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A Note on Evolution Equation on Manifold

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Abstract. In the present paper, considering the differential equations on smooth closed manifolds, we investigate and establish the well-posedness of boundary value problems nonlocal type for parabolic equations and also hyperbolic equations in Hölder spaces. Furthermore, in various Hölder norms we establish new coercivity estimates for the solutions of such type parabolic boundary value problems on manifolds and hyperbolic boundary value problems on manifolds as well.

1. Introduction

The role played by the coercivity inequalities (well-posedness) is well known (see, e.g. [1–3]) in the study of boundary value problems involving partial differential equations. The well-posedness of partial differential equations of nonlocal parabolic and hyperbolic types in the Euclidean space has been well studied (see, for example [4–9],[11–20] and also the references therein).

The present article considers the differential equations on smooth closed manifolds, investigates and establishes in Hölder spaces the well posedness of nonlocal type boundary value problems on manifolds. In addition, for the solutions of such boundary value problems for parabolic equations and hyperbolic equations on manifolds it establishes new coercivity inequlities in various Hölder norms.

2. Laplacian on Riemannian manifolds

In this section, we will recall the basic definitions and fact about the Laplacian on Riemannian manifolds. For more information and unexplained subjects, the reader is referred to [22, 23], and the references therein.

A pair $(\mathcal{M}, g)$ is a *Riemannian manifold* if $\mathcal{M}$ is a smooth manifold and g is a map assigning to each $x \in \mathcal{M}$ a symmetric positive definite non degenerate bilinear form $\langle \cdot, \cdot \rangle_{g(x)} : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ so that for all smooth vector fields $X, Y \in \Gamma_{C^\infty}(T\mathcal{M})$, the map $x \longmapsto \langle X(x), Y(x) \rangle_{g(x)}$ is smooth.

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For the local coordinates $(x_1, \dots, x_n)$, let us consider (U, φ) corresponding natural basis $\left\{\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x\right\}$ of tangent space $T_x\mathcal{M}$, then by g_{ij} we denote $\left\langle\left(\frac{\partial}{\partial x_i}\right)_x, \left(\frac{\partial}{\partial x_j}\right)_x\right\rangle_{g(x)}$. We also denote by g^{ij} the entries of the inverse matrix of (g_{ij}).

Recall that the *gradient* $\nabla_g : \mathscr{C}^\infty(\mathcal{M}) \to \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$ is defined by

$$\left\langle \nabla_g \varphi, X \right\rangle_g = d\varphi(X)$$

for all $\varphi \in \mathscr{C}^\infty(\mathcal{M})$, $X \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$. In the local coordinates $(x_1, \dots, x_n)$, the gradient vector field has the form

$$\nabla_g \varphi = \sum_{i,j=1}^{n} g^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Since the differential operator d is linear and satisfies the Leibniz rule, we have $\nabla_g(\varphi + \psi) = \nabla_g\varphi + \nabla_g\psi$ and $\nabla_g(\varphi \cdot \psi) = \varphi \cdot \nabla_g\psi + \psi \cdot \nabla_g\varphi$ for all $\varphi, \psi \in \mathscr{C}^1(\mathcal{M})$.

Let $\omega \in \Omega^n(\mathcal{M})$ be an n−form and X be a vector field on $\mathcal{M}$. The $(n-1)$−form $\iota_X\omega \in \Omega^{n-1}(\mathcal{M})$ is defined by

$$\iota_X\omega(X_1, \dots, X_{n-1}) = \omega(X, X_1, \dots, X_{n-1}),$$

where $X_1, \dots, X_{n-1}$ are any vector fields on $\mathcal{M}$. For $d(\iota_X\omega)$ being an n−form, there exits a number $\operatorname{div}_\omega(X)$ so that $d(\iota_X\omega) = \operatorname{div}_\omega(X)\omega$.

Divergence $\operatorname{div}_g : \Gamma_{\mathscr{C}^\infty}(T\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M})$ is the operator defined by

$$d\left(\iota_X\omega_g\right) = \operatorname{div}_g(X)\omega_g \text{ for all } X \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M}).$$

Here, $\omega_g \in \Omega^n(\mathcal{M})$ is the natural volume element obtained from the metric g on $\mathcal{M}$. In the local coordinates $(x_1, \dots, x_n)$, for $X = \sum\limits_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \in \Gamma_{\mathscr{C}^\infty}(T\mathcal{M})$ we have

$$\operatorname{div}_g(X) = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(b_i \sqrt{\det g}\right). \tag{1}$$

For all $X, Y \in \Gamma_{C^\infty}(T\mathcal{M})$ and $\omega \in \Omega^n(\mathcal{M})$ we get $\iota_{X+Y}\omega = \iota_X\omega + \iota_Y\omega$. This yields $\operatorname{div}_g(X+Y) = \operatorname{div}_g(X) + \operatorname{div}_g(Y)$ Furthermore, by (1), we have for every $\varphi \in \mathscr{C}^\infty(M)$

$$\operatorname{div}_g(\varphi X) = \varphi \operatorname{div}_g X + \left\langle \nabla_g\varphi, X \right\rangle_g. \tag{2}$$

The *Laplace-Beltrami* operator $\Delta_g : \mathscr{C}^\infty(\mathcal{M}) \to \mathscr{C}^\infty(\mathcal{M})$ on $(\mathcal{M}, g)$ is the operator defined by

$$\Delta_g = -\operatorname{div}_g \circ \nabla_g.$$

For both ∇_g and div_g being linear, we have

$$\Delta_g(\varphi + \psi) = \Delta_g\varphi + \Delta_g\psi$$

for every $\varphi, \psi \in \mathscr{C}^\infty(\mathcal{M})$. We also have

$$\Delta_g(\varphi \cdot \psi) = \psi\Delta_g\varphi + \varphi\Delta_g\psi - 2\left\langle \nabla_g\varphi, \nabla_g\psi \right\rangle_g.$$

Considering the local coordinates $(x_1, \dots, x_n)$, the following equality holds:

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\frac{\partial}{\partial x_j}\right).$$

For instance, let us consider the $n-$spere

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

in geodesic polar coordinates namely $\xi : (0, \pi)^{n-1} \times (0, 2\pi) \to \mathbb{S}^n$,

$$\begin{aligned} x_1 &= \cos\theta_1 \\ x_2 &= \sin\theta_1 \cos\theta_2 \\ x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ &\vdots \\ x_n &= \sin\theta_1 \sin\theta_2 \cdots \cos\theta_n \\ x_{n+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_n \end{aligned} \tag{3}$$

where $0 < \theta_1, \theta_2, \ldots, \theta_{n-1} < \pi$, $0 < \theta_n < 2\pi$. Then we have

$$g_{\mathbb{S}^n} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & \sin^2\theta_1 & 0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & \sin^2\theta_1 \sin^2\theta_2 & 0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & \ddots & 0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & \sin^2\theta_1 \cdots \sin^2\theta_{n-1} \end{bmatrix},$$

$$\sqrt{\det g_{\mathbb{S}^n}} = \prod_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell}.$$

Thus, in these coordinates the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ is

$$-\frac{1}{\prod_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell}} \sum_{j=1}^{n} \frac{\partial}{\partial\theta_j}\left(a_j(\theta_1, \ldots, \theta_n)\frac{\partial}{\partial\theta_j}\right), \tag{4}$$

where $a_1 = 1$ and for $j = 2, \ldots, n$, $a_j = \dfrac{\prod_{\ell=1}^{n-1} (\sin\theta_\ell)^{n-\ell}}{\prod_{i=1}^{j-1} \sin^2\theta_i}$.

Before finishing this section, let us recall Stokes' Theorem and Divergence Theorem for manifolds.

Theorem 2.1 (Stokes' Theorem). *If $\mathcal{M}$ is an oriented smooth compact n-manifold with boundary $\partial\mathcal{M}$ and if $\alpha \in \Omega^{n-1}(\mathcal{M})$ have compact support, then*

$$\int_{\mathcal{M}} \iota^*\alpha = \int_{\mathcal{M}} d\alpha, \text{ or for short, } \int_{\mathcal{M}} \alpha = \int_{\mathcal{M}} d\alpha.$$

Here, $\iota : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion map.

Theorem 2.2 (Divergence Theorem). *If $\mathcal{M}$ is a Riemannian manifold and X is a C^1-vector field on $\mathcal{M}$, then*

$$\int_{\mathcal{M}} \mathrm{div}_g(X)\, dV_g = \int_{\partial\mathcal{M}} \langle X, \nu\rangle_g\, d\sigma_g.$$

Here, div_g is the divergence operator on $(\mathcal{M}, g)$, dV_g is the natural volume element on $(\mathcal{M}, g)$, and ν is the unit vector normal to $\partial\mathcal{M}$.

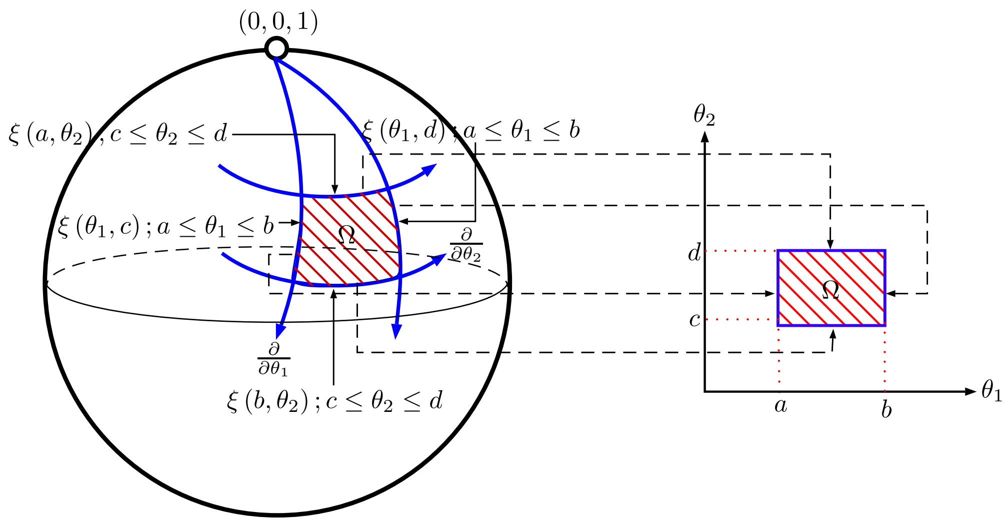

Figure 1: Topological rectangle on 2−sphere $\mathbb{S}^2$

These theorems yield

Theorem 2.3 (Green's Theorem). *Suppose* $(\mathcal{M}, g)$ *is a compact Riemannian manifold with boundary. If* $\psi \in \mathscr{C}^1\left(\overline{\mathcal{M}}\right)$ *and* $\varphi \in \mathscr{C}^2\left(\overline{\mathcal{M}}\right)$*, then*

$$\int_{\mathcal{M}} \psi \Delta_{\mathcal{M}} \phi \, dV_g = \int_{\mathcal{M}} \left\langle \nabla_g \psi, \nabla_g \phi \right\rangle dV_g - \int_{\partial \mathcal{M}} \psi \frac{\partial \phi}{\partial \nu} d\sigma_g.$$

Here, ∇_g *is the gradient operator on the Riemannian manifold* $(\mathcal{M}, g)$*.*

From Green's Theorem it follows that

Theorem 2.4. *[23] Suppose* $(\mathcal{M}, g)$ *is a closed (i.e. compact with empty boundary) Riemannian manifold, then*

1. *(Formal self-adjointness):* $\left\langle \psi, \Delta_{\mathcal{M}} \varphi \right\rangle_{\mathscr{L}_2(\mathcal{M}, dV_g)} = \left\langle \varphi, \Delta_{\mathcal{M}} \psi \right\rangle_{\mathscr{L}_2(\mathcal{M}, dV_g)}$
2. *(Positivity):* $\left\langle \Delta_{\mathcal{M}} \varphi, \varphi \right\rangle_{\mathscr{L}_2(\mathcal{M}, dV_g)} \geq 0.$

Here, $\mathscr{L}_2(\mathcal{M}, dV_g)$ *is the Hilbert space*

$$\left\{ \varphi : \mathcal{M} \to \mathbb{R}; \langle \varphi, \varphi \rangle_{\mathscr{L}_2(\mathcal{M}, dV_g)} := \int_{\mathcal{M}} \varphi^2(x) \, dV_g(x) < \infty \right\}.$$

Finally, let us recall that eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ on the n−sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are restrictions of harmonic polynomials to the sphere. The eigenvalues are $\lambda_\ell = \ell(\ell + n - 1)$, where $\ell = 0, 1, 2, \ldots$.

3. Parabolic differential equations on manifolds

3.1. Nonlocal parabolic differential equation

Let $(\mathcal{M}, g)$ be a smooth closed orientable Riemannian manifold (such as n−sphere $\mathbb{S}^n$, n−torus $\mathbb{T}^n$). Let us consider the nonlocal parabolic differential equation

$$\begin{cases} u_t\,(t, x) + \Delta_{\mathcal{M}} u(t, x) + \delta u(t, x) = f(t, x), \quad 0 \leq t \leq 1, x \in \mathcal{M}, \\ u(0, x) = \sum_{i=1}^{p} \alpha_i u(\lambda_i, x) + \mu(x), \quad x \in \mathcal{M}, \quad 0 < \lambda_1 < \cdots < \lambda_p \leq 1. \end{cases} \tag{5}$$

Here, $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on the manifold $(\mathcal{M}, g)$, $\delta > 0$, and $\sum_{i=1}^{p} |\alpha_i| \leq 1$.

We have

Theorem 3.1. *The solutions of (5) satisfy the following stability inequality*

$$\| u_t \|_{\mathscr{C}_0^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))} + \| \mathbf{L}u \|_{\mathscr{C}_0^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))} \leq K(\lambda_1, \delta)\left(\frac{\|f\|_{\mathscr{C}_0^{\alpha}(\mathscr{L}_2(\mathcal{M},dV_g))}}{\alpha(1-\alpha)} + \|\mathbf{L}\mu\|_{\mathscr{L}_2(\mathcal{M},dV_g)}\right),$$

where $K(\lambda_1, \delta)$ does not depend on $\mu(x)$, $f(t,x)$.

We consider problem (5) as the nonlocal boundary value problem

$$\begin{cases} U'(t) + \mathbf{L}U(t) = F(t), \quad 0 \leq t \leq 1, \\ U(0) = \sum_{i=1}^{p} \alpha_i U(\lambda_i) + \mu, \quad 0 < \lambda_1 < \cdots < \lambda_p \leq 1 \end{cases} \tag{6}$$

in the Hilbert space $H = \mathscr{L}_2\left(\mathcal{M}, dV_g\right)$ with the self-adjoint positive definite operator $\mathbf{L} = \Delta_{\mathcal{M}} + \delta I$. Here, I denotes the identity operator, $\|\psi\|_{\mathscr{L}_2(\mathcal{M},dV_g)} = \left(\int_{\mathcal{M}} \psi^2(x) dV_g(x)\right)^{1/2}$, and dV_g is the natural volume element of $\mathcal{M}$ obtained from metric tensor g.

The proof of Theorem 3.1 is based on the following theorem.

Theorem 3.2. *Problem (6) is well-posed in $\mathscr{C}_0^{\alpha}(H)$ and the following coercivity inequality holds:*

$$\|U'\|_{\mathscr{C}_0^{\alpha}(H)} + \|\mathbf{L}U\|_{\mathscr{C}_0^{\alpha}(H)} \leq K(\lambda_1, \delta)\left(\frac{1}{\alpha(1-\alpha)} \|F\|_{\mathscr{C}_0^{\alpha}(H)} + \|\mathbf{L}\mu\|_H\right),$$

where $K(\lambda_1, \delta)$ is independent of α, μ, and F. Here, $\mathscr{C}_0^{\alpha}(H)$ $(0 < \alpha < 1)$ is the Banach space which is the completion of of smooth functions $U : [0, 1] \to H$ in the norm

$$\|U\|_{\mathscr{C}_0^{\alpha}(H)} = \|U\|_{\mathscr{C}(H)} + \sup_{0 \leq t < t+\tau \leq 1} \frac{t^{\alpha} \|U(t+\tau) - U(t)\|_H}{\tau^{\alpha}}$$

and $\|U\|_{\mathscr{C}(H)}$ equals to $\max_{0 \leq t \leq 1} \|U(t)\|_H$.

3.2. Nonlocal BVP for parabolic equation on a relatively compact domain $\Omega \subset \mathbb{S}^n$

Let $(a_i, b_i) \subset (0, \pi)$, $i = 1, \ldots, n-1$ and $(a_n, b_n) \subset (0, 2\pi)$. Let us consider the domain

$$\Omega = \xi((a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1}) \times (a_n, b_n)) \subset \mathbb{S}^n. \tag{7}$$

Here, $\xi : (0, \pi)^{n-1} \times (0, 2\pi) \to \mathbb{S}^n$ is the geodesic polar parametrization (3).

Clearly, $\Omega \subset \mathbb{S}^n$ is a normal domain with the property $\mathbb{S}^n - \overline{\Omega}$ is open in $\mathbb{S}^n$.

We consider the nonlocal boundary value problem for parabolic equation

$$\begin{cases} u_t(t,x) + \Delta_{\mathbb{S}^n} u(t,x) + \delta u(t,x) = f(t,x), \quad 0 \leq t \leq 1, \quad x \in \Omega, \\ u(0,x) = \sum_{i=1}^{p} \alpha_i u(\lambda_i, x) + \mu(x), \quad x \in \Omega, \quad 0 < \lambda_1 < \cdots < \lambda_p \leq 1, \\ u(t,x) = 0, \quad 0 \leq t \leq 1, \quad x \in \partial\Omega. \end{cases} \tag{8}$$

Here, $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on $\mathbb{S}^n$, $\delta > 0$, and $\sum_{i=1}^{p} |\alpha_i| \leq 1$.

We have

Theorem 3.3. *The following stability inequality*

$$\begin{aligned} \| u_t \|_{\mathscr{C}_0^\alpha(\mathscr{L}_2(\Omega, dV_g))} &+ \| u \|_{\mathscr{C}_0^\alpha(\mathscr{W}_2^2(\Omega, dV_g))} \\ &\leq K(\lambda_1, \delta) \left(\frac{\|f\|_{\mathscr{C}_0^\alpha(\mathscr{L}_2(\Omega, dV_g))}}{\alpha(1-\alpha)} + \|\mu\|_{\mathscr{W}_2^2(\Omega, dV_g)} \right) \end{aligned}$$

is valid for the solutions of (8), where $K(\lambda_1, \delta)$ does not depend on $\mu(x)$, $f(t, x)$.

We consider problem (8) as the nonlocal boundary value problem (6) in the Hilbert space $H = \mathscr{L}_2\left(\Omega, dV_g\right)$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathbb{S}^n} + \delta I$.

Theorem 3.2 with $H = \mathscr{L}_2\left(\Omega, dV_g\right)$ and the following result (Theorem 3.4) which is about the coercivity inequality for the solution of the elliptic differential problem in $\mathscr{L}_2(\Omega, dV_g)$ prove Theorem 3.3.

Theorem 3.4. *If we consider the following elliptic differential problem*

$$\begin{cases} \Delta_{\mathbb{S}^n} u(\xi(\overrightarrow{\theta})) = \omega(\xi(\overrightarrow{\theta})), \overrightarrow{\theta} = (\theta_1, \ldots, \theta_n) \in (a_1, b_1) \times \cdots \times (a_n, b_n), \\ u(\xi(\overrightarrow{\theta})) = 0, \ \overrightarrow{\theta} \ in\ boundary\ of\ [a_1, b_1] \times \cdots \times [a_n, b_n], \end{cases}$$

then we have the coercivity inequality

$$\sum_{i=1}^{n} \left\| u_{\theta_i \theta_i} \right\|_{\mathscr{L}_2(\Omega, dV_g)} \leq K_1 \|\omega\|_{\mathscr{L}_2(\Omega, dV_g)}.$$

The proof of Theorem 3.4 is based on the following theorem.

Theorem 3.5. *[4] For the solutions of the elliptic differential problem*

$$\begin{cases} A^\xi u(\xi) = \omega(\xi), \quad \xi \in (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n), \\ u(\xi) = 0, \quad \xi \ in\ boundary\ [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \end{cases}$$

the coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| u_{\xi_r \xi_r} \right\|_{\mathscr{L}_2((\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n))} \leq K_2 \|\omega\|_{\mathscr{L}_2((\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n))}.$$

Here, $A^\xi = \sum_{r=1}^{n} \frac{\partial}{\partial \xi_r} \left(a_r(\xi) \frac{\partial}{\partial \xi_r} \right)$ and $a_r(\xi) \geq a > 0$, $r = 1, \ldots, n$.

Proof. [Proof of Theorem 3.4] Note that boundary of Ω is the image $\xi(\overrightarrow{\theta})$ of the boundary of $[a_1, b_1] \times \cdots \times [a_n, b_n]$. Note also that this parametrization sends the interior of $[a_1, b_1] \times \cdots \times [a_n, b_n]$ to the interior of Ω. If $u : \Omega \to \mathbb{R}$ and vanishes on the boundary of Ω, then $v = u \circ \xi : [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ and v vanishes on the boundary of $[a_1, b_1] \times \cdots \times [a_n, b_n]$.

There exist $k, K > 0$ so that on Ω we have $0 < k \leq \prod_{\ell=1}^{n-1} (\sin \theta_\ell)^{n-\ell} \leq K$.

From equation (4) and Theorem 3.5 it follows

$$\begin{aligned}
\int_\Omega \left|\Delta_{s^n} u(x)\right|^2 dV_g(x) &= \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{\left\{\sum_{j=1}^n \frac{\partial}{\partial\theta_j}\left(a_j(\overrightarrow{\theta})\frac{\partial u\circ\xi(\overrightarrow{\theta})}{\partial\theta_j}\right)\right\}^2}{\left(\prod\limits_{\ell=1}^{n-1}\sin\theta_\ell\right)^{n-\ell}} d\theta_n\cdots d\theta_1 \\
&\geq \frac{1}{K}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\left\{\sum_{j=1}^n \frac{\partial}{\partial\theta_j}\left(a_j(\overrightarrow{\theta})\frac{\partial u\circ\xi(\overrightarrow{\theta})}{\partial\theta_j}\right)\right\}^2 d\theta_n\cdots d\theta_1 \\
&= \frac{1}{K}\left\|A^{(\theta_1,\dots,\theta_n)}u\circ\xi\right\|^2_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} \\
&= \frac{1}{K}\left\|A^{(\theta_1,\dots,\theta_n)}v\right\|^2_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} \\
&\geq \frac{1}{K\cdot K_2^2}\left(\sum_{i=1}^n \left\|v_{\theta_i\theta_i}\right\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))}\right)^2.
\end{aligned}$$

Hence, we get

$$\left(\int\limits_\Omega \left|\Delta_{s^n} u(x)\right|^2 dV_g(x)\right)^{1/2} \geq \frac{1}{\sqrt{K}K_2}\sum_{i=1}^n \left\|v_{\theta_i\theta_i}\right\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))}. \tag{9}$$

Let us now note that for $i = 1,\dots,n$, we have

$$\begin{aligned}
&\left\|v_{\theta_i\theta_i}\right\|_{\mathscr{L}_2((a_1,b_1)\times\cdots\times(a_n,b_n))} = \left(\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\left|v_{\theta_i\theta_i}(\theta_1,\dots,\theta_n)\right|^2 d\theta_n\cdots d\theta_1\right)^{1/2} \\
&\geq \left(\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\left|v_{\theta_i\theta_i}(\theta_1,\dots,\theta_n)\right|^2 \frac{\prod\limits_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell}}{K} d\theta_n\cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}}\left(\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\left|v_{\theta_i\theta_i}(\theta_1,\dots,\theta_n)\right|^2 \prod_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell} d\theta_n\cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}}\left(\int\limits_{a_1}^{b_1}\cdots\int\limits_{a_n}^{b_n}\left|(u\circ\xi)_{\theta_i\theta_i}(\theta_1,\dots,\theta_n)\right|^2 \prod_{\ell=1}^{n-1}(\sin\theta_\ell)^{n-\ell} d\theta_n\cdots d\theta_1\right)^{1/2} \\
&= \frac{1}{\sqrt{K}}\left\|u_{\theta_i\theta_i}\right\|_{\mathscr{L}_2(\Omega,dV_g)}.
\end{aligned} \tag{10}$$

Equations (9) and (10) yield

$$\left(\int\limits_\Omega \left|\Delta_{s^n} u(x)\right|^2 dV_g(x)\right)^{1/2} \geq \frac{1}{K\cdot K_2}\sum_{i=1}^n \left\|u_{\theta_i\theta_i}\right\|_{\mathscr{L}_2(\Omega,dV_g)}.$$

This ends the proof of Theorem 3.4. □

3.3. Nonlocal reverse parabolic differential equation

Let $(\mathcal{M}, g)$ be a smooth closed orientable Riemannian manifold. Let us consider the boundary value problem of nonlocal parabolic type

$$\begin{cases} u_t(t,x) - \Delta_{\mathcal{M}} u(t,x) - \delta u(t,x) = f(t,x), \ \ 0 \le t \le 1, \ \ x \in \mathcal{M}, \\ u(1,x) = \sum_{i=1}^{p} \alpha_i u(\lambda_i, x) + \varphi(x), \ \ x \in \mathcal{M}, \ \ 0 \le \lambda_1 < \cdots < \lambda_p < 1. \end{cases} \tag{11}$$

Here, $\Delta_{\mathcal{M}}$ denotes the Laplace-Beltrami operator on the Riemannian manifold $(\mathcal{M}, g)$, $\delta > 0$, and $\sum_{i=1}^{p} |\alpha_i| \le 1$.

We prove

Theorem 3.6. *For the solutions of (11), the following stability inequality*

$$\begin{aligned} &\| u_t \|_{\mathscr{C}_1^{\alpha}(\mathscr{L}_2(\mathcal{M}, dV_g))} + \| \mathbf{L} u \|_{\mathscr{C}_1^{\alpha}(\mathscr{L}_2(\mathcal{M}, dV_g))} \\ &\le K(\lambda_p, \delta) \left(\frac{\|f\|_{\mathscr{C}_1^{\alpha}(\mathscr{L}_2(\mathcal{M}, dV_g))}}{\alpha(1-\alpha)} + \|\mathbf{L}\varphi\|_{\mathscr{L}_2(\mathcal{M}, dV_g)} \right) \end{aligned}$$

is valid. Here, $K(\lambda_p, \delta)$ is independent of $\varphi(x)$, $f(t,x)$.

Problem (11) is considered as the nonlocal boundary value problem

$$\begin{cases} U'(t) - \mathbf{L} U(t) = F(t), \ \ 0 \le t \le 1, \\ U(1) = \sum_{i=1}^{p} \alpha_i U(\lambda_i) + \varphi, \ \ 0 \le \lambda_1 < \cdots < \lambda_p < 1 \end{cases} \tag{12}$$

in $H = \mathscr{L}_2\left(\mathcal{M}, dV_g\right)$ with self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathcal{M}} + \delta I$.

The proof of Theorem 3.6, we use the following result.

Theorem 3.7. *[21] If A is a self-adjoint positive definite operator on a Hilbert space H, $\varphi \in \mathscr{D}(A)$, $F(t) \in \mathscr{C}_1^{\alpha}(H)$ and $\sum_{i=1}^{p} |\alpha_i| \le 1$, then problem*

$$\begin{cases} v'(t) - A v(t) = g(t), \ \ 0 \le t \le 1, \\ v(1) = \sum_{i=1}^{p} \alpha_i v(\lambda_i) + \varphi, \ \ 0 \le \lambda_1 < \cdots < \lambda_p < 1 \end{cases} \tag{13}$$

is well-posed in $\mathscr{C}_1^{\alpha}(H)$ and the coercivity estimate holds:

$$\|v'\|_{\mathscr{C}_1^{\alpha}(H)} + \|Av\|_{\mathscr{C}_1^{\alpha}(H)} \le K(\lambda_p, \delta) \left(\frac{1}{\alpha(1-\alpha)} \|g\|_{\mathscr{C}_1^{\alpha}(H)} + \|A\varphi\|_H \right),$$

where $K(\lambda_p, \delta)$ is independent of φ and $g(t)$, $t \in [0,1]$. Here, $\mathscr{C}_1^{\alpha}(H)$ is the Banach space which is the completion of smooth funtions $v : [0,1] \to H$ with the norm

$$\|v\|_{\mathscr{C}_1^{\alpha}(H)} = \|v\|_{\mathscr{C}(H)} + \sup_{0 \le t < t+\tau \le 1} \frac{(1-t)^{\alpha} \|v(t+\tau) - v(t)\|_H}{\tau^{\alpha}}.$$

3.4. Nonlocal reverse parabolic differential equation on a relatively compact domain $\Omega \subset \mathbb{S}^n$

We consider the normal domain $\Omega \subset \mathbb{S}^n$ in (7).

We consider the nonlocal boundary value problem of parabolic type

$$\begin{cases} u_t(t,x) - \Delta_{\mathbb{S}^n} u(t,x) - \delta u(t,x) = f(t,x), \quad 0 \le t \le 1, \quad x \in \Omega, \\ u(1,x) = \sum_{i=1}^{p} \alpha_i u(\lambda_i, x) + \varphi(x), \quad x \in \Omega, \quad \lambda_1 < \cdots < \lambda_p \le 1, \\ u(t,x) = 0, \quad x \in \partial\Omega, \quad 0 \le t \le 1. \end{cases} \tag{14}$$

Here, $\Delta_{\mathbb{S}^n}$ denotes the Laplace-Beltrami operator on the Riemannian manifold $(\mathbb{S}^n, g_{\mathbb{S}^n})$, $\delta > 0$, and $\sum_{i=1}^{p} |\alpha_i| \le 1$.

We have

Theorem 3.8. *For the solutions of (14), we have the following stability estimate*

$$\| u_t \|_{\mathscr{C}_1^{\alpha}(\mathscr{L}_2(\Omega,dV_g))} + \| u \|_{\mathscr{C}_1^{\alpha}(\mathscr{W}_2^2(\Omega,dV_g))} \le K(\lambda_p,\delta)\left(\tfrac{1}{\alpha(1-\alpha)} \| f \|_{\mathscr{C}_1^{\alpha}(\mathscr{L}_2(\Omega,dV_g))} + \|\varphi\|_{\mathscr{W}_2^2(\Omega,dV_g)}\right).$$

Here, $K(\lambda_p,\delta)$ does not depend on $\varphi(x)$, $f(t,x)$.

Theorem 3.9. *If $\sum_{\ell=1}^{n}(a_\ell(x)\varphi_{x_\ell}(x))_{x_\ell} - \delta\varphi = f(1,x) - \sum_{i=1}^{p} \alpha_i f(\lambda_i, x)$, then the solutions of problem (14) satisfy the stability inequality:*

$$\| u_t \|_{\mathscr{C}^{\alpha}(\mathscr{L}_2(\Omega,dV_g))} + \| u \|_{\mathscr{C}^{\alpha}(\mathscr{W}_2^2(\Omega,dV_g))} \le \frac{K(\lambda_p,\delta)}{\alpha(1-\alpha)} \| f \|_{\mathscr{C}^{\alpha}(\mathscr{L}_2(\Omega,dV_g))}.$$

Here, $K(\lambda_p,\delta)$ is independent of $\varphi(x)$, $f(t,x)$.

We consider problem (14) as the boundary value problem of nonlocal type (12) in $H = \mathscr{L}_2\left(\Omega, dV_g\right)$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathbb{S}^n} + \delta I$.

The proofs of Theorem 3.8 and Theorem 3.9, we use the symmetry properties of the operator $\mathbf{L}$ defined by (14), Theorem 3.4, Theorem 3.7, and also the following result.

Theorem 3.10. *[21] If $g(t) \in \mathscr{C}^{\alpha}(H)$, $g(1) - \sum_{i=1}^{p} \alpha_i g(\lambda_i) + A\varphi \in H_\alpha$ and $\sum_{i=1}^{p} |\alpha_i| \le 1$, then problem (13) is well-posed in $\mathscr{C}^{\alpha}(H)$ and moreover for the solutions the following coercivity estimate*

$$\| v'\|_{\mathscr{C}^{\alpha}(H)} + \|Av\|_{\mathscr{C}^{\alpha}(H)} + \|v'\|_{\mathscr{C}(H_\alpha)} \le K\left(\frac{1}{\alpha}\left\|g(1) - \sum_{i=1}^{p} \alpha_i g(\lambda_i) + A\varphi\right\|_{H_\alpha} + \frac{K(\lambda_p,\delta)}{\alpha(1-\alpha)}\|g\|_{\mathscr{C}^{\alpha}(H)}\right)$$

is valid, where K is independent of φ and $g(t)$, $t \in [0,1]$. Here, $H_\alpha = H_{\alpha,\infty}(H,A)$ is the fractional space consisting all $v \in H$ for which the following norm $\|v\|_{H_\alpha} = \|v\|_H + \sup_{\lambda>0} \|\lambda^{1-\alpha} A e^{-\lambda A} v\|_H$ is finite.

4. Hyperbolic differential equations on manifolds

4.1. Nonlocal hyperbolic differential equation

Suppose $(\mathcal{M}, g)$ is a smooth closed orientable Riemannian manifold. Consider the mixed boundary value problem for hyperbolic equations

$$\begin{cases} u_{tt}(t,x) + \Delta_{\mathcal{M}} u(t,x) + \delta u(t,x) = f(t,x), \quad (t,x) \in [0,1] \times \mathcal{M}, \\ u(0,x) = \sum_{j=1}^{p} \alpha_j u\left(\lambda_j, x\right) + \varphi(x), \quad x \in \mathcal{M}, \\ u_t(0,x) = \sum_{j=1}^{p} \beta_k u_t(\lambda_k, x) + \psi(x), \quad x \in \mathcal{M}, \\ 0 < \lambda_1 \leq \cdots \leq \lambda_p \leq 1. \end{cases} \tag{15}$$

Here, $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on the manifold $(\mathcal{M}, g)$, $\delta > 0$. We assume

$$\sum_{j=1}^{p} |\alpha_j + \beta_j| + \sum_{j=1}^{p} |\alpha_j| \sum_{m=1,\, m\neq j}^{p} |\beta_m| < \left| 1 + \sum_{j=1}^{p} \alpha_j \beta_j \right|. \tag{16}$$

We have

Theorem 4.1. *The solutions of (15) satisfy the stability inequalities*

$$\max_{0\leq t\leq 1} \| u(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)} \leq K\Big[\| \varphi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \| \mathbf{L}^{-1/2}\psi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \max_{0\leq t\leq 1} \| \mathbf{L}^{-1/2} f(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)}\Big],$$

$$\max_{0\leq t\leq 1} \| \mathbf{L}^{1/2} u(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)} \leq K\Big[\| \mathbf{L}^{1/2}\varphi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \| \psi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \max_{0\leq t\leq 1} \| f(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)}\Big],$$

$$\max_{0\leq t\leq 1} \| u_{tt}(t,\cdot) \|_{L_2(\mathcal{M},dV_g)} + \max_{0\leq t\leq 1} \| \mathbf{L}u(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)} \leq K\Big[\| \mathbf{L}\varphi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \| \mathbf{L}^{1/2}\psi \|_{\mathscr{L}_2(\mathcal{M},dV_g)} + \| f(0,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)} \int_0^t \| f_t(t,\cdot) \|_{\mathscr{L}_2(\mathcal{M},dV_g)}\, dt\Big],$$

where K is independent of $f(t,x)$, $\varphi(x)$, and $\psi(x)$.

We consider problem (15) as the following problem

$$\begin{cases} U_{tt}(t) + \mathbf{L}U(t) = F(t), \quad 0 \leq t \leq 1, \\ U(0) = \sum_{j=1}^{p} \alpha_j U\left(\lambda_j\right) + \varphi, \\ U_t(0) = \sum_{j=1}^{p} \beta_k U_t(\lambda_k) + \psi, \\ 0 < \lambda_1 \leq \cdots \leq \lambda_p \leq 1 \end{cases} \tag{17}$$

in $H = \mathscr{L}_2(\mathcal{M}, dV_g)$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathcal{M}} + \delta I$. Here, I denotes the identity operator, $\|U\|_{\mathscr{L}_2(\mathcal{M},dV_g)} = \left(\int_{\mathcal{M}} U^2(x)dV_g(x)\right)^{1/2}$, and dV_g is the natural volume element of $\mathcal{M}$ obtained from metric tensor g.

The proof of Theorem 4.1 is based on the following result.

Theorem 4.2. *([8, 9]) If A is a self-adjoint positive definite operator on a Hilbert space H, $\varphi \in D(A)$, $\psi \in D(A^{1/2})$, and $g(t)$ is in $C^1([0,1],H)$, and the assumption (16) is valid, then there is a unique solution of*

$$\begin{cases} v_{tt}(t) + Av(t) = g(t), \quad 0 \le t \le 1, \\ v(0) = \sum\limits_{j=1}^{p} \alpha_j v\left(\lambda_j\right) + \varphi, \\ v_t(0) = \sum\limits_{j=1}^{p} \beta_k v_t\left(\lambda_k\right) + \psi, \\ 0 < \lambda_1 \le \cdots \le \lambda_p \le 1 \end{cases}$$

and the following stability inequalities hold:

$$\begin{aligned} \| v \|_{C(H)} &\le K\left[\| \varphi \|_H + \| A^{-1/2}\psi \|_H + \| A^{-1/2} g \|_{C(H)}\right], \\ \| A^{1/2} v \|_{C(H)} &\le K\left[\| A^{1/2}\varphi \|_H + \| \psi \|_H + \| g \|_{C(H)}\right], \\ \| v'' \|_{C(H)} + \| Av \|_{C(H)} &\le K\left[\| A\varphi \|_H + \| A^{1/2}\psi \|_H + \| g(0) \|_H \int_0^t \| g'(t) \|_H \, dt\right], \end{aligned}$$

where K is independent of $g(t), t \in [0,1]$, and φ, ψ. Here, $\|v\|_{C(H)}$ is equal to $\max\limits_{0\le t\le 1} \|v(t)\|_H$.

4.2. *Nonlocal hyperbolic differential equation on a relatively compact domain $\Omega \subset \mathbb{S}^n$*

Let us consider the domain $\Omega \subset \mathbb{S}^n$ in (7). We consider the mixed boundary value problem for hyperbolic equations

$$\begin{cases} u_{tt}(t,x) + \Delta_{\mathbb{S}^n} u(t,x) = f(t,x), \quad (t,x) \in [0,1] \times \Omega, \\ u(0,x) = \sum\limits_{j=1}^{p} \alpha_j u\left(\lambda_j, x\right) + \varphi(x), \quad x \in \Omega, \\ u_t(0,x) = \sum\limits_{j=1}^{p} \beta_k u_t\left(\lambda_k, x\right) + \psi(x), \quad x \in \Omega, \\ 0 < \lambda_1 \le \cdots \le \lambda_p \le 1, \\ u(t,x) = 0, \quad x \in \partial\Omega, \quad 0 \le t \le 1 \end{cases} \tag{18}$$

under the assumption

$$\sum_{j=1}^{p} |\alpha_j + \beta_j| + \sum_{j=1}^{p} |\alpha_j| \sum_{m=1,\, m\neq j}^{p} |\beta_m| < \left|1 + \sum_{j=1}^{p} \alpha_j \beta_j\right|. \tag{19}$$

Here, $\Delta_{\mathbb{S}^n}$ denotes the Laplace-Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

We have

Theorem 4.3. *For the solutions of (18), the following stability inequalities*

$$\begin{aligned} &\max_{0\le t\le 1}\Big(\| u_\phi(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)} + \| u_\theta(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)}\Big) \\ \le\ & K\Big[\max_{0\le t\le 1}\| f(t,\cdot\|_{\mathscr{L}_2(\Omega,dV_g)} + \Big(\|\varphi_\phi\|_{\mathscr{L}_2(\Omega,dV_g)} + \|\varphi_\theta\|_{\mathscr{L}_2(\Omega,dV_g)}\Big) + \|\psi\|_{\mathscr{L}_2(\Omega,dV_g)}\Big], \end{aligned}$$

$$\begin{aligned} &\max_{0\le t\le 1}\Big(\| u_{\phi\phi}(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)} + \| u_{\theta\theta}(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)}\Big) + \max_{0\le t\le 1}\| u_{tt}(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)} \\ &\le K\Big[\max_{0\le t\le 1}\| f_t(t,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)} + \| f(0,\cdot)\|_{\mathscr{L}_2(\Omega,dV_g)} \\ &+\Big(\|\varphi_{\phi\phi}\|_{\mathscr{L}_2(\Omega,dV_g)} + \|\varphi_{\theta\theta}\|_{\mathscr{L}_2(\Omega,dV_g)}\Big) + \Big(\|\psi_\phi\|_{\mathscr{L}_2(\Omega,dV_g)} + \|\psi_\theta\|_{\mathscr{L}_2(\Omega,dV_g)}\Big)\Big] \end{aligned}$$

are valid, where K does not depend on $f(t,x)$, $\varphi(x)$, and $\psi(x)$.

Equation (18) can be considered as problem (17) in $H = \mathscr{L}_2(\Omega, dV_g)$ with the self-adjoint and positive definite operator $\mathbf{L} = \Delta_{\mathbb{S}^n}$.

To prove Theorem 4.3, we use Theorem 3.4 and Theorem 4.2 with $H = \mathscr{L}_2(\Omega, dV_g)$.

5. Conclusion

In this article, we investigate the differential equations on smooth closed manifolds. We prove the well-posedness of boundary value problems nonlocal type for parabolic equations and also hyperbolic equations in Hölder spaces. Moreover, in various Hölder norms we obtain new coercivity estimates for the solutions of such type parabolic boundary value problems on manifolds and hyperbolic boundary value problems on manifolds as well. Some statements without proof were published in [10]. In future works, following the techniques introduced in [11], we will investigate difference of equations associated to the differential equations.

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