty\right]
\end{aligned}$$

where

$$\tau(\alpha_n,\beta_n;\zeta) = \left(\frac{\alpha_n}{\beta_n} - 1\right)^2\zeta^2 + \frac{1}{\beta_n}\left\{\alpha_n\left(4 + \frac{1}{a-1}\right) - (\beta_n + 2)\right\}\zeta + \frac{\beta_n + 2}{\beta_n^2},$$

and $M > 0$ is a constant.

Proof. We have

$$f(t) - f(\zeta) = f(t) - g(t) + g(t) - g(\zeta) + g(\zeta) - f(\zeta).$$

Then using the linearity property of $\mathbb{D}_n^*$, we get

$$\begin{aligned}&|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|\\&\quad\leq|\mathbb{D}_n^*(f-g;\zeta,a)|+|\mathbb{D}_n^*(g;\zeta,a)-g(\zeta)|+|f(\zeta)-g(\zeta)|.\end{aligned}$$

Consider the function $g\in C_B^2[0,\infty)$. By Theorem 4.4, we have

$$\begin{aligned}|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)| &\leq 2\|f-g\|+\tau(\alpha_n,\beta_n;\zeta)\|g\|_{C_B^2}\\&\leq 2K\left(f;\frac{1}{2}\tau(\alpha_n,\beta_n;\zeta)\right),\end{aligned}$$

and using (10), we obtain the result. □

Let $f\in\tilde{C}[0,\infty)$, the space of uniformly continuous functions on $[0,\infty)$. Then

$$\tilde{w}(f,\delta)=\sup_{\zeta\in[0,\infty),|\zeta-y|\leq\delta}|f(\zeta)-f(y)|.$$

Theorem 4.6. *If $f\in\tilde{C}[0,\infty)$ and $|f(\zeta)|\leq Me^{A\zeta}$, then*

$$|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|\leq\left\{1+\sqrt{\left(1+\frac{1}{a-1}\right)\zeta+\frac{2}{\beta_n}}\right\}\tilde{\omega}\left(f;\frac{1}{\sqrt{\beta_n}}\right)\tag{12}$$

for sufficiently large n.

Proof. According to Lemma 2.1 (i) we have

$$\begin{aligned}&|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|\\&\quad\leq e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left|f\left(\frac{k}{\beta_n}\right)-f(\zeta)\right|\end{aligned}$$

$$\begin{aligned}&|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|\\&\leq\left\{1+\frac{1}{\delta}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left|\frac{k}{\beta_n}-\zeta\right|\right\}\tilde{\omega}(f;\delta).\end{aligned}\tag{13}$$

By using the Cauchy-Schwarz inequality and Lemma 2.1 and (5), we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left|\frac{k}{\beta_n}-\zeta\right|\\&\quad\leq\left(e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left(\frac{k}{\beta_n}-\zeta\right)^2\right)^{\frac{1}{2}}\\&\quad=e\left(1-\frac{1}{a}\right)^{-(a-1)\alpha_n\zeta}\sqrt{\frac{1}{\beta_n}\left(1+\frac{1}{a-1}\right)\zeta+\frac{2}{\beta_n^2}}.\end{aligned}$$

By choosing $\delta=\delta_n=\frac{1}{\sqrt{\beta_n}}$, (13) leads to

$$\begin{aligned}|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)| &\leq \left\{1+\frac{1}{\delta}\sqrt{\frac{1}{\beta_n}\left(1+\frac{1}{a-1}\right)\zeta+\frac{2}{\beta_n^2}}\right\}\tilde{\omega}(f;\delta)\\&\leq\left\{1+\sqrt{\left(1+\frac{1}{a-1}\right)\zeta+\frac{2}{\beta_n}}\right\}\tilde{\omega}\left(f;\frac{1}{\sqrt{\beta_n}}\right).\end{aligned}$$

This completes the proof. $\square$

We define a weighted modulus of continuity $\Omega(f;\delta)$ for the function $f \in C_\rho^k$ by

$$\Omega_n(f;\delta) = \sup_{|t-\zeta|\leq\delta;\ \zeta,t\in[0,\infty)} \frac{|f(t)-f(\zeta)|}{[1+(t-\zeta)^2]\rho(\zeta)}.$$

Note that $\lim_{\delta\to 0}\Omega(f;\delta)=0$.

Lemma 4.7. *Let $f \in C_\rho^k$. Then we have*

(*i*) *$\Omega(f;\delta)$ is a monotonically increasing function of $\delta \geq 0$.*
(*ii*) *For each positive value of λ*

$$\Omega_n(f;\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f;\delta). \tag{14}$$

From (14), we get

$$|f(t)-f(\zeta)| \leq 2\left(1+\frac{|t-\zeta|}{\delta}\right)(1+\delta^2)(1+\zeta^2)\left(1+(t-\zeta)^2\right)\Omega(f;\delta) \tag{15}$$

for every $f \in C_\rho^k$ and $\zeta, t \in [0,\infty)$.

Theorem 4.8. *If $f \in C_\rho^k$, then*

$$\sup_{\zeta\geq 0}\frac{|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|}{(1+\zeta^2)^3} \leq K\Omega\left(f;\frac{1}{\sqrt{\beta_n}}\right)$$

holds for a sufficiently large n, where K is a constant independent of α_n, β_n.

Proof. From (15), we get

$$\begin{aligned}
&|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)| \\
&\quad\leq 2(1+\delta_n^2)\Omega(f;\delta_n)(1+\zeta^2)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta} \\
&\qquad\times\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left(1+\frac{\left|\frac{k}{\beta_n}-\zeta\right|}{\delta_n}\right)\left(1+\left(\frac{k}{\beta_n}-\zeta\right)^2\right) \\
&\quad\leq 4\Omega(f;\delta_n)(1+\zeta^2) \\
&\qquad\times\left\{1+\frac{1}{\delta_n}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left|\frac{k}{\beta_n}-\zeta\right|\right. \\
&\qquad+e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left(\frac{k}{\beta_n}-\zeta\right)^2 \\
&\qquad\left.+\frac{1}{\delta_n}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left|\frac{k}{\beta_n}-\zeta\right|\left(\frac{k}{\beta_n}-\zeta\right)^2\right\}
\end{aligned}$$

for any $\delta_n > 0$. Applying Cauchy-Schwartz inequality, we obtain

$$|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)| \leq 4\Omega(f;\delta_n)(1+\zeta^2)\left\{1+\frac{2}{\delta_n}\sqrt{B_1}+B_1+\frac{1}{\delta_n}B_2\right\} \tag{16}$$

where

$$\begin{aligned} B_1 &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left(\frac{k}{\beta_n}-\zeta\right)^2, \\ B_2 &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)\alpha_n\zeta}\sum_{k=0}^{\infty}\frac{\mathfrak{C}_k^{(a)}(-(a-1)\alpha_n\zeta)}{k!}\left(\frac{k}{\beta_n}-\zeta\right)^4. \end{aligned}$$

From Lemma 2.2 (ii)-(iii) and using condition (5) in Lemma 2.2 (ii)-(iii), we can write

$$\begin{aligned} B_1 &= O\left(\frac{1}{\beta_n}\right)(\zeta^2+\zeta), \\ B_2 &= O\left(\frac{1}{\beta_n}\right)(\zeta^4+\zeta^3+\zeta^2+\zeta). \end{aligned}$$

Substituting these results in (16), we have

$$\begin{aligned} &|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)| \\ &\leq 4\Omega(f;\delta_n)(1+\zeta^2)\left\{1+\frac{2}{\delta_n}\sqrt{O\left(\frac{1}{\beta_n}\right)(\zeta^2+\zeta)}+O\left(\frac{1}{\beta_n}\right)(\zeta^2+\zeta)\right. \\ &\quad \left.+\frac{1}{\delta_n}O\left(\frac{1}{\beta_n}\right)(\zeta^4+\zeta^3+\zeta^2+\zeta)\right\}. \end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{\beta_n}}$, for sufficiently large n, we obtain

$$\sup_{x\geq 0}\frac{|\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)|}{(1+\zeta^2)^3} \leq K\Omega\left(f;\frac{1}{\sqrt{\beta_n}}\right)$$

where K is a constant independent of α_n, β_n. □

5. Voronoskaja type theorem

Theorem 5.1. *If $f \in C_B^2[0,\infty)$, then*

$$\lim_{n\to\infty}\alpha_n[\mathbb{D}_n^*(f;\zeta,a)-f(\zeta)] = f'(\zeta)+\zeta f''(\zeta)$$

holds for every $\zeta \in [0,a]$.

Proof. For a fixed point $\zeta_0 \in [0,\infty)$ and for all $\zeta \in [0,\infty)$, by the Taylor formula we have

$$f(\zeta)-f(\zeta_0) = (\zeta-\zeta_0)f'(\zeta_0)+\frac{1}{2}(\zeta-\zeta_0)^2 f''(\zeta_0)+\varphi(\zeta,\zeta_0)(\zeta-\zeta_0)^2$$

where $\varphi(\zeta,\zeta_0) \in C_B[0,\infty)$ and $\lim_{\zeta\to\zeta_0}\varphi(\zeta,\zeta_0) = 0$.

$$\begin{aligned} &\alpha_n\Big[\mathbb{D}_n^*(f;\zeta_0,a)-f(\zeta_0)\Big] \\ &= \alpha_n\mathbb{D}_n^*\Big((e_1-\zeta_0);\zeta_0,a\Big)f'(\zeta_0)+\frac{1}{2}\alpha_n\mathbb{D}_n^*\Big((e_1-\zeta_0)^2;\zeta_0,a\Big)f''(\zeta_0) \\ &\quad +\alpha_n\mathbb{D}_n^*\Big(\varphi(\zeta,\zeta_0)(\zeta-\zeta_0)^2;\zeta_0,a\Big) \end{aligned}$$

holds for $\zeta_0 \in [0,\infty)$. Using Lemma 2.2 (i) and (ii), we have

$$\lim_{n\to\infty}\alpha_n\mathbb{D}_n^*\Big((e_1-\zeta_0);\zeta_0,a\Big) = 1,$$

$$\lim_{n\to\infty} \alpha_n \mathbb{D}_n^*\Big((e_1-\zeta_0)^2;\zeta_0,a\Big)=\zeta_0.$$

Now, we consider the $\mathbb{D}_n^*\Big(\varphi(\zeta,\zeta_0)(\zeta-\zeta_0)^2;\zeta_0,a\Big)$. Use the Cauchy-Schwarz inequality to get

$$\alpha_n \mathbb{D}_n^*\Big(\varphi(\zeta,\zeta_0)(\zeta-\zeta_0)^2;\zeta_0,a\Big)\le \sqrt{\alpha_n^2 \mathbb{D}_n^*\Big((e_1-\zeta_0)^4;\zeta_0,a\Big)\mathbb{D}_n^*\Big(\varphi^2(\zeta,\zeta_0);\zeta_0,a\Big)}.$$

One has from Lemma 2.2 (iii) that

$$\lim_{n\to\infty} \alpha_n^2 \mathbb{D}_n^*\Big((e_1-\zeta_0)^4;\zeta_0,a\Big)=4\zeta_0^3.$$

Since for the function $\psi(\zeta,\zeta_0)=\varphi^2(\zeta,\zeta_0),\ \zeta\ge 0$, we have $\psi(\zeta,\zeta_0)\in C_B[0,\infty)$ and $\lim_{\zeta\to\zeta_0}\psi(\zeta,\zeta_0)=0$.

$$\lim_{n\to\infty} \mathbb{D}_n^*\Big(\varphi^2(\zeta,\zeta_0);\zeta_0,a\Big)=\varphi^2(\zeta,\zeta_0),$$

$$\lim_{n\to\infty} \mathbb{D}_n^*\Big(\psi(\zeta,\zeta_0);\zeta_0,a\Big)=\psi(\zeta_0,\zeta_0)=0$$

holds. Consequently, we have

$$\lim_{n\to\infty} \alpha_n \mathbb{D}_n^*\Big(\varphi(\zeta,\zeta_0)(\zeta-\zeta_0)^2;\zeta_0,a\Big)=0.$$

Now, taking the limit as $n\to\infty$, we have

$$\lim_{n\to\infty} \alpha_n[\mathbb{D}_n^*(f;\zeta_0,a)-f(\zeta_0)]=f'(\zeta_0)+\zeta_0 f''(\zeta_0).$$

This completes the proof. □

6. Approximation via q-statistical convergence

The q-integer ($q>0$) of any positive integer n is defined by

$$[n]=[n]_q=\begin{cases}\dfrac{1-q^n}{1-q}, & q\ne 1,\\ n, & q=1.\end{cases}$$

Recently, Aktuğlu and Bekar [3] studied the notion of density and statistical convergence via q-calculus. Let E be subset of the set of natural numbers $\mathbb{N}$. Then

$$\delta_q(E)=\delta_{C_1^q}(E)=\lim\inf_{n\to\infty}(C_1^q\chi_E)_n,\ q\ge 1,$$

defines the q-density, where $C^1(q)=(c_{nk}^1(q^k))_{n,k=0}^{\infty}$ is the q-Cesàro matrix (see [1], [3]) defined by

$$c_{nk}^1(q^k)=\begin{cases}\frac{q^k}{[n+1]_q} & \text{if } k\le n,\\ 0 & \text{otherwise.}\end{cases}.$$

A sequence $x=(x_k)$ is said to be q-statistically convergent to the number l if $\delta_q(\mathcal{L}_\varepsilon)=0$, where $\mathcal{L}_\varepsilon=\{k\le n:|x_k-l|\ge\varepsilon\}$ for every $\varepsilon>0$ and we write $St_q-\lim x_k=l$.

Note that for an infinite set E, $\delta(E)=0$ implies $\delta_q(E)=0$. Hence statistical convergence [6] implies q-statistical convergence but not conversely (c.f. [3, Example 15]).

Theorem 6.1. *For $f\in C_B[0,\infty)$, we have*

$$St_q-\lim_n \|\mathbb{D}_n^*(f;\zeta,a)-f\|_\infty=0.$$

Proof. From Lemma 2.1(i), it is immediate that

$$\lim_{n\to\infty} \|\mathbb{D}_n^*(1;\zeta,a) - 1\|_\infty = 0. \tag{17}$$

Using Lemma 2.1(ii) and (5), we have

$$\|\mathbb{D}_n^*(t;\zeta,a) - \zeta\|_\infty = \Big(\frac{\alpha_n}{\beta_n} - 1\Big)\sup_{x\geq 0}\frac{\zeta}{1+\zeta^2} + \frac{1}{\beta_n}\sup_{x\geq 0}\frac{1}{1+\zeta^2}.$$

Now, for a given positive $\epsilon > 0$, let

$$D_1 := \quad \Big\{n : \|\mathbb{D}_n^*(t;\zeta,a) - \zeta\|_\infty \geq \epsilon\Big\},$$

$$D_2 := \quad \Big\{n : \frac{\alpha_n}{\beta_n} - 1 \geq \frac{\epsilon}{2}\Big\},$$

$$D_3 := \quad \Big\{n : \frac{1}{\beta_n} \geq \frac{\epsilon}{2}\Big\}.$$

Then $D_1 \subseteq D_2 \bigcup D_3$ which implies that $\delta_q(D_1) \leq \delta_q(D_2) + \delta_q(D_3)$. Hence, we have

$$St_q - \lim_{n\to\infty} \|\mathbb{D}_n^*(t;\zeta,a) - \zeta\|_\infty = 0. \tag{18}$$

By means of Lemma 2.1(iii) and (5), we get

$$\|\mathbb{D}_n^*(t^2;\zeta,a) - \zeta^2\|_\infty$$
$$= \left(\frac{\alpha_n^2}{\beta_n^2} - 1\right)\sup_{\zeta\geq 0}\frac{\zeta^2}{1+\zeta^2} + \frac{\alpha_n}{\beta_n^2}\left(3 + \frac{1}{a-1}\right)\sup_{\zeta\geq 0}\frac{\zeta}{1+\zeta^2} + \frac{2}{\beta_n^2}\sup_{\zeta\geq 0}\frac{1}{1+\zeta^2},$$

For a given positive $\epsilon > 0$, let

$$E_1 := \quad \left\{n : \left\|\mathbb{D}_n^*(t^2;\zeta,a) - \zeta^2\right\|_\infty \geq \epsilon\right\},$$

$$E_2 := \quad \left\{n : \left(\frac{\alpha_n^2}{\beta_n^2} - 1\right) \geq \frac{\epsilon}{3}\right\},$$

$$E_3 := \quad \left\{n : \frac{\alpha_n}{\beta_n}\left(3 + \frac{1}{a-1}\right) \geq \frac{\epsilon}{3}\right\},$$

$$E_4 := \quad \left\{n : \frac{2}{\beta_n^2} \geq \frac{\epsilon}{3}\right\},$$

Then $E_1 \subseteq E_2 \bigcup E_3 \bigcup E_4$, which implies that $\delta_q(E_1) \leq \delta_q(E_2) + \delta_q(E_3) + \delta_q(E_4)$. Hence

$$St_q - \lim_{n\to\infty} \|\mathbb{D}_n^*(t^2;\zeta,a) - \zeta^2\|_\infty = 0. \tag{19}$$

From (17), (18) and (19), for $i \in \{0,1,2\}$, we have

$$St_q - \lim_{n\to\infty} \|\mathbb{D}_n^*(t^i;\zeta,a) - \zeta^i\|_\infty = 0.$$

Using Theorem 3.1, we get the desired result.

□

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