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New Topologies From Obtained Operators via Weak Semi-Local Function and Some Comparisons

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Abstract. In this paper, the local function, the weak semi-local function and the local closure function are compared with each other according to the inclusion relation. We define a new operator by using the weak semi-local function and investigate its properties. Thanks to this operator, we obtain two new topologies which are finer than some previously defined topologies.

1. Introduction

In topological spaces, ideals were studied by Kuratowski [12] and Vaidyanathaswamy [21]. In these studies, the concept of local function was defined by using the concept of ideal. Also, a Kuratowski closure operator was obtained by using the local function. Ideals in topological spaces was applied in different branches of mathematics. One of these works was written by Freud who generalized the Cantor-Bendixson Theorem [6]. Janković and Hamlett developed well-known results in ideal topological spaces and obtained new results [9]. Apart from all these studies, many special ideal topological spaces were defined by using the concepts of ideal and local function such as $\mathcal{J}$-Baire spaces [14], $\mathcal{J}$-Alexandroff and $\mathcal{J}_g$-Alexandroff spaces [5], $\mathcal{J}$-Extremally disconnected spaces [10], $\mathcal{J}$-Resolvable spaces and $\mathcal{J}$-Hyperconnected spaces [4], $\mathcal{J}$-Rothberger spaces [7].

In 1963, Levine defined the concept of semi-open set [13]. Although the family of all semi-open sets in a topological space includes the family of all open sets, it is not a topology. However, the family of semi-open sets in extremally disconnected topological spaces forms a topology [15]. In [22], the concepts of θ-open set and θ-closed set were given by Veličko. In any given topological space, the family of θ-open sets forms a topology and this topology is coarser than the given topology.

In recent years, new local operators have been defined apart from the well-known local function. Some of these are the semi-local function [11], the local closure function [1], the semi-closure local function [8] and the weak semi-local funtion [24]. In [1], the local closure function and the operator Ψ_Γ were defined by Al-Omari and Noiri. Two new topologies was obtained with this operator Ψ_Γ. In [18], Pavlović gave an example that one of these two topologies is strictly finer than the other. The concept of weak semi-local

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function was defined and its basic properties were investigated in [24]. Also, it was shown with examples that the weak semi-local function is different from the other local operators.

In this study, we deal with concepts of local function, local closure function and weak semi-local function. We compare them according to the inclusion relation and define two different topologies which are finer than defined topologies by Al-Omari and Noiri in [1].

2. Definitions and Notations

Let (U, τ) be any topological space. We show the interior and the closure of subset M as $Int(M)$ and $Cl(M)$, respectively. The family of open neighborhoods for any point u is denoted by τ_u. The family of all subsets of U is denoted by $\mathcal{P}(U)$. The set of natural numbers containing 0 is denoted by ω.

Definition 2.1. ([12]) Let $U \neq \emptyset$ and $\mathcal{J} \subseteq \mathcal{P}(U)$. If $\mathcal{J}$ satifies the following conditions, it is called an ideal on U:

a) $\emptyset \in \mathcal{J}$.
b) If $M \in \mathcal{J}$ and $K \subseteq M$, then $K \in \mathcal{J}$.
c) If $M, K \in \mathcal{J}$, then $M \cup K \in \mathcal{J}$.

To define an ideal on U, there is no need any topology on U. The family of finite subsets of U and the family of countable subsets of U form an ideal on the nonempty set U. These ideals denoted by $\mathcal{J}_{fin}$ and $\mathcal{J}_{co}$, respectively. Let (U, τ) be a topological space and $M \subseteq U$. A subset M is called nowhere dense, if $Int(Cl(M)) = \emptyset$. A subset M is called discrete set if $M \cap M^d = \emptyset$ (where M^d is derived set of M). A subset of U is called meager, if it can be written as a countable union of nowhere dense subsets of U. A subset of U is called relatively compact, if its closure is compact. For a topological space (U, τ), family of nowhere dense subsets ($\mathcal{J}_{nw}$), family of closed-discrete subsets ($\mathcal{J}_{cd}$), family of meager subsets ($\mathcal{J}_{mg}$) and family of relatively compact subsets ($\mathcal{J}_K$) are an ideal on U. If (U, τ) is a topological space with an ideal $\mathcal{J}$ on U, this space is called an ideal topological space or briefly $\mathcal{J}$-space.

Definition 2.2. ([12]) Let (U, τ) be a $\mathcal{J}$-space and $M \subseteq U$. An operator $(.)^* : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by

$$M^*(\mathcal{J}, \tau) = \{u \in U : O \cap M \notin \mathcal{J} \text{ for every } O \in \tau_u\}$$

and is called the local function of M with respect to $\mathcal{J}$ and τ.

Sometimes we use the notations M^* or $M^*(\mathcal{J})$ instead of $M^*(\mathcal{J}, \tau)$.

Let (U, τ) be a topological space. A subset M of U is called θ-open [22], if each point of M has a open neighborhood O such that $Cl(O) \subseteq M$. The complement of θ-open set is called θ-closed set. The θ-closure [22] of a subset M is defined by $Cl_\theta(M) = \{u \in U : Cl(O) \cap M \neq \emptyset \text{ for every } O \in \tau_u\}$. The family of all θ-open sets in the topological space (U, τ) is denoted by τ_θ and it is a topology on U. This topology is coaser than τ. In other words, every θ-open set is an open set [22].

Definition 2.3. ([1]) Let (U, τ) be a $\mathcal{J}$-space and $M \subseteq U$. An operator $\Gamma : \mathcal{P}(U) \to \mathcal{P}(U)$ is defined by

$$\Gamma(M)(\mathcal{J}, \tau) = \{u \in U : Cl(O) \cap M \notin \mathcal{J} \text{ for every } O \in \tau_u\}$$

and is called the local closure function of M with respect to $\mathcal{J}$ and τ.

Sometimes we use the notations $\Gamma(M)$ or $\Gamma(M)(\mathcal{J})$ instead of $\Gamma(M)(\mathcal{J}, \tau)$.

In [1], Al-Omari and Noiri defined an operator $\Psi_\Gamma : \mathcal{P}(U) \to \tau$ in $\mathcal{J}$-space (U, τ) as follows: For a subset M of U,

$$\Psi_\Gamma(M) = \{u \in U : \text{there exists } O \in \tau_u \text{ such that } Cl(O) \setminus M \in \mathcal{J}\}.$$

It is clear that $\Psi_\Gamma(M) = U \setminus \Gamma(U \setminus M)$. Moreover, these authors in [1] defined two topologies on U by using this operator as follows:

$$\sigma = \{M \subseteq U : M \subseteq \Psi_\Gamma(M)\} \text{ and } \sigma_0 = \{M \subseteq U : M \subseteq Int(Cl(\Psi_\Gamma(M)))\}.$$

Elements of this topologies are said to be σ-open sets and σ_0-open sets, respectively. Every σ-open set is σ_0-open set and moreover, $\tau_\theta \subseteq \sigma \subseteq \sigma_0$. In [1], a question was asked that "Is there an example that shows $\sigma \subsetneq \sigma_0$?" It was answered by Pavlović with an example in which $\sigma \subsetneq \sigma_0$ [18]. The following diagram was obtained in [1].

$$\begin{array}{ccc} \theta\text{-open} & \rightarrow & \text{open} \\ \downarrow & & \\ \sigma\text{-open} & \rightarrow & \sigma_0\text{-open} \end{array}$$

Diagram I

Lemma 2.4. ([1]) *In any $\mathcal{J}$-space (U, τ) , the local closure function of a subset M includes its local function i.e. $M^*(\mathcal{J}, \tau) \subseteq \Gamma(M)(\mathcal{J}, \tau)$.*

"When do these two functions coincide?" The following theorem answers this question.

Theorem 2.5. ([18]) *In any $\mathcal{J}$-space (U, τ), each of the following conditions implies that $M^* = \Gamma(M)$ for any subset M of U:*

a) *τ has a clopen base.*
b) *τ is a T_3-space on U.*
c) $\mathcal{J} = \mathcal{J}_{cd}$.
d) $\mathcal{J} = \mathcal{J}_K$.
e) $\mathcal{J}_{nw} \subseteq \mathcal{J}$.
f) $\mathcal{J} = \mathcal{J}_{mg}$.

Definition 2.6. ([13]) In any topological space, the subset M is called semi-open set if there exists an open set W such that $W \subseteq M \subseteq Cl(W)$.

Theorem 2.7. ([13]) *In any topological space, a subset M is semi-open if and only if $M \subseteq Cl(Int(M))$.*

The family of all semi-open sets in a topological space (U, τ) is denoted by $SO(U)$. Every open set is semi-open i.e. $\tau \subseteq SO(U)$ [13]. The union of all semi-open subsets contained of M is called the [2] semi-interior of M and is denoted by $sInt(M)$. The family of all semi-open neighborhood of a point $u \in U$ is denoted by $SO(U, u)$. Obviously, it is $\tau_u \subseteq SO(U, u)$ for every $u \in U$. A subset M is called semi-closed set, if the complement of M is semi-open [2]. The semi-closure of a subset M is defined as the intersection of all semi-closed sets containing M and it is denoted by $sCl(M)$ [2]. For every subset M,

$$Int(M) \subseteq sInt(M) \subseteq M \subseteq sCl(M) \subseteq Cl(M).$$

Definition 2.8. ([24]) Let (U, τ) be $\mathcal{J}$-space and $M \subseteq U$. An operator $\xi : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ is defined by

$$\xi(M)(\mathcal{J}, \tau) = \{u \in U : sCl(O) \cap M \notin \mathcal{J} \text{ for every } O \in SO(U, u)\}$$

and is called the weak semi-local function of M with respect to $\mathcal{J}$ and τ.

Sometimes we use the notations $\xi(M)$ or $\xi(M)(\mathcal{J})$ instead of $\xi(M)(\mathcal{J}, \tau)$.

In [24], differences between local, local closure and weak semi-local functions are shown in examples defined on finite sets. Let us give the following examples with some well-known topologies and ideals defined on the infinite set.

Example 2.9. Let $(\mathbb{R}, \tau)$ be $\{\emptyset\}$-space with usual topology τ. For subset $M = (a, b)$ where $a, b \in \mathbb{R}$, $\xi(M) = (a, b)$ and $\Gamma(M) = M^* = [a, b]$.

Example 2.10. Let $(\mathbb{R}, \tau_L)$ be $\mathcal{J}_{fin}$-space where real numbers set $\mathbb{R}$ with left-ray topology τ_L i.e. $\tau_L = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, and let $\mathcal{J}_{fin}$ be the ideal of all finite subsets of $\mathbb{R}$. For subset $M = [a, b]$ where $a, b \in \mathbb{R}$, $M^* = [a, \infty)$ and $\xi(M) = \Gamma(M) = \mathbb{R}$.

Proposition 2.11. ([24]) *Let M and K be two subsets in a topological space (U, τ). Also, let $\mathcal{J}$ and $\mathcal{I}$ be two ideals on U. The operator ξ satisfies the following properties:*

a) *If $M \subseteq K$, then $\xi(M) \subseteq \xi(K)$.*
b) *If $\mathcal{J} \subseteq \mathcal{I}$, then $\xi(M)(\mathcal{I}) \subseteq \xi(M)(\mathcal{J})$.*
c) *$\xi(M) = sCl(\xi(M)) \subseteq Cl_\theta(M)$. That is, $\xi(M)$ is semi-closed set. 2*
d) *If $M \in \mathcal{J}$, then $\xi(M) = \emptyset$.*
e) *$\xi(M \cup K) = \xi(M) \cup \xi(K)$.*

Let (U, τ) be a topological space and $M \subseteq U$. The semi-θ-closure of M is defined [3] by $sCl_\theta(M) = \{u \in U : sCl(O) \cap M \neq \emptyset$ for every $O \in SO(U, u)\}$. If (U, τ) is a $\{\emptyset\}$-space, obviously $\xi(M) = sCl_\theta(M)$. Moreover, in any $\mathcal{J}$-space (U, τ), from Proposition 2.11 b), $\xi(M)(\mathcal{J}) \subseteq sCl_\theta(M)$.

Lemma 2.12. ([24]) *In any $\mathcal{J}$-space (U, τ), the local closure function of a subset M includes its weak semi-local function i.e. $\xi(M)(\mathcal{J}, \tau) \subseteq \Gamma(M)(\mathcal{J}, \tau)$.*

Lemma 2.13. ([24]) *Let (U, τ) be $\mathcal{J}$-space and $M, K \subseteq U$. If $K \in \mathcal{J}$, then $\xi(M \setminus K) = \xi(M)$.*

3. Some Comparisons

In different ideal topological spaces, it was shown to be $M^* \subsetneq \xi(M)$ or $\xi(M) \subsetneq M^*$ (Example 2.9, Example 2.10 and also Example 2.3, Example 2.4 in [24]). Moreover, even for two different subsets M, K in the same ideal topological space, it is possible that $M^* \subsetneq \xi(M)$ and $\xi(K) \subsetneq K^*$. We give an example for this case:

Example 3.1. Let $\tau = \{U, \emptyset, \{p\}, \{s\}, \{p, s\}, \{p, t\}, \{p, s, t\}, \{p, r, s\}\}$ be a topology on $U = \{p, r, s, t\}$ and $\mathcal{J} = \{\emptyset, \{r\}\}$. For subsets $M = \{t\}$ and $K = \{r, s\}$, $M^* = \{t\} \subsetneq \xi(M) = \{r, t\}$, $\xi(K) = \{s\} \subsetneq K^* = \{r, s\}$.

In any $\mathcal{J}$-space , local function and weak semi-local function may not be compared for each subset. Let us give an example for this:

Example 3.2. Let $\tau = \{U, \emptyset, \{x\}, \{t\}, \{x, y\}, \{x, z\}, \{p, s\}, \{x, t\}, \{x, y, z\}, \{p, s, t\}, \{x, p, s\}, \{x, z, t\}, \{x, y, t\}, \{p, r, s, t\}, \{x, p, s, t\}, \{x, y, p, s\}, \{x, z, p, s\}, \{x, y, z, t\}, \{x, y, z, p, s\}, \{x, p, r, s, t\}, \{x, y, p, s, t\}, \{x, z, p, s, t\}, \{x, z, p, r, s, t\}, \{x, y, p, r, s, t\}, \{x, y, z, p, s, t\}\}$ be a topology on $U = \{x, y, z, p, r, s, t\}$ and $\mathcal{J} = \{\emptyset, \{x\}, \{p\}, \{x, p\}\}$. For subset $M = \{x, y, p, t\}$, $M^* = \{y, r, t\}$ and $\xi(M) = \{x, y, z, t\}$.

In this section, we answer the questions such as "when is $M^* \subseteq \xi(M)$?", "when is $\xi(M) \subseteq M^*$?" and "when is $\Gamma(M) = \xi(M)$?". Obviously, in any $\mathcal{J}$-space , if $M \in \mathcal{J}$, then $M^* = \xi(M) = \Gamma(M) = \emptyset$. If $\mathcal{J} = \mathcal{P}(U)$, again $M^* = \xi(M) = \Gamma(M) = \emptyset$.

A topological space is extremally disconnected [23] if the closure of every open set is open.

Theorem 3.3. ([16], [19]) *The following conditions are equivalent:*

a) *(U, τ) is extremally disconnected.*
b) *For every $O \in SO(U)$, $sCl(O) = Cl(O)$.*
c) *For every $O \in SO(U)$, $Cl(O)$ is open set.*

Theorem 3.4. *Let (U, τ) be a $\mathcal{J}$-space and $M \subseteq U$. If (U, τ) is extremally disconnected space, then*

$$M^* \subseteq \xi(M) = \Gamma(M).$$

Proof. We will show that $\Gamma(M) \subseteq \xi(M)$. Because, we have $\xi(M) \subseteq \Gamma(M)$ from Lemma 2.12. Suppose that $x \notin \xi(M)$. So, there exists $O \in SO(U,u)$ such that $sCl(O) \cap M \in \mathcal{J}$. From Theorem 3.3 c), $Cl(O) \in \tau_u$. Moreover, $x \notin \Gamma(M)$, since $Cl(Cl(O)) \cap M = Cl(O) \cap M = sCl(O) \cap M \in \mathcal{J}$ from Theorem 3.3 b). This implies that $\Gamma(M) \subseteq \xi(M)$. Furthermore, $M^* \subseteq \Gamma(M)$ from Lemma 2.4. Consequently, $M^* \subseteq \xi(M) = \Gamma(M)$. □

We give an example in which we show that the relation $M^* \subseteq \xi(M) = \Gamma(M)$ strictly holds.

Example 3.5. Let $\tau = \{U, \emptyset, \{r\}, \{s\}, \{p,r\}, \{r,s\}\}$ be a topology on $U = \{p,r,s\}$ and $\mathcal{J} = \{\emptyset, \{r\}\}$. Then, (U,τ) is extremally disconnected space and for a subset $M = \{p,r\}$, $M^* = \{p\} \subsetneq \xi(M) = \Gamma(M) = \{p,r\}$.

A topological space is hyperconnected [20] if every nonempty open set is dense.

Theorem 3.6. *Let (U,τ) be a $\mathcal{J}$-space and $M \subseteq U$. If (U,τ) is hyperconnected space and $M \notin \mathcal{J}$, then*

$$M^* \subseteq \xi(M) = \Gamma(M) = U.$$

Proof. Since every hyperconnected space is extremally disconnected space, $M^* \subseteq \xi(M) = \Gamma(M)$ from Theorem 3.4. Let $u \in U \setminus \Gamma(M)$ and $M \notin \mathcal{J}$. Since $u \notin \Gamma(M)$ and (U,τ) is hyperconnected, there exists $O \in \tau_u$ such that $Cl(O) \cap M = U \cap M = M \in \mathcal{J}$. This is a contradiction. Consequently, $M^* \subseteq \xi(M) = \Gamma(M) = U$. □

In the following examples, we show that the relation $M^* \subseteq \xi(M) = \Gamma(M) = U$ strictly hold.

Example 3.7. Let $\tau = \{U, \emptyset, \{p\}\}$ be a topology on $U = \{p,r,s\}$ and $\mathcal{J} = \{\emptyset, \{p\}\}$. Then, (U,τ) is hyperconnected space and for a set $M = U$, $M^* = \{r,s\} \subsetneq \xi(M) = \Gamma(M) = U$.

Example 3.8. In Example 2.10, the topology $(\mathbb{R}, \tau_L)$ is hyperconnected space and $M^* \subsetneq \xi(M) = \Gamma(M)$ for subset $M = [a,b]$.

Lemma 3.9. *Let (U,τ) be a $\mathcal{J}$-space and $u \in M^*$ for a subset $M \subseteq U$. If $u \in Int(O)$ for every $O \in SO(U,u)$, then $u \in \xi(M)$.*

Proof. Suppose that $u \notin \xi(M)$. Then, there exists $O \in SO(U,u)$ such that $sCl(O) \cap M \in \mathcal{J}$. According to hypothesis, $Int(O) \in \tau_u$. From $Int(O) \cap M \subseteq sCl(O) \cap M$ and the definition of ideal, $Int(O) \cap M \in \mathcal{J}$. This implies that $u \notin M^*$. This is a contradiction. Consequently, $u \in \xi(M)$. □

Theorem 3.10. *Let (U,τ) be a $\mathcal{J}$-space and $M \subseteq U$. Then $M^* \subseteq \xi(M)$ if the following condition holds for every $u \in M^*$: $u \in Int(O)$ for every $O \in SO(U,u)$.*

Proof. It is obvious, from Lemma 3.9. □

In the following example, we show that the converse of Theorem 3.10 is not true.

Example 3.11. Consider the $\mathcal{J}$-space in Example 3.1. If $M = \{r,s,t\} \subseteq U$, then $M^* = \{r,s,t\} \subseteq \xi(M) = U$. Although $O = \{p,r\} \in SO(U,r)$ and $r \in M^*$, $r \notin Int(O)$.

Now let us give an example which satisfies hypothesis of Theorem 3.10 and $M^* \subsetneq \xi(M)$.

Example 3.12. Consider the $\mathcal{J}$-space in Example 3.1. If we choose $M = \{t\} \subseteq U$, then $t \in M^* = \{t\}$ and $t \in int(O)$ for every $O \in SO(U,t)$. Furthermore, $M^* = \{t\} \subsetneq \xi(M) = \{p,t\}$.

Theorem 3.13. *Let (U,τ) be a $\mathcal{J}$-space and $M \subseteq U$. Each of conditions in Theorem 2.5 implies that*

$$\xi(M) \subseteq M^* = \Gamma(M).$$

Proof. From Theorem 2.5, we have $M^* = \Gamma(M)$ and from Lemma 2.12, we have $\xi(M) \subseteq \Gamma(M)$. Consequently, $\xi(M) \subseteq M^* = \Gamma(M)$. □

Even if any condition in Theorem 2.5 is satisfied, it is not necessary that $\xi(M) = M^* = \Gamma(M)$ for a subset M of any $\mathcal{J}$-space. We illustrate this case in the following examples.

Example 3.14. Let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be $\mathcal{J}_K$-space where the topology $\tau_{\mathbb{Q}}$ on the set $\mathbb{Q}$ rational numbers is induced by the usual topology on set $\mathbb{R}$ real numbers and $\mathcal{J}_K$ is the ideal of relatively compact subsets of $\mathbb{Q}$. For the subset $M = \mathbb{Q} \cap [0,1]$, $\Gamma(M)(\mathcal{J}_K) = M^*(\mathcal{J}_K) = \mathbb{Q} \cap [0,1]$. Since $M \cap sCl(O) = \{0\} \in \mathcal{J}_K$ for $O = \mathbb{Q} \cap (-1, 0] \in SO(\mathbb{Q}, 0)$, $0 \notin \xi(M)(\mathcal{J}_K)$. Furthermore, this topology on rational numbers has clopen base.

Consequently, this example showed that even if an ideal topological space satisfies conditions a) and d) of Theorem 2.5, it can be $\xi(M) \subsetneq M^* = \Gamma(M)$ for a subset M in this space.

Example 3.15. Let us consider a subset $M = [1, 2]$ of $\mathbb{R}$ with usual topology. $\Gamma(M)(\mathcal{J}_{cd}) = \Gamma(M)(\mathcal{J}_{mg}) = \Gamma(M)(\mathcal{J}_{nw}) = M^*(\mathcal{J}_{cd}) = M^*(\mathcal{J}_{mg}) = M^*(\mathcal{J}_{nw}) = [1, 2]$ and $\xi(M)(\mathcal{J}_{cd}) = \xi(M)(\mathcal{J}_{mg}) = \xi(M)(\mathcal{J}_{nw}) = (1, 2)$. Also, this topology is T_3-space.

Consequently, this example showed that even if an ideal topological space satisfies the conditions b), c), e), f) of Theorem 2.5, it can be $\xi(M) \subsetneq M^* = \Gamma(M)$ for a subset M in this space.

4. A New Operator via Ideal

Definition 4.1. Let (U, τ) be a $\mathcal{J}$-space. For any subset M of U, an operator $\Psi_\xi : \mathcal{P}(U) \to SO(U)$ is defined as follows:

$$\Psi_\xi(M) = \{u \in U : \text{there exists } O \in SO(U, u) \text{ such that } sCl(O) \setminus M \in \mathcal{J}\}.$$

It is also obvious that $\Psi_\xi(M) = U \setminus \xi(U \setminus M)$.

Now, we give basic properties of this operator.

Theorem 4.2. *Let (U, τ) be a $\mathcal{J}$-space. The operator Ψ_ξ satisfies the following properties:*

a) *For every subset $M \subseteq U$, $\Psi_\xi(M)$ is semi open set.*
b) *If $M \subseteq K$, then $\Psi_\xi(M) \subseteq \Psi_\xi(K)$.*
c) *If $M, K \subseteq U$, then $\Psi_\xi(M \cap K) = \Psi_\xi(M) \cap \Psi_\xi(K)$.*
d) *For every subset $M \subseteq U$, $\Psi_\xi(\Psi_\xi(M)) = U \setminus \xi(\xi(U \setminus M))$.*
e) *For every subset $M \subseteq U$, $\Psi_\xi(M) = \Psi_\xi(\Psi_\xi(M)) \Leftrightarrow \xi(U \setminus M) = \xi(\xi(U \setminus M))$.*
f) *If $M \in \mathcal{J}$, then $\Psi_\xi(M) = U \setminus \xi(U)$.*
g) *If $M \subseteq U$ and $K \in \mathcal{J}$, then $\Psi_\xi(M \setminus K) = \Psi_\xi(M)$.*
h) *If $M \subseteq U$ and $K \in \mathcal{J}$, then $\Psi_\xi(M \cup K) = \Psi_\xi(M)$.*
i) *If $(M \setminus K) \cup (K \setminus M) \in \mathcal{J}$, then $\Psi_\xi(M) = \Psi_\xi(K)$.*

Proof. a) From Proposition 2.11 c), $\Psi_\xi(M) = U \setminus \xi(U \setminus M)$ is semi open.

b) It is obvious from Proposition 2.11 a).

c) Using Proposition 2.11 e),

$$\begin{aligned}\Psi_\xi(M \cap K) &= U \setminus \xi(U \setminus (M \cap K)) \\ &= U \setminus [\xi(U \setminus M) \cup \xi(U \setminus K)] \\ &= [U \setminus \xi(U \setminus M)] \cap [U \setminus \xi(U \setminus K)] \\ &= \Psi_\xi(M) \cap \Psi_\xi(K).\end{aligned}$$

d) From definition of the operator Ψ_ξ,

$$\begin{aligned}\Psi_\xi(\Psi_\xi(M)) &= \Psi_\xi(U \setminus \xi(U \setminus M)) \\ &= U \setminus \xi(U \setminus (U \setminus \xi(U \setminus M)) \\ &= U \setminus \xi(\xi(U \setminus M))\end{aligned}$$

e) The previous feature 4),

$$\begin{aligned}\Psi_\xi(\Psi_\xi(M)) = \Psi_\xi(M) &\Leftrightarrow U \setminus \xi(\xi(U \setminus M)) = U \setminus \xi(U \setminus M)\\ &\Leftrightarrow \xi(\xi(U \setminus M)) = \xi(U \setminus M)\end{aligned}$$

f) Since $M \in \mathcal{J}$, $U \setminus \xi(U \setminus M) = U \setminus \xi(U)$ from Lemma 2.13. So, $\Psi_\xi(M) = U \setminus \xi(U)$.
g) Using Proposition 2.11 e) and d),

$$\begin{aligned}\Psi_\xi(M \setminus K) &= U \setminus \xi(U \setminus (M \setminus K))\\ &= U \setminus \xi((U \setminus M) \cup K)\\ &= U \setminus [\xi(U \setminus M) \cup \xi(K)]\\ &= U \setminus \xi(U \setminus M)\\ &= \Psi_\xi(M).\end{aligned}$$

h) Using Lemma 2.13,

$$\begin{aligned}\Psi_\xi(M \cup K) &= U \setminus \xi(U \setminus (M \cup K))\\ &= U \setminus \xi((U \setminus M) \cap (U \setminus K))\\ &= U \setminus \xi((U \setminus M) \setminus K)\\ &= U \setminus \xi(U \setminus M)\\ &= \Psi_\xi(M)\end{aligned}$$

i) Suppose that $(M \setminus K) \cup (K \setminus M) \in \mathcal{J}$. From the definition of ideal, $(M \setminus K) \in \mathcal{J}$ and $(K \setminus M) \in \mathcal{J}$. Using g) and h),

$$\begin{aligned}\Psi_\xi(M) &= \Psi_\xi(M \setminus (M \setminus K))\\ &= \Psi_\xi((M \setminus (M \setminus K)) \cup (K \setminus M))\\ &= \Psi_\xi((M \cap K) \cup (K \setminus M))\\ &= \Psi_\xi(K).\end{aligned}$$

□

Theorem 4.3. *Let (U, τ) be a $\mathcal{J}$-space. Then, the family $\sigma_\xi = \{M \subseteq U : M \subseteq \Psi_\xi(M)\}$ is a topology on U.*

Proof. It is obvious that $\emptyset, U \in \sigma_\xi$. Let $M, K \in \sigma_\xi$. Since $M \subseteq \Psi_\xi(M)$ and $K \subseteq \Psi_\xi(K)$, $M \cap K \subseteq \Psi_\xi(M) \cap \Psi_\xi(K) = \Psi_\xi(M \cap K)$ from Theorem 4.2 c). Therefore, $M \cap K \in \sigma_\xi$. Let $\{M_\alpha\}_{\alpha\in I}$ be a family of subsets of σ_ξ for any index set I. Since $M_\alpha \subseteq \Psi_\xi(M_\alpha)$ for every $\alpha \in I$, $M_\alpha \subseteq \Psi_\xi(M_\alpha) \subseteq \Psi_\xi(\cup_{\alpha\in I} M_\alpha)$. Then, $\cup_{\alpha\in I} M_\alpha \subseteq \Psi_\xi(\cup_{\alpha\in I} M_\alpha)$. Hence $\cup_{\alpha\in I} M_\alpha \in \sigma_\xi$. Consequently, σ_ξ is a topology on U. □

The elements of topology σ_ξ are called σ_ξ-open sets.

Lemma 4.4. ([17]) *Let (U, τ) be a topological space and let M, K be subsets of U. If either $M \in SO(U)$ or $K \in SO(U)$, then*

$$Int(Cl(M \cap K)) = Int(Cl(M)) \cap Int(Cl(K)).$$

Theorem 4.5. *Let (U, τ) be a $\mathcal{J}$-space. Then, the family $\sigma_{\xi_0} = \{M \subseteq U : M \subseteq Int(Cl(\Psi_\xi(M)))\}$ is a topology on U.*

Proof. It is obvious that $\emptyset, U \in \sigma_{\xi_0}$. Let $M, K \in \sigma_{\xi_0}$. So, $M \subseteq Int(Cl(\Psi_\xi(M)))$ and $K \subseteq Int(Cl(\Psi_\xi(K)))$. From Theorem 4.2 a), c) and Lemma 4.4, we have

$$\begin{aligned}M \cap K &\subseteq Int(Cl(\Psi_\xi(M))) \cap Int(Cl(\Psi_\xi(K)))\\ &= Int(Cl(\Psi_\xi(M) \cap \Psi_\xi(K)))\\ &= Int(Cl(\Psi_\xi(M \cap K))).\end{aligned}$$

Therefore, $M \cap K \in \sigma_{\xi_0}$. Let $\{M_\alpha\}_{\alpha\in I}$ be a family of subsets of σ_{ξ_0} for any index set I. Since $M_\alpha \subseteq Int(Cl(\Psi_\xi(M_\alpha)))$ for every $\alpha \in I$, $M_\alpha \subseteq Int(Cl(\Psi_\xi(M_\alpha))) \subseteq Int(Cl(\Psi_\xi(\cup_{\alpha\in I} M_\alpha)))$. Then $\cup_{\alpha\in I} M_\alpha \subseteq Int(Cl(\Psi_\xi(\cup_{\alpha\in I} M_\alpha)))$. Therefore, $\cup_{\alpha\in I} M_\alpha \in \sigma_\xi$. Consequently, σ_{ξ_0} is a topology on U. □

The elements of topology σ_{ξ_0} are called σ_{ξ_0}-open sets.

Lemma 4.6. *Let (U, τ) be a $\mathcal{J}$-space. Then, $\Psi_\Gamma(M) \subseteq \Psi_\xi(M)$ for every subset $M \subseteq U$.*

Proof. From Lemma 2.12, we have $\xi(U \setminus M) \subseteq \Gamma(U \setminus M)$. Hence, $\Psi_\xi(M) = U \setminus \xi(U \setminus M) \supseteq U \setminus \Gamma(U \setminus M) = \Psi_\Gamma(M)$. □

Theorem 4.7. *Let (U, τ) be a $\mathcal{J}$-space. Every σ-open set is σ_ξ-open set. In other words, the topology σ_ξ is finer than the topology σ.*

Proof. Let M be σ-open set. Therefore, $M \subseteq \Psi_\Gamma(M) \subseteq \Psi_\xi(M)$ from Lemma 4.6. □

We give an example in which a set is σ_ξ-open set but not σ-open set.

Example 4.8. Let $U = \{p, r, s, t\}$, $\tau = \{U, \emptyset, \{t\}, \{p, s\}, \{p, s, t\}\}$ and $\mathcal{J} = \{\emptyset, \{s\}, \{t\}, \{s, t\}\}$. For a subset $M = \{r, t\}$, $\Psi_\Gamma(M) = \{t\}$ and $\Psi_\xi(M) = \{r, t\}$. Although M is σ_ξ-open set, M is not σ-open set.

Theorem 4.9. *Let (U, τ) be a $\mathcal{J}$-space. Every σ_0-open set is σ_{ξ_0}-open set. In other words, the topology σ_{ξ_0} is finer than the topology σ_0.*

Proof. Let M be σ-open set. Therefore, $M \subseteq Int(Cl(\Psi_\Gamma(M))) \subseteq Int(Cl(\Psi_\xi(M)))$ from Lemma 4.6. □

Example 4.10. Consider the $\mathcal{J}$-space in Example 4.8. For a subset $M = \{p, t\}$ of U, since $Int(Cl(\Psi_\Gamma(M))) = \emptyset$, M is not σ_0-open set. On the other hand, we have $Int(Cl(\Psi_\xi(M))) = U$. This shows that M is σ_{ξ_0}-open set.

Corollary 4.11. *The following corollaries are obtained from Example 4.12 and Example 4.13.*

a) *The concepts of σ_{ξ_0}-open set and σ_ξ-open set are independent from each other.*
b) *The concepts of σ_0-open set and σ_ξ-open set are independent from each other.*

Example 4.12. Let $U = \omega + 1 = \omega \cup \{\omega\}$ and the topology on it be the order topology. This topology can be expressed [18] as:

$$\tau = P(\omega) \cup \{\{\omega\} \cup (\omega \setminus K) : K \subseteq \omega \text{ and } K \text{ is finite}\}.$$

Let (U, τ) be a $\mathcal{J}_{fin}$-space and $M = \{\omega\}$. Except for the point ω, every point in this space has a semi-open neighborhood such that its semi-closure is finite. Therefore,

$$\Psi_\xi(M) = U \setminus \xi(U \setminus M) = U \setminus \xi(\omega) = U \setminus \{\omega\} = \omega.$$

So, $M \nsubseteq \Psi_\xi(M) = \omega$ and $M \subseteq Int(Cl(\Psi_\xi(M))) = U$. As a result, M is not σ_ξ-open set despite being σ_{ξ_0}-open set. Moreover, M is σ_0-open set that is shown in [18].

Example 4.13. Consider the $\mathcal{J}$-space in Example 4.8. It has been shown that subset $M = \{r, t\}$ is σ_ξ-open set in this space. Since $M \nsubseteq Int(Cl(\Psi_\Gamma(M))) = Int(Cl(\Psi_\xi(M))) = \{t\}$, M is neither σ_0-open set nor σ_{ξ_0}-open set.

Corollary 4.14. *The following corollaries are obtained from Example 4.15 and Example 4.16.*

a) *The concepts of σ_ξ-open set and open set are independent from each other.*
b) *The concepts of σ_{ξ_0}-open set and open set are independent from each other.*

Example 4.15. Consider the $\mathcal{J}$-space in Example 3.1. $M = \{p, r, s\}$ is open set but neither σ_ξ-open set nor σ_{ξ_0}-open set because $M \nsubseteq \Psi_\xi(M) = \{r, s\}$ and $M \nsubseteq Int(Cl(\Psi_\xi(M))) = \{s\}$.

Example 4.16. Consider the $\mathcal{J}$-space in Example 4.8. $M = \{p\}$ is not open set but it is both σ_ξ-open set and σ_{ξ_0}-open set because $M \subseteq \Psi_\xi(M) = \{p, s, t\}$ and $M \subseteq Int(Cl(\Psi_\xi(M))) = U$.

From Theorem 4.7, Example 4.8, Theorem 4.9, Example 4.10, Corollary 4.11 (Example 4.12 and 4.13), Corollary 4.14 (Example 4.15 and 4.16) and Diagram I, the following diagram is obtained.

$$\begin{array}{ccc} \theta\text{-open} & \rightarrow & \text{open} \\ \downarrow & & \\ \sigma\text{-open} & \rightarrow & \sigma_0\text{-open} \\ \downarrow & & \downarrow \\ \sigma_\xi\text{-open} & & \sigma_{\xi_0}\text{-open} \end{array}$$

Diagram II

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