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On the Matrix Classes (c_0, c_0) and $(c_0, c_0; P)$ over Complete Ultrametric Fields

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Abstract. Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Sequences, infinite series and infinite matrices have entries in K. In this paper, we record some interesting properties about the matrix classes (c_0, c_0) and $(c_0, c_0; P)$.

1. Introduction

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Sequences, infinite series and infinite matrices have entries in K.

c_0 denotes the ultrametric (or non-archimedean) Banach space of all null sequences with entries in K. If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$, is an infinite matrix, we write $A \in (c_0, c_0)$ if

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots$$

is defined and the sequence $A(x) = \{(Ax)_n\} \in c_0$, whenever $x = \{x_k\} \in c_0$.

The following result is well-known (see [1]).

Theorem 1.1. $\sum_{k=0}^{\infty} x_k$ *converges if and only if* $\lim_{k\to\infty} x_k = 0$.

In view of Theorem 1.1, if $\{x_n\} \in c_0$, then $\sum_{k=0}^{\infty} x_k$ converges and so that following is relevant. We write $A = (a_{nk}) \in (c_0, c_0; P)$ if $A \in (c_0, c_0)$ and

$$\sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k, x = \{x_k\} \in c_0.$$

The following result can be easily proved.

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Theorem 1.2. *$A = (a_{nk}) \in (c_0, c_0)$ if and only if*

$$\sup_{n,k} |a_{nk}| < \infty; \tag{1}$$

and

$$\lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots. \tag{2}$$

Further, $A \in (c_0, c_0; P)$ if and only if (1) and (2) hold and

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots. \tag{3}$$

The matrix classes (c_0, c_0) and $(c_0, c_0; P)$ were studied by the author in [5] in the context of Steinhaus type theorems.

2. Main Results

In this section, we prove the main results of the paper.

Theorem 2.1. *(c_0, c_0) is a Banach algebra, with identity, under the usual matrix product.*

Proof. It is clear that (c_0, c_0) is a normed linear space under the norm

$$||A|| = \sup_{n,k} |a_{nk}|, A = (a_{nk}) \in (c_0, c_0). \tag{4}$$

Let, now, $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0)$. Let, for convenience, $C = (c_{nk}) = AB$ and $x = \{x_k\} \in c_0$. Now,

$$\begin{aligned}(Cx)_n &= \sum_{k=0}^{\infty} c_{nk} x_k \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right) x_k.\end{aligned}$$

Consider $\sum_{i=0}^{\infty} a_{ni} \left(\sum_{k=0}^{\infty} b_{ik} x_k \right)$. Note that $(Bx)_i = \sum_{k=0}^{\infty} b_{ik} x_k$ and $\{(Bx)_i\} \in c_0$, since $B \in (c_0, c_0)$. Since $A \in (c_0, c_0)$,

$$\sum_{i=0}^{\infty} a_{ni} (Bx)_i \to 0, n \to \infty.$$

We know that, in ultrametric fields, unconditional convergence and convergence are equivalent (see [6]) and so

$$\begin{aligned}&\sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right) x_k \\ &\qquad = \sum_{i=0}^{\infty} a_{ni} \left(\sum_{k=0}^{\infty} b_{ik} x_k \right).\end{aligned}$$

Thus

$$\begin{aligned}(Cx)_n &= \sum_{i=0}^{\infty} a_{ni} \left(\sum_{k=0}^{\infty} b_{ik} x_k \right) \\ &\to 0, n \to \infty,\end{aligned}$$

as noted above. Hence $C \in (c_0, c_0)$ and so (c_0, c_0) is closed under matrix product. Also

$$\begin{aligned}||AB|| &= \sup_{n,k} |c_{nk}| \\ &= \sup_{n,k} \left| \sum_{i=0}^{\infty} a_{ni} b_{ik} \right| \\ &\leq \left(\sup_{n,k} |a_{nk}| \right) \left(\sup_{n,k} |b_{nk}| \right) \\ &= ||A|| ||B||.\end{aligned}$$

We have proved above that

$$(AB)(x) = A(B(x)), x \in c_0,$$

using which, we can prove the associative law

$$(AB)C = A(BC), A, B, C \in (c_0, c_0).$$

We can check the other algebraic laws to conclude that (c_0, c_0) is an algebra. The unit matrix I,

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c_0)$$

is the identity element of the algebra (c_0, c_0). Finally, we conclude the proof by proving that (c_0, c_0) is complete under the norm defined by (4). To this end, let $\{A^{(n)}\}$ be a Cauchy sequence in (c_0, c_0), where

$$A^{(n)} = (a_{ij}^{(n)}), i, j = 0, 1, 2, \dots; n = 0, 1, 2, \dots.$$

Since $\{A^{(n)}\}$ is Cauchy, for $\epsilon > 0$, there exists a positive integer n_0 such that

$$\begin{aligned}&||A^{(m)} - A^{(n)}|| < \epsilon, m, n \geq n_0, \\ &\textit{i.e.}, \sup_{i,j} |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0.\end{aligned}$$

Thus, for all $i, j = 0, 1, 2, \dots,$

$$|a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0. \tag{5}$$

So $\{a_{ij}^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in K, $i, j = 0, 1, 2, \dots$. Since K is complete,

$$a_{ij}^{(n)} \to a_{ij}, n \to \infty \text{ in } K, i, j = 0, 1, 2, \dots.$$

Consider the infinite matrix $A = (a_{ij})$, $i, j = 0, 1, 2, \ldots$. Using (5), for all $n \geq n_0$, allowing $m \to \infty$, we get

$$\begin{aligned} & |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, i, j = 0, 1, 2, \ldots, \\ \textit{i.e.,}\ & \sup_{i,j} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, \\ \textit{i.e.,}\ & \|A^{(n)} - A\| \leq \epsilon, n \geq n_0, \\ \textit{i.e.,}\ & A^{(n)} \to A, n \to \infty. \end{aligned} \tag{6}$$

We now claim that $A \in (c_0, c_0)$. Now, in view of (6),

$$|a_{ij} - a_{ij}^{(n_0)}| \leq \epsilon, i, j = 0, 1, 2, \ldots. \tag{7}$$

Since $A = (a_{ij}^{(n_0)}) \in (c_0, c_0)$,

$$\sup_{i,j} |a_{ij}^{(n_0)}| = M < \infty. \tag{8}$$

Now, for all $i, j = 0, 1, 2, \ldots,$

$$\begin{aligned} |a_{ij}| &= \left| \left\{ a_{ij} - a_{ij}^{(n_0)} \right\} + a_{ij}^{(n_0)} \right| \\ &\leq \max \left[\left| a_{ij} - a_{ij}^{(n_0)} \right|, \left| a_{ij}^{(n_0)} \right| \right] \\ &< \max[\epsilon, M], \text{ using (7) and (8)} \\ &< \infty, \end{aligned}$$

so that

$$\sup_{i,j} |a_{ij}| < \infty.$$

Also,

$$\lim_{i \to \infty} a_{ij}^{(n_0)} = 0, j = 0, 1, 2, \ldots,$$

since $A = (a_{ij}^{(n_0)}) \in (c_0, c_0)$. For $j = 0, 1, 2, \ldots$, taking limit as $i \to \infty$ in (7), we get

$$\begin{aligned} & \left| \lim_{i \to \infty} a_{ij} - 0 \right| \leq \epsilon, \\ \textit{i.e.,}\ & \left| \lim_{i \to \infty} a_{ij} \right| \leq \epsilon, \text{ for every } \epsilon > 0, \\ \textit{i.e.,}\ & \lim_{i \to \infty} a_{ij} = 0, j = 0, 1, 2, \ldots. \end{aligned}$$

Consequently

$$A \in (c_0, c_0),$$

completing the proof of the theorem. □

Theorem 2.2. *$(c_0, c_0; P)$, as a subset of (c_0, c_0), is a closed K-convex semigroup with identity.*

Proof. Let $A = (a_{nk}), B = (b_{nk}), C = (c_{nk}) \in (c_0, c_0; P)$. Let λ, μ, γ be such that $|\lambda|, |\mu|, |\gamma| \leq 1$ and $\lambda + \mu + \gamma = 1$. Now,

$$(\lambda A + \mu B + \gamma C)_{nk} = \lambda a_{nk} + \mu b_{nk} + \gamma c_{nk},$$

from which we have

$$\lim_{n\to\infty}(\lambda A + \mu B + \gamma C)_{nk} = 0, \text{ since } \lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} = \lim_{n\to\infty} c_{nk} = 0,$$
$$A, B, C \in (c_0, c_0; P).$$

Also, since $|\lambda|, |\mu|, |\gamma| \leq 1$ and $A, B, C \in (c_0, c_0; P)$,

$$\begin{aligned}\sup_{n,k} |\lambda A + \mu B + \gamma C|_{nk} \\ &\leq \max\left[|\lambda| \sup_{n,k} |a_{nk}|, |\mu| \sup_{n,k} |b_{nk}|, |\gamma| \sup_{n,k} |c_{nk}|\right] \\ &\leq \max\left[\sup_{n,k} |a_{nk}|, \sup_{n,k} |b_{nk}|, \sup_{n,k} |c_{nk}|\right] \\ &< \infty.\end{aligned}$$

So

$$\lambda A + \mu B + \gamma C \in (c_0, c_0),$$

using Theorem 1.2.
Also, since $A, B, C \in (c_0, c_0; P)$, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda A + \mu B + \gamma C)_{nk} &= \lambda \sum_{n=0}^{\infty} a_{nk} + \mu \sum_{n=0}^{\infty} b_{nk} + \gamma \sum_{n=0}^{\infty} c_{nk} \\ &= \lambda(1) + \mu(1) + \gamma(1) \\ &= \lambda + \mu + \gamma \\ &= 1.\end{aligned}$$

Hence $\lambda A + \mu B + \gamma C \in (c_0, c_0; P)$, proving that $(c_0, c_0; P)$ is a K-convex subset of (c_0, c_0) (for the definition of K-convexity, one can refer to [5]).

We next claim that $(c_0, c_0; P)$ is closed. Let

$$A = (a_{nk}) \in \overline{(c_0, c_0; P)}.$$

There exist $A^{(m)} = (a_{nk}^{(m)}) \in (c_0, c_0; P)$, $m = 0, 1, 2, \ldots$ such that $A^{(m)} \to A$, $m \to \infty$. So, given $\epsilon > 0$, there exists a positive integer N such that

$$||A^{(m)} - A|| < \epsilon, m \geq N. \tag{9}$$

Now, for $n, k = 0, 1, 2, \ldots,$

$$\begin{aligned}|a_{nk}| &= \left|\left\{a_{nk} - a_{nk}^{(N)}\right\} + a_{nk}^{(N)}\right| \\ &\leq \max\left[\left|a_{nk} - a_{nk}^{(N)}\right|, \left|a_{nk}^{(N)}\right|\right] \qquad (10) \\ &\leq \max\left[\sup_{n,k}\left|a_{nk} - a_{nk}^{(N)}\right|, \sup_{n,k}\left|a_{nk}^{(N)}\right|\right] \\ &= \max[||A^{(N)} - A||, ||A^{(N)}||] \\ &< \max[\epsilon, ||A^{(N)}||], \text{ using (9)},\end{aligned}$$

and thus

$$\sup_{n,k} |a_{nk}| < \infty.$$

From (10), for $k = 0, 1, 2, \ldots,$

$$|a_{nk}| \le \max \left[\left\| A^{(N)} - A \right\|, \left| a_{nk}^{(N)} \right| \right]. \tag{11}$$

Since $\lim_{n\to\infty} a_{nk}^{(N)} = 0$, there exists a positive integer N' such that

$$\left| a_{nk}^{(N)} \right| < \epsilon, n \ge N'. \tag{12}$$

Using (9) and (12) in (11), we get, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} |a_{nk}| &\le \max[\epsilon, \epsilon] \\ &= \epsilon, n \ge N', \end{aligned}$$

$$i.e., \lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots.$$

Thus $A \in (c_0, c_0)$. Again, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \left| \sum_{n=0}^{\infty} a_{nk} - 1 \right| &= \left| \sum_{n=0}^{\infty} a_{nk} - \sum_{n=0}^{\infty} a_{nk}^{(N)} \right|, \text{since} \\ &\qquad\qquad A^{(N)} \in (c_0, c_0; P) \\ &= \left| \sum_{n=0}^{\infty} (a_{nk} - a_{nk}^{(N)}) \right| \\ &\le \sup_{n,k} \left| a_{nk} - a_{nk}^{(N)} \right| \\ &= \|A - A^{(N)}\| \\ &< \epsilon, \text{ using (9), for every } \epsilon > 0. \end{aligned}$$

It now follows that

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots.$$

Consequently, $A \in (c_0, c_0; P)$ and hence $(c_0, c_0; P)$ is closed. It remains to check closure under matrix product. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0; P)$. We have already proved that $AB \in (c_0, c_0)$. Since $\sum_{n=0}^{\infty} a_{nk} = \sum_{n=0}^{\infty} b_{nk} = 1$, $k = 0, 1, 2, \ldots$ and using the fact that convergence and unconditional convergence are equivalent in K (see [6]), for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \sum_{n=0}^{\infty} (AB)_{nk} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik} \right) \\ &= \sum_{i=0}^{\infty} b_{ik} \left(\sum_{n=0}^{\infty} a_{ni} \right) \\ &= \sum_{i=0}^{\infty} b_{ik} \\ &= 1, \end{aligned}$$

proving that $A \in (c_0, c_0; P)$. The identity of the semi-group $(c_0, c_0; P)$ is the unit matrix I,

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c_0; P),$$

completing the proof of the theorem. □

Note 2.3. *In the context of Theorem 2.2, we can check that $(c_0, c_0; P)$ is not a group. We can give an example of a matrix in $(c_0, c_0; P)$, which does not have an inverse.*

In the classical set up, the author defined the convolution product $\circ$ in [4]. We retain the same definition in the ultrametric set up too.

Definition 2.4. *For $A = (a_{nk})$, $B = (b_{nk})$, define*

$$(A \circ B)_{nk} = \sum_{i=0}^{n} a_{ik} b_{n-i,k}, n, k = 0, 1, 2, \dots.$$

$A \circ B = ((A \circ B)_{nk})$ is called the convolution product of A and B.

We keep the usual norm structure in (c_0, c_0) so that (c_0, c_0) is a Banach space. We replace the usual matrix product by the convolution product $\circ$ and prove the next result.

Theorem 2.5. *(c_0, c_0) is a commutative Banach algebra with identity under the convolution product $\circ$.*

Proof. We will prove closure under the convolution product $\circ$. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0)$. Since $\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} = 0$, $k = 0, 1, 2, \dots$, using Theorem 1 of [3],

$$\begin{aligned}(A \circ B)_{nk} &= \sum_{i=0}^{n} a_{ik} b_{n-i,k} \\ &= a_{0k} b_{n,k} + a_{1k} b_{n-1,k} + \dots + a_{nk} b_{0,k} \\ &\to 0, n \to \infty.\end{aligned}$$

Now, since $A, B \in (c_0, c_0)$,

$$\begin{aligned}\sup_{n,k} |(A \circ B)_{nk}| &= \sup_{n,k} \left| \sum_{i=0}^{n} a_{ik} b_{n-i,k} \right| \\ &\le \left(\sup_{n,k} |a_{nk}| \right) \left(\sup_{n,k} |b_{nk}| \right) \qquad (13) \\ &< \infty.\end{aligned}$$

Thus $A \circ B \in (c_0, c_0)$. Also,

$$\|A \circ B\| \le \|A\| \|B\|, \text{ using (13).}$$

It is clear that $\circ$ is commutative. The identity element of (c_0, c_0) under the convolution product $\circ$ is the matrix $E = (e_{nk})$, whose first row consists of 1's and which has 0's elsewhere, i.e.,

$$\begin{aligned} e_{0k} &= 1, k = 0, 1, 2, \dots; \\ e_{nk} &= 0, n = 1, 2, \dots; k = 0, 1, 2, \dots. \end{aligned}$$

Note also that $\|E\| = 1$ and $E \in (c_0, c_0; P)$. It remains to prove that $(c_0, c_0; P)$ is closed under the convolution product $\circ$. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c_0; P)$. Since $\sum_{n=0}^{\infty} a_{nk} = \sum_{n=0}^{\infty} b_{nk} = 1$, $k = 0, 1, 2, \ldots,$

$$\begin{aligned}\sum_{n=0}^{\infty}(A \circ B)_{nk} &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_{ik} b_{n-i,k}\right)\\ &= \left(\sum_{n=0}^{\infty} b_{nk}\right)\left(\sum_{n=0}^{\infty} a_{nk}\right)\\ &= 1, k = 0, 1, 2, \ldots.\end{aligned}$$

Hence $A \circ B \in (c_0, c_0; P)$. This completes the proof of the theorem. □

Corollary 2.6. *$(c_0, c_0; P)$, as a subset of the algebra (c_0, c_0) under the convolution product $\circ$, is a semigroup without identity.*

The classical analogous of the above results for conservative and regular matrices were studied by Maddox in [2] and those for (ℓ_1, ℓ_1) and $(\ell_1, \ell_1; P)$ matrices were studied by the author in [4].

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