Filomat 35:2 (2021), 515–534
https://doi.org/10.2298/FIL2102515B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Measure Pseudo Almost Periodic Solution for a Class of Nonlinear Delayed Stochastic Evolution Equations Driven by Brownian Motion

Nadia Belmabrouk[a], Mondher Damak[a], Mohsen Miraoui[b]

[a]*Faculty of Sciences of Sfax, University of Sfax, Tunisia.*
[b]*Preparatory Institute for Engineering Studies of Kairouan, University of Kairouan, Tunisia.*

Abstract. In this work, we present a new concept of measure-ergodic process to define the space of measure pseudo almost periodic process in the p-th mean sense. We show some results regarding the completeness, the composition theorems and the invariance of the space consisting in measure pseudo almost periodic process. Motivated by above mentioned results, the Banach fixed point theorem and the stochastic analysis techniques, we prove the existence, uniqueness and the global exponential stability of doubly measure pseudo almost periodic mild solution for a class of nonlinear delayed stochastic evolution equations driven by Brownian motion in a separable real Hilbert space. We provide an example to illustrate the effectiveness of our results.

1. Introduction

The qualitative theory of differential equations, involving almost periodicity, has been an attractive topic because of its significance and applications in areas such as physics, mathematical biology, and control theory. The concept of almost periodicity was first introduced in the literature by Bohr in 1923, for more details about this topic we refer the reader to the recent book of N'Guérékata [16] where the author gave an important overview about the theory of almost periodic functions and their applications to differential equations. The notion of $\mu-$pseudo almost periodicity, which was introduced and developed by Ezzinbi et al. [3, 8, 10, 13–15], is a generalization of the almost periodicity and pseudo almost periodicity introduced by Zhang [18, 19]; it is also a generalization of weighted pseudo almost periodicity firstly introduced by Diagana [9].

In recent years, stochastic differential systems have been extensively studied since stochastic modeling plays an important role in physics, engineering, finance, social science and so on. Qualitative properties such as existence, uniqueness and stability for stochastic differential systems have attracted more and more researchers attention. The existence of almost periodic, pseudo almost periodic and measure pseudo almost periodic solutions for stochastic differential equations was obtained. We refer the reader to [1, 2, 6] and references therein.

2010 *Mathematics Subject Classification*. Primary 60H15; Secondary 60G51, 34C27, 35R60
Keywords. pseudo almost periodic solution; measure theory; stochastic processes; stochastic evolution equations.
Received: 24 June 2019; Accepted: 01 February 2021
Communicated by Miljana Jovanović
Email addresses: nadiabenmabrouk09@gmail.com (Nadia Belmabrouk), Mondher_damak@yahoo.com (Mondher Damak), miraoui.mohsen@yahoo.fr (Mohsen Miraoui)

In this work, we consider the following stochastic evolution equation driven by Brownian motion in a separable Hilbert space $\mathbb{H}$:

$$dx(t) = Ax(t)dt + f(t, x(t-\tau))dt + \varphi(t, x(t-\tau))d\mathbb{W}(t), \ t \in \mathbb{R}, \tag{1}$$

where A is the infinitesimal generator of a C_0−semi-group $(T(t))_{t\geq 0}$ exponentially stable, f, φ are two stochastic processes and $\mathbb{W}(t)$ is a two-sided standard Brownian motion with values in $\mathbb{H}$.
In [7], F. Chérif investigates the existence of the quadratic-mean pseudo almost periodic solutions of Eq. (1).
In [11], the authors obtained sufficient condition for the existence of p-th mean μ-pseudo almost periodic mild solutions to the following class of nonlinear stochastic evolution equations driven by a fractional Brownian motion in a separable Hilbert space $\mathbb{H}$:

$$dx(t) = A(t)x(t)dt + f(t, x(t))dt + \theta(t, x(t))d\mathbb{W}(t) + \psi(t)dB^H(t), \ t \in \mathbb{R}, \tag{2}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying Acquistapace–Terreni conditions; f, θ are two stochastic processes and ψ a function deterministic. The concept of almost automorphy was first introduced in the literature by Bochner in the earlier sixties, it is a natural generalization of the almost periodicity. In [12], the authors gave some results for the existence of p-th mean μ−pseudo almost automorphic (or measure pseudo almost automorphic) mild solutions from equation (1) without delay. Motivated by the above discussion, we introduce the concept of doubly measure pseudo almost periodicity, we give some fundamental properties and we investigate the existence, uniqueness and stability of (μ, ν)-pseudo almost periodic mild solutions in p-th mean sense for Eq. (1).
The organization of the work is as follows : In section 2, we introduce the concept of doubly measure pseudo almost periodicity. In section 3, we give some new developments on the completeness and composition of measure pseudo almost automorphic functions. In section 4, we study the existence and uniqueness of (μ, ν)-pseudo almost periodic mild solutions in p-th mean sense for Eq. (1). Section 5 states the stability of the (μ, ν)-pseudo almost periodic mild solutions for Eq.(1). Finally, in section 6, we provide an example to illustrate the basic theory of this work.

2. (μ, ν)-pseudo-almost periodic processes

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by M the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a < b)$. We introduce the following new space of (μ, ν)-ergodic functions :

Definition 2.1. *Let $\mu, \nu \in$ M. A bounded continuous function $f : \mathbb{R} \to \mathbb{H}$ is said to be (μ, ν)-ergodic in p-th $(p \geq 2)$ mean sense, if*

$$\lim_{r\to+\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \| f(t) \|^p \, d\mu(t) = 0.$$

We denote the space of such all functions by $\xi(\mathbb{R}, \mathbb{H}, \mu, \nu)$.

We give the following hypothesis.
$(\mathbf{H}_1)$ Let $\mu, \nu \in M$,

$$\limsup_{r\to+\infty} \frac{\mu([-r, r])}{\nu([-r, r])} = \alpha < \infty.$$

Proposition 2.2. *Let $\mu, \nu \in$ M satisfy $(\boldsymbol{H}_1)$. Then $(\xi(\mathbb{R}, \mathbb{H}, \mu, \nu), \| \,.\, \|_\infty)$ is a Banach space, where* $\| f \|_\infty = \sup_{t\in\mathbb{R}} \| f(t) \|$.

Proof. It is enough to prove that $\xi(\mathbb{R},\mathbb{H},\mu,\nu)$ is closed in $\mathcal{B}_C$. Let $(f_n)_n$ be a sequence in $\xi(\mathbb{R},\mathbb{H},\mu,\nu)$ such that $\lim\limits_{n\to+\infty}\| f_n - f \|_\infty = 0$. Since $\| f \|^p$ is a convex function, then for $r>0$, we have

$$\int_{-r}^{r} \| f(t) \|^p \, d\mu(t) \leqslant 2^{p-1}\int_{-r}^{r} \| f_n(t) - f(t) \|^p \, d\mu(t) + 2^{p-1}\int_{-r}^{r} \| f_n(t) \|^p \, d\mu(t).$$

It implies

$$\frac{1}{\nu([-r,r])}\int_{-r}^{r} \| f(t) \|^p \, d\mu(t) \leqslant \frac{2^{p-1}}{\nu([-r,r])}\int_{-r}^{r} \| f_n(t) - f(t) \|^p \, d\mu(t) + \frac{2^{p-1}}{\nu([-r,r])}\int_{-r}^{r} \| f_n(t) \|^p \, d\mu(t).$$

It follows that,

$$\limsup_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{-r}^{r} \| f(t) \|^p \, d\mu(t) \leqslant \limsup_{r\to+\infty} 2^{p-1}\frac{\mu([-r,r])}{\nu([-r,r])}\sup_{t\in\mathbb{R}} \| f_n(t) - f(t) \|^p .$$

Moreover,

$$\sup_{t\in\mathbb{R}} \| f_n(t) - f(t) \|^p \leqslant \Big(\sup_{t\in\mathbb{R}} \| f_n(t) - f(t) \| \Big)^p = \| f_n - f \|_\infty^p,$$

and from $(\mathbf{H}_1)$, we obtain

$$\limsup_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{-r}^{r} \| f(t) \|^p \, d\mu(t) \leqslant \alpha 2^{p-1} \| f_n - f \|_\infty^p .$$

Since $\lim\limits_{n\to+\infty}\| f_n - f \|_\infty = 0$, we deduce that

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{-r}^{r} \| f(t) \|^p \, d\mu(t) = 0.$$

□

Definition 2.3. *[11] A stochastic process $x(t) : \mathbb{R} \to \mathcal{L}^p(\Omega,\mathbb{H})$ is said to be stochastically bounded in p-th mean sense, if there exists $C > 0$ such that*

$$\mathbb{E} \| x(t) \|^p \leqslant C, \quad \forall t \in \mathbb{R},$$

and a stochastic process $x(t) : \mathbb{R} \to \mathcal{L}^p(\Omega,\mathbb{H})$ is said to be stochastically continuous in p-th mean sense, if

$$\lim_{t\to s} \mathbb{E} \| x(t) - x(s) \|^p = 0, \quad \forall\ t,\ s \in \mathbb{R}.$$

Denote by $\mathcal{B}_C(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$ the collection of all the stochastically bounded continuous processes. We can verify that $\mathcal{B}_C(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\| . \|_\infty)$ is a Banach space, where

$$\| x \|_\infty = \sup_{t\in\mathbb{R}} \Big(\mathbb{E} \| x(t) \|^p \Big)^{1/p}.$$

Definition 2.4. *Let $\mu,\nu \in$ M. A stochastic process x is said to be (μ,ν)-ergodic in p-th mean sense, if $x \in \mathcal{B}_C(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H})$ and it satisfies*

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{-r}^{r} \mathbb{E} \| x(t) \|^p \, d\mu(t) = 0.$$

Denote by $\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ the set of all such stochastic processes.

Proposition 2.5. *Let $\mu,\nu \in$ M satisfy $(\boldsymbol{H}_1)$. Then $\Big(\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu),\| . \|_\infty \Big)$ is a Banach space.*

To prove this proposition, we have just to use the same arguments in the proof of Proposition 2.2. The following lemma gives some properties of ergodicity.

Lemma 2.6. *Let μ, $\nu \in$ M, satisfy ($\boldsymbol{H}_1$), and let I be a bounded interval (eventually $I = \emptyset$). Suppose that $x \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$. Then the following assertions are equivalent :*
(i) $x \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$;
(ii) $\lim_{r\to+\infty} \frac{1}{\nu([-r,r]\backslash I)} \int_{[-r,r]\backslash I} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) = 0$;
(iii) *for any* $\epsilon > 0$,

$$\lim_{r\to+\infty} \frac{\mu\{t \in [-r,r]\backslash I : \mathbb{E} \parallel x(t) \parallel^p > \epsilon\}}{\nu\{t \in [-r,r]\backslash I\}} = 0.$$

Proof. The proof uses the same arguments of the proof of Theorem 2.22 in [10].
(i) $\Longleftrightarrow$ **(ii)** We denote by $\mathcal{A} = \nu(I)$, $\mathcal{B} = \int_I \mathbb{E} \parallel x(t) \parallel^p d\mu(t)$ and $C = \mu(I)$. Since the interval I is bounded and $x \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$, then $\mathcal{A}$, $\mathcal{B}$ *and* C are finite.
For $r > 0$ such that $I \subset [-r, r]$ and $\nu([-r,r]\backslash I) > 0$,
we have

$$\begin{aligned}
\frac{1}{\nu([-r,r]\backslash I)} \int_{[-r,r]\backslash I} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) &= \frac{1}{\nu([-r,r]) - A}\Big(\int_{-r}^{r} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) - B\Big) \\
&= \frac{\nu([-r,r])}{\nu([-r,r]) - A}\Big(\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) - \frac{B}{\nu([-r,r])}\Big).
\end{aligned}$$

Since $\nu(\mathbb{R}) = +\infty$, we obtain that,

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) = 0.$$

Thus, **(i)** and **(ii)** are equivalent.
(ii) $\Rightarrow$ **(iii)** Let $A_r^\varepsilon = \{t \in [-r,r]\backslash I : \mathbb{E} \parallel x(t) \parallel^p > \varepsilon\}$ and $B_r^\varepsilon = \{t \in [-r,r] : \mathbb{E} \parallel x(t) \parallel^p \leqslant \varepsilon\}$.
Assume that **(ii)** holds. Then, we have

$$\begin{aligned}
\frac{1}{\nu([-r,r]\backslash I)} \int_{[-r,r]\backslash I} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) &\geq \frac{1}{\nu([-r,r]\backslash I)} \int_{A_r^\varepsilon} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) \\
&\geq \varepsilon \frac{\mu(A_r^\varepsilon)}{\nu([-r,r]\backslash I)}.
\end{aligned}$$

Therefore, for r large enough, we obtain **(iii)**.
(iii) $\Rightarrow$ **(ii)** Assume that **(iii)** holds, then

$$\begin{aligned}
\frac{1}{\nu([-r,r]\backslash I)} \int_{[-r,r]\backslash I} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) &= \int_{A_r^\varepsilon} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) + \int_{B_r^\varepsilon} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) \\
&\leqslant \parallel x \parallel_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r,r]\backslash I)} + \varepsilon \frac{\mu(B_r^\varepsilon)}{\nu([-r,r]\backslash I)} \\
&\leqslant \parallel x \parallel_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r,r]\backslash I)} + \varepsilon \frac{\mu([-r,r]\backslash I)}{\nu([-r,r]\backslash I)} \\
&\leqslant \parallel x \parallel_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r,r]\backslash I)} + \varepsilon \frac{\mu([-r,r]) - C}{\nu([-r,r]) - A} \\
&\leqslant \parallel x \parallel_\infty \frac{\mu(A_r^\varepsilon)}{\nu([-r,r]\backslash I)} + \varepsilon \frac{\mu([-r,r])}{\nu([-r,r])} \frac{1 - \frac{C}{\mu([-r,r])}}{1 - \frac{A}{\nu([-r,r])}}.
\end{aligned}$$

Since, $\mu(\mathbb{R}) = \nu(\mathbb{R}) = +\infty$, then from ($\mathbf{H}_1$), we get that

$$\limsup_{r\to+\infty} \frac{1}{\nu([-r,r]\backslash I)} \int_{[-r,r]\backslash I} \mathbb{E} \parallel x(t) \parallel^p d\mu(t) \leqslant Cst.\varepsilon.$$

In conclusion, **(ii)** holds. □

Definition 2.7. *Let $\mu,\ \nu \in$ M. A function $f : \mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}) \to \mathcal{L}^p(\Omega,\mathbb{H})$, $(t,x) \mapsto f(t,x)$ is said to be (μ,ν)-ergodic in p-th sense in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathcal{K}$, if $f \in \mathcal{B}_C(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}))$ and it satisfies*

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel f(t,x)\parallel^p d\mu(t)=0,$$

where $\mathcal{K}\subset\mathcal{L}^p(\Omega,\mathbb{H})$ is compact.

We denote by

$$\xi_p(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)=\{f(.,x)\in\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H})\mu,\nu)\ for\ any\ x\in\mathcal{L}^p(\Omega,\mathbb{H})\}$$

the set of all such functions.

Definition 2.8. *[4] Let $x : \mathbb{R} \to \mathcal{L}^p(\Omega,\mathbb{H})$ be a continuous stochastic process. x is said to be almost periodic process in p-th mean sense if for each $\varepsilon > 0$ there exists $l > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha,\alpha+l]$ satisfying*

$$\sup_{t\in\mathbb{R}}\mathbb{E}\parallel x(t+\tau)-x(t)\parallel^p<\varepsilon.$$

We denote by $\mathbf{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$ the space of all such stochastic processes. It is easy to verify that $\left(\mathbf{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H})),\|\ .\ \|_\infty\right)$ is a Banach space.

Definition 2.9. *[11] Let $f : \mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}) \to \mathcal{L}^p(\Omega,\mathbb{H})$ be continuous. f is said to be almost periodic in p-th mean sense in $t \in \mathbb{R}$ uniformly in $x \in \mathcal{K}$, where $\mathcal{K}\subset\mathcal{L}^p(\Omega,\mathbb{H})$ is a compact, if for each $\varepsilon > 0$, there exists $l(\varepsilon,\mathcal{K}) > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha,\alpha+l(\varepsilon,\mathcal{K})]$ satisfying*

$$\sup_{t\in\mathbb{R}}\mathbb{E}\parallel f(t+\tau,x)-f(t,x)\parallel^p<\varepsilon,$$

for each stochastic process $x : \mathbb{R}\to\mathcal{K}$.
We denote by $\boldsymbol{AP}(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}))=\{f(.,x)\in\boldsymbol{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))\ for\ any\ x\in\mathcal{L}^p(\Omega,\mathbb{H})\}$ the space of such stochastic processes.

Definition 2.10. *Let $\mu,\nu \in M$. A continuous stochastic process x is said to be (μ,ν)-pseudo almost periodic in p-th mean sense, if it can be written as*

$$x = x_1 + x_2$$

where $x_1 \in \boldsymbol{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$ and $x_2 \in \xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.

Denote by $\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ the set of all such stochastic processes.
We can verify that

$$\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)\subset\mathcal{B}_C(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H})).$$

We introduce the following new space of doubly measure pseudo almost periodic functions :

Definition 2.11. *Let $\mu,\nu \in M$. A continuous function $f : \mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}) \to \mathcal{L}^p(\Omega,\mathbb{H})$ is said to be $(\mu,\ \nu)$-pseudo almost periodic in p-th mean sense, if it can be written as*

$$f = g + h$$

where $g \in \boldsymbol{AP}(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}))$ and $h \in \xi_p(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.

Denote by $\mathbf{PAP}(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ the set of all such functions.

Definition 2.12. *[3] Let $\mu_1,\ \mu_2 \in M$. If there exist positive constants $\alpha,\ \beta$ and a bounded interval I (eventually $I = \emptyset$) such that*

$$\alpha\mu_1(A) \leqslant \mu_2(A) \leqslant \beta\mu_1(A)$$

for $A \in \mathcal{B}$ satisfying $A \cap I = \emptyset$, then we say that μ_1 and μ_2 are equivalent $\mu_1 \sim \mu_2$.

Proposition 2.13. *Let $\mu_1,\ \mu_2,\ \nu_1$ and $\nu_2 \in M$. If μ_1 and ν_1 are equivalent respectively to μ_2 and ν_2, then*

$$\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2), \tag{3}$$

and

$$\boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2). \tag{4}$$

Proof. We prove in the first (3).
Since $\mu_1 \sim \mu_2, \nu_1 \sim \nu_2$ and $\mathcal{B}$ is a Lebesgue σ-field, then there exist $\alpha, \beta, \gamma, \theta > 0$, such that

$$\alpha\mu_1 \leqslant \mu_2 \leqslant \beta\mu_1,$$

and

$$\gamma\nu_1 \leqslant \nu_2 \leqslant \theta\nu_1.$$

It implies that

$$\begin{aligned} \frac{\alpha}{\theta}\frac{\mu_1\{t \in [-r,r]\backslash I : \mathbb{E} \parallel f(t) \parallel^p > \varepsilon\}}{\nu_1\{t \in [-r,r]\backslash I\}} &\leqslant \frac{\mu_2\{t \in [-r,r]\backslash I : \mathbb{E} \parallel f(t) \parallel^p > \varepsilon\}}{\nu_2\{t \in [-r,r]\backslash I\}} \\ &\leqslant \frac{\beta}{\gamma}\frac{\mu_1\{t \in [-r,r]\backslash I : \mathbb{E} \parallel f(t) \parallel^p > \varepsilon\}}{\nu_1\{t \in [-r,r]\backslash I\}}. \end{aligned}$$

From Lemma 2.6, we get that

$$\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2).$$

Using the definition of (μ, ν)-pseudo almost periodicity, we conclude that,

$$\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_1, \nu_1) = \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu_2, \nu_2). \qquad \square$$

For $\mu \in M$ and $\sigma \in \mathbb{R}$, we define the positive measure μ_σ on $(\mathbb{R}, \mathcal{B})$ by

$$\mu_\sigma(A) = \mu(a + \sigma : a \in A), \quad A \in \mathcal{B}.$$

We give the following hypothesis :
$(\mathbf{H}_2)$ For all $\sigma \in \mathbb{R}$, there exist $\alpha > 0$ and a bounded interval I such that

$$\mu_\sigma(A) \leqslant \alpha\mu(A),$$

where $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Lemma 2.14. *[3] Let $\mu \in M$. Then μ satisfies $(\boldsymbol{H}_2)$ if and only if μ is equivalent to μ_σ for all $\sigma \in \mathbb{R}$.*

Lemma 2.15. *[3] It follows from hypothesis $(\boldsymbol{H}_2)$ that,*

$$\forall\ \delta > 0, \quad \limsup_{r \to +\infty} \frac{\mu([-r - \delta, r + \delta])}{\mu([-r, r])} < +\infty.$$

Let $f \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$. For all $\alpha \in \mathbb{R}$, we define f_α by

$$f_\alpha(t) = f(t + \alpha).$$

We say that the subset $\mathcal{S}$ of $\mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ is translation invariant, if $\forall\ f \in \mathcal{S}$, we have $f_\alpha \in \mathcal{S}$.

Theorem 2.16. *Let $\mu, \nu \in M$ satisfy ($\boldsymbol{H}_2$). Then $\boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ is translation invariant.*

Proof. Firstly, we need to prove that $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ is translation invariant. In other words, if $f \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$, then $f_\tau \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$, $\forall\, \tau \in \mathbb{R}$.
Let $\mu_\tau = \mu(\{t + \tau;\ t \in A\})\ \forall A \in \mathcal{B}$.

$$\begin{aligned}
\frac{1}{\nu([-r, r])} \int_{-r}^{r} \mathbb{E} \parallel f(t+\tau) \parallel^p d\mu(t) &= \frac{\nu[-r+\tau, r+\tau]}{\nu([-r, r])} . \frac{1}{\nu[-r+\tau, r+\tau]} \int_{-r}^{r} \mathbb{E} \parallel f(t+\tau) \parallel d\mu(t) \\
&= \frac{\nu[-r+\tau, r+\tau]}{\nu([-r, r])} . \frac{1}{\nu[-r+\tau, r+\tau]} \int_{-r+\tau}^{r+\tau} \mathbb{E} \parallel f(t) \parallel d\mu_{-\tau}(t) \\
&\leqslant \frac{\nu[-r-|\tau|, r+|\tau|]}{\nu([-r, r])} . \frac{1}{\nu[-r+\tau, r+\tau]} \int_{-r+\tau}^{r+\tau} \mathbb{E} \parallel f(t) \parallel d\mu_{-\tau}(t).
\end{aligned}$$

Since μ and ν satisfy ($\mathbf{H}_2$) and according to Lemma 2.15 we obtain that

$$\frac{1}{\nu([-r, r])} \int_{-r}^{r} \mathbb{E} \parallel f(t+\tau) \parallel^p d\mu(t) \leqslant Cte. \frac{\nu[-r-|\tau|, r+|\tau|]}{\nu([-r, r])} . \frac{1}{\nu([-r+\tau, r+\tau])} \int_{-r+\tau}^{r+\tau} \mathbb{E} \parallel f(t) \parallel d\mu(t).$$

We get that

$$\lim_{t \to +\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \mathbb{E} \parallel f(t+\tau) \parallel^p d\mu(t) = 0.$$

Therefore, $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ is translation invariant. Since $\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ is translation invariant, then $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ is also translation invariant. □

3. Completeness and composition theorem

Theorem 3.1. *Let $\mu, \nu \in M$. Assume that $f \in \boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ can be written as $f = g + h$, where $g \in \boldsymbol{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ and $h \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$. If μ and ν satisfy ($\boldsymbol{H}_2$), then*

$$\{g(t),\ t \in \mathbb{R}\} \subset \overline{\{f(t),\ t \in \mathbb{R}\}}. \tag{5}$$

Proof. For the proof, we use the same arguments given in [3]. Assume that (5) does not hold. Then, there exists $t_0 \in \mathbb{R}$ such that

$$g(t_0) \notin \overline{\{f(t),\ t \in \mathbb{R}\}}.$$

Since μ, ν satisfy ($\mathbf{H}_2$), and from Theorem2.16, we deduce that $\mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ and $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ are translation invariants. We can suppose that $t_0 = 0$, then there exists $\varepsilon > 0$ such that

$$\mathbb{E} \parallel f(t) - g(0) \parallel^p \geq 2^p \varepsilon \quad \forall t \in \mathbb{R}.$$

Note,

$$\mathbb{E} \parallel f(t) - g(0) \parallel^p \leqslant 2^{p-1} \mathbb{E} \parallel f(t) - g(t) \parallel^p + 2^{p-1} \mathbb{E} \parallel g(t) - g(0) \parallel^p .$$

For all $t \in C_\varepsilon := \left\{ t \in \mathbb{R} : \mathbb{E} \parallel g(t) - g(0) \parallel^p < \varepsilon \right\}$, we obtain that,

$$\begin{aligned}
\mathbb{E} \parallel h(t) \parallel^p = \mathbb{E} \parallel f(t) - g(t) \parallel^p &\geq 2^{1-p} \mathbb{E} \parallel f(t) - g(0) \parallel^p - \mathbb{E} \parallel g(t) - g(0) \parallel^p \\
&\geq 2^{1-p} \mathbb{E} \parallel f(t) - g(0) \parallel^p \\
&\geq \varepsilon.
\end{aligned}$$

For all $i \in \{1, 2, ..., n\}$ and for all $t \in \alpha_i + C_\varepsilon$, where $\alpha_1, ..., \alpha_n \in \mathbb{R}$ such that

$$\mathbb{R} = \bigcup_{i=1}^{n} (\alpha_i + C_\varepsilon),$$

we have,

$$\mathbb{E} \parallel h(t-\alpha_i) \parallel^p \geq \varepsilon. \tag{6}$$

Let ψ be the function defined by

$$\psi(t) = \sum_{i=1}^{n} \mathbb{E} \parallel h(t-\alpha_i) \parallel^p . \tag{7}$$

(6) and (7) imply that

$$\psi(t) \geq \varepsilon \quad \forall t \in \mathbb{R}. \tag{8}$$

Since $\xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ is translation invariant, then we get that

$$[t \to h(t-\alpha_i)] \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu), \ \forall\, i \in \{1, ..., n\}.$$

Therefore, $\psi \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$, which is absurd by (8). □

Theorem 3.2. *Let $\mu, \nu \in M$ satisfy $(\boldsymbol{H}_2)$, then the decomposition of (μ, ν)-pseudo almost periodic function in the form $f = g + h$, where $g \in \boldsymbol{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ and $h \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$, is unique.*

This theorem can be proved with the same steps developed in the proof of Theorem 2 in [7].

Theorem 3.3. *Let $\mu, \nu \in M$, satisfy $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_2)$. Then $(\boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu, \parallel . \parallel_\infty)$ is a Banach space.*

Proof. We use the same arguments of the proof of Theorem 2.19 in [10], where we take the Banach space $X := \mathcal{L}^p(\Omega, \mathbb{H})$ equipped with the norm $\parallel x \parallel_{\mathcal{L}^p} = (\mathbb{E} \parallel x \parallel^p)^{1/p}$. □

Theorem 3.4. *[5] Let $f : \mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}) \to \mathcal{L}^p(\Omega, \mathbb{H})$, $(t, x) \mapsto f(t, x)$, be an almost periodic process in t uniformly in $x \in \mathcal{K}$, where $\mathcal{K} \subset \mathcal{L}^p(\Omega, \mathbb{H})$ is a compact. Assume that f satisfies the Lipschitz condition : there exists $L > 0$ such that for any $x, y \in \mathcal{L}^p(\Omega, \mathbb{H})$,*

$$\mathbb{E} \parallel f(t,x) - f(t,y) \parallel^p \leqslant L.\mathbb{E} \parallel x - y \parallel^p .$$

Then $t \mapsto f(t, x(t)) \in \boldsymbol{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H})$ for any $x \in \boldsymbol{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$.

Theorem 3.5. *Let $\mu, \nu \in M$, satisfy $\boldsymbol{H}_2$. Assume that $f \in \boldsymbol{PAP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$. If f satisfies the Lipschitz condition in the second variable, that is, there exists $L > 0$ such that, for any $x, y \in \mathcal{L}^p(\Omega, \mathbb{H})$,*

$$\mathbb{E} \parallel f(t,x) - f(t,y) \parallel^p \leqslant L.\mathbb{E} \parallel x - y \parallel^p, \quad \forall t \in \mathbb{R}.$$

Then $t \mapsto f(t, x(t)) \in \boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ for any $x \in \boldsymbol{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$.

Proof. Let $f \in \mathbf{PAP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$ and $x \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$.
Then, we can write $f = g + h$, where $g \in \mathbf{AP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}))$ and $h \in \xi_p(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathbb{H}), \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$.
And

$$x = x_1 + x_2,$$

with $x_1 \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H})$, and $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$. We decomposed f as

$$\begin{aligned} f(t, x(t)) &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + [f(t, x_1(t)) - g(t, x_1(t))] \\ &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + h(t, x_1(t)). \end{aligned}$$

To prove this Theorem, we need to verify
(i) $g(., x_1(.)) \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$.
(ii) $f(., x(.)) - f(., x_1(.)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$.

(iii) $h(.,x_1(.)) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.
To demonstrate **(i)**, we use the similar arguments of **Step (1)** in the proof of Theorem 5.7 in [11].
(ii) Let $x,\ x_1 \in \mathcal{L}^p(\Omega,\mathbb{H})$. By using the Lipschitz condition, we obtain

$$\begin{aligned}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel f(t,x(t))-f(t,x_1(t)\parallel^p d\mu(t) &\leqslant \frac{1}{\nu([-r,r])}.L\int_{-r}^{r}\mathbb{E}\parallel x(t)-x_1(t)\parallel^p d\mu(t)\\ &\leqslant \frac{1}{\nu([-r,r])}.L\int_{-r}^{r}\mathbb{E}\parallel x_2(t)\parallel^p d\mu(t).\end{aligned}$$

Since $x_2 \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$, then $\lim\limits_{r\to+\infty}\dfrac{1}{\nu([-r,r])}\displaystyle\int_{-r}^{r}\mathbb{E}\parallel x_2(t)\parallel^p d\mu(t)=0$.
We deduce that,

$$\limsup_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel f(t,x(t))-f(t,x_1(t)\parallel^p d\mu(t)=0.$$

Therefore,

$$f(.,x(.)-f(.,x_1(.))\in\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu).$$

(iii) It remains to demonstrate the ergodicity of $h(.,x_1(.))$. First, we have

$$\begin{aligned}\parallel h(t,x)-h(t,y)\parallel^p &= \parallel f(t,x)-g(t,x)-f(t,y)+g(t,y)\parallel^p\\ &\leqslant 2^{p-1}\parallel f(t,x)-f(t,y)\parallel^p+2^{p-1}\parallel g(t,x)-g(t,y)\parallel^p .\end{aligned}$$

By using the Lipschitz condition, we obtain that

$$\begin{aligned}\mathbb{E}\parallel h(t,x)-h(t,y)\parallel^p &\leqslant 2^{p-1}\mathbb{E}\parallel f(t,x)-f(t,y)\parallel^p+2^{p-1}\mathbb{E}\parallel g(t,x)-g(t,y)\parallel^p\\ &\leqslant 2^p.L\mathbb{E}\parallel x-y\parallel^p .\end{aligned}$$

Since $\mathcal{K}=\overline{\{x_1(t),\ t\in\mathbb{R}\}}$ is a compact. Then, for $\varepsilon>0$ there exists $x_1,...,x_m\in\mathcal{K}$, such that $\mathcal{K}\subset\cup_{i=1}^m\Big(Bx_i,\ \dfrac{\varepsilon}{2^{2p-1}L}\Big)$, where $B(x_i,\ \frac{\varepsilon}{2^{2p-1}L})=\{x\in\mathcal{K};\ \parallel x_i-x\parallel^p\leqslant\frac{\varepsilon}{2^{2p-1}L}\}$.
It implies that, $\mathcal{K}\subset\cup_{i=1}^m\Big\{x\in\mathcal{K},\forall\, t\in\mathbb{R},\mathbb{E}\parallel h(t,x)-h(t,x_i)\parallel^p\leqslant\dfrac{\varepsilon}{2^{p-1}}\Big\}$.
Let $t\in\mathbb{R}$ and $x\in\mathcal{K}$. Then, there exists $i_0\in\{1,...,m\}$ such that

$$\mathbb{E}\parallel h(t,x)-h(t,x_{i_0})\parallel^p\leqslant\frac{\varepsilon}{2^{p-1}}.$$

We get that

$$\begin{aligned}\mathbb{E}\parallel h(t,x_1(t))\parallel^p &\leqslant 2^{p-1}\mathbb{E}\parallel h(t,x_1(t))-h(t,x_{i_0})\parallel^p+2^{p-1}\mathbb{E}\parallel h(t,x_{i_0})\parallel^p\\ &\leqslant \varepsilon+2^{p-1}\sum_{i=1}^m\mathbb{E}\parallel h(t,x_i)\parallel^p .\end{aligned}$$

Since $\forall i\in\{1,...,m\}$ we have

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t,x_i)\parallel^p d\mu(t)=0.$$

It follows that

$$\limsup_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t,x_1(t))\parallel^p d\mu(t)\leqslant\varepsilon,\quad\forall\varepsilon>0.$$

We deduce that

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t,x_1(t))\parallel^p d\mu(t)=0.$$

Finally,

$$t\mapsto h(t,x_1(t))\in\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu).$$

It ends the proof. □

4. Existence of (μ,ν)-pseudo almost periodic mild solutions for nonlinear stochastic differential equations with delay

This section is devoted to the existence of (μ,ν)-pseudo almost periodic solution for the nonlinear stochastic delayed evolution equation (1) in the space $\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$, where $\mu,\ \nu\in M$. • $A: D(A)\subset\mathcal{L}^p(\Omega,\mathbb{H})\to\mathcal{L}^p(\Omega,\mathbb{H})$ is an infinitesimal generator, generating a C_0- semi group exponentially stable, denoted by $(T(t)_{t\geq 0})$ such that $\forall t\in\mathbb{R}$, there exists $K>0$ and $\omega>0$ satisfying

$$(\mathbf{H}_0)\ \ \| T(t) \|\leqslant Ke^{-\omega t}.$$

• $f:\ \mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H})\to\mathcal{L}^p(\Omega,\mathbb{H}),\ \ \varphi\ :\ \mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H})\to\mathcal{L}^p(\Omega,\mathbb{H})$ are two stochastic processes.
• $W(.)$ is a two-sided and standard one-dimensional Brownian motion with values in $\mathbb{H}$.
• $\tau>0$ is a constant delay.

Definition 4.1. *An $\mathcal{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of Equation* (1), *if it satisfies the corresponding stochastic integral equation*

$$x(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s,x(s-\tau))ds+\int_a^t T(t-s)\varphi(s,x(s-\tau))dW(s), \tag{9}$$

for all $t,a\in\mathbb{R}$ such that $t\geq a$.

Lemma 4.2. *[17] Let $S:[0,T]\times\Omega\to\ell(\mathcal{L}^p(\Omega,\mathbb{H}))$ be an $\mathcal{F}_t$-adapted measurable stochastic process satisfying*

$$\int_0^T \mathbb{E}\parallel S(t)\parallel^2 dt<\infty\ \ a.s,$$

where $\ell(\mathcal{L}^p(\Omega,\mathbb{H}))$ designate the space of all continuous linear operators from $\mathcal{L}^p(\Omega,\mathbb{H})$ to itself. Then $\forall p\geq 1$, there exist a constant $C_p>0$ such that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\parallel\int_0^T S(s)dW(s)\parallel^p\leqslant C_p\mathbb{E}\Big(\int_0^T\parallel S(s)\parallel^2 ds\Big)^{p/2},\ \ T>0.$$

Theorem 4.3. *If $f\in\boldsymbol{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$, then*
(i) $t\mapsto\int_0^t T(t-s)f(s-\tau)ds\in\boldsymbol{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.
(ii) $t\mapsto\int_0^t T(t-s)f(s-\tau)dW(s)\in\boldsymbol{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.

Proof. **(i)** We know that $f\in\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$, then it can be decomposed as $f=g+h$ where $g\in\mathbf{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$, and $h\in\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.
Denote by $(\Lambda x)(t)=\int_{-\infty}^t T(t-s)g(s-\tau)ds$ and $(\Gamma x)(t)=\int_{-\infty}^t T(t-s)h(s-\tau)ds$.
We need to verify that $(\Lambda x)(t)\in\mathbf{AP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$ and $(\Gamma x)(t)\in\xi_p(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$.
• Our first step consists to proving that $(\Lambda x)(t)$ *and* $(\Gamma x)(t)$ are stochastically continuous.

Let $t_0 \in \mathbb{R}$ fixed. Assume that $\alpha = s - t + t_0$, and from Holder's inequality, we get that

$$\begin{aligned}
&\mathbb{E} \parallel (\Lambda x)(t) - (\Lambda x)(t_0) \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t} T(t-s)g(s-\tau)ds - \int_{-\infty}^{t_0} T(t_0-s)g(s-\tau)ds \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t_0} T(t_0-\alpha)g(\alpha+t-t_0, x(\alpha+t-t_0-\tau))d\alpha - \int_{-\infty}^{t_0} T(t_0-s)g(s-\tau)ds \parallel^p \\
&\leqslant \mathbb{E}\Big[\int_{-\infty}^{t_0} \parallel T(t_0-s) \parallel\parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel ds\Big]^p \\
&\leqslant \mathbb{E}\Big[\int_{-\infty}^{t_0} \parallel T(t_0-s) \parallel^{\frac{p-1}{p}} \parallel T(t_0-s) \parallel^{\frac{1}{p}} \parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel ds\Big]^p \\
&\leqslant \mathbb{E}\Big[\Big(\int_{-\infty}^{t_0} (\parallel T(t_0-s) \parallel^{\frac{p-1}{p}})^{\frac{p}{p-1}} ds\Big)^{\frac{p-1}{p}} \times \\
&\Big(\int_{-\infty}^{t_0} \Big(\parallel T(t_0-s) \parallel^{\frac{1}{p}} \parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel \Big)^p ds\Big)^{\frac{1}{p}}\Big]^p \\
&\leqslant \Big(\int_{-\infty}^{t_0} \parallel T(t_0-s) \parallel ds\Big)^{p-1} \times \\
&\int_{-\infty}^{t_0} \parallel T(t_0-s) \parallel .\mathbb{E} \parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel^p ds \\
&\leqslant \frac{K^p}{\omega^{p-1}} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} \mathbb{E} \parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel^p ds.
\end{aligned}$$

Let $\{t_n\}$ be a real arbitrary sequence such that $t_n \to t_0$ as $n \to +\infty$. Since $g \in \mathcal{B}_C(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$, we have

$$e^{-\omega(t_0-s)} \mathbb{E} \parallel g(s+t_n-t_0-\tau) - g(s-\tau) \parallel^p \to 0, \ n \to +\infty.$$

Consequently, for n large enough one has

$$e^{-\omega(t_0-s)} \mathbb{E} \parallel g(s+t_n-t_0-\tau) - g(s-\tau) \parallel^p \leqslant 2^p e^{-\omega(t_0-s)} \parallel g \parallel_\infty^p .$$

Furthermore,

$$\int_{-\infty}^{t_0} 2^p e^{-\omega(t_0-s)} \parallel g \parallel_\infty^p ds < \infty.$$

Then, from the Lebesgue's Dominated Convergence Theorem, we get that

$$\lim_{n\to+\infty} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} \mathbb{E} \parallel g(s+t_n-t_0-\tau) - g(s-\tau) \parallel^p ds = 0.$$

Hence,

$$\lim_{t\to t_0} \int_{-\infty}^{t_0} e^{-\omega(t_0-s)} \mathbb{E} \parallel g(s+t-t_0-\tau) - g(s-\tau) \parallel^p ds = 0.$$

Therefore

$$\lim_{t\to t_0} \mathbb{E} \parallel (\Lambda x)(t) - (\Lambda x)(t_0) \parallel^p = 0.$$

In this way we have shown that $(\Lambda x)(t)$ is stochastically continuous. In the same way we demonstrate that $(\Gamma x)(t)$ is a continuous process.

Thus, we conclude that $\int_{-\infty}^{t} T(t-s)f(s-\tau)ds$ is stochastically continuous.

• We now prove the almost periodicity of Λx. Note that the integral

$$\int_{-\infty}^{t} T(t-s)g(s-\tau)ds$$

is absolutely convergent, and

$$\| \int_{-\infty}^{t} T(t-s)g(s-\tau)ds \| \leqslant \frac{K}{\omega} \| g \|_{\infty} .$$

Since $g \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$, then $\forall \varepsilon > 0,\ \exists\, l > 0,\ \ \forall\, \alpha \in \mathbb{R},$ *there exists* $\delta \in [\alpha, \alpha + l]$ satisfying

$$\sup_{t\in\mathbb{R}} \mathbb{E} \| g(t+\delta) - g(t) \|^p < \varepsilon\Big(\frac{\omega}{K}\Big)^p.$$

In other hand, assume that $\sigma = s - \delta$, then we obtain :

$$\begin{aligned}
&\mathbb{E} \| (\Lambda x)(t+\delta) - (\Lambda x)(t) \|^p \\
&= \mathbb{E} \| \int_{-\infty}^{t+\delta} T(t+\delta-s)g(s-\tau)ds - \int_{-\infty}^{t} T(t-s)g(s-\tau)ds \|^p \\
&= \mathbb{E} \| \int_{-\infty}^{t} T(t-\sigma)g(\sigma+\delta-\tau)d\sigma - \int_{-\infty}^{t} T(t-s)g(s-\tau)ds \|^p \\
&= \mathbb{E} \| \int_{-\infty}^{t} T(t-s)\Big(g(s+\delta-\tau) - g(s-\tau)\Big)ds \|^p .
\end{aligned}$$

Using the same steps developed above and the condition of exponential C_0−semi-group$(T(t))_{t\geq 0}$, we get that

$$\begin{aligned}
&\mathbb{E} \| (\Lambda x)(t+\delta) - (\Lambda x)(t) \|^p \\
&= \mathbb{E} \| \int_{-\infty}^{t} T(t-s)\Big(g(s+\delta-\tau) - g(s-\tau)\Big)ds \|^p \\
&\leqslant \Big(\int_{-\infty}^{t} \| T(t-s) \| ds\Big)^{p-1} \times \int_{-\infty}^{t} \| T(t-s) \| .\mathbb{E} \| g(s+\delta-\tau) - g(s-\tau) \|^p ds \\
&\leqslant \Big(\frac{K}{\omega}\Big)^{p-1} . \frac{K}{\omega} .\varepsilon. \Big(\frac{\omega}{K}\Big)^p \\
&\leqslant \varepsilon.
\end{aligned}$$

• It remains to verify that $(\Gamma x) \in \xi_p(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$. First, we have

$$\begin{aligned}
\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \| (\Gamma x)(t) \|^p d\mu(t) &= \frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \| \int_{-\infty}^{t} T(t-s)h(s-\tau)ds \|^p d\mu(t) \\
&\leqslant \frac{1}{\nu([-r,r])} \int_{-r}^{r} \Big[\mathbb{E} \int_{-\infty}^{t} \| T(t-s)h(s-\tau) \| ds\Big]^p d\mu(t).
\end{aligned}$$

From Holder's inequality and Fubini's theorem, we get that

$$\begin{aligned}
&\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \| (\Gamma x)(t) \|^p d\mu(t) \\
&\leqslant \frac{K^p}{\nu([-r,r])} \int_{-r}^{r} \Big(\int_{-\infty}^{t} e^{-\omega(t-s)} ds\Big)^{p-1} \times \Big(\int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E} \| h(s-\tau) \|^p ds\Big) d\mu(t) \\
&\leqslant \frac{K^p}{\omega^{p-1}} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E} \| h(s-\tau) \|^p ds d\mu(t) \\
&\leqslant \frac{K^p}{\omega^{p-1}} . \frac{1}{\nu([-r,r])} \int_{-r}^{r} \int_{\mathbb{R}} \mathbf{1}_{]-\infty,t]}(s) e^{-\omega(t-s)} \mathbb{E} \| h(s-\tau) \|^p ds d\mu(t) \\
&\leqslant \frac{K^p}{\omega^{p-1}} . \frac{1}{\nu([-r,r])} \int_{\mathbb{R}} \int_{-r}^{r} \mathbf{1}_{]-\infty,t]}(s) e^{-\omega(t-s)} \mathbb{E} \| h(s-\tau) \|^p d\mu(t) ds.
\end{aligned}$$

Let $v = t - s$, then we obtain

$$\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel(\Gamma x)(t)\parallel^{p}d\mu(t) \leqslant \frac{K^{p}}{\omega^{p-1}}\int_{0}^{+\infty}\frac{e^{-\omega v}}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t-v-\tau)\parallel^{p}d\mu(t)dv$$

Moreover, we have

$$|\frac{e^{-\omega v}}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t-v-\tau)\parallel^{p}d\mu(t)|\leqslant e^{-\omega v}\parallel h\parallel_{\infty}^{p}.$$

Since $h\in\xi_{p}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}),\mu,\nu)$ which is translation invariant, and from the Lebesgue's dominated convergence theorem, we deduce that

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel(\Gamma x)(t)\parallel^{p}d\mu(t)\\
&\leqslant\frac{K^{p}}{\omega^{p-1}}\int_{0}^{+\infty}\Big(e^{-\omega v}\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel h(t-v-\tau)\parallel^{p}d\mu(t)\Big)dv\\
&=0.
\end{aligned}$$

Thus, $(\Gamma x)(t)\in\xi_{p}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}),\mu,\nu)$. Finally, **(i)** holds.

(ii) Similarly to **(i)**, since $f\in\mathbf{PAP}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}),\mu,\nu)$, then it can be written as $f=\Theta+\varphi$ where $\Theta\in\mathbf{AP}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}))$, and $\varphi\in\xi_{p}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}),\mu,\nu)$.
Let $\mathcal{S}_1x(t)=\int_{-\infty}^{t}T(t-s)\Theta(s-\tau)dW(t)$ and $\mathcal{S}_2x(t)=\int_{-\infty}^{t}T(t-s)\varphi(s-\tau)dW(t)$. We must demonstrate that $\mathcal{S}_1x\in\mathbf{AP}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}))$ and $\mathcal{S}_2x\in\xi_{p}(\mathbb{R},\mathcal{L}^{p}(\Omega,\mathbb{H}),\mu,\nu)$.
•• In the first time we verify that $(\mathcal{S}_1x)(t)$ is stochastically continuous. We take an arbitrary number $t_0\in\mathbb{R}$. Let $\alpha=s-t+t_0$, then we obtain

$$\begin{aligned}
&\mathbb{E}\parallel(\mathcal{S}_1x)(t)-(\mathcal{S}_1x)(t_0)\parallel^{p}\\
&=\mathbb{E}\parallel\int_{-\infty}^{t}T(t-s)\Theta(s-\tau)dW(s)-\int_{-\infty}^{t_0}T(t_0-s)\Theta(s-\tau)dW(s)\parallel^{p}\\
&=\mathbb{E}\parallel\int_{-\infty}^{t_0}T(t_0-\alpha)\Theta(\alpha+t-t_0-\tau)dW(\alpha+t-t_0)-\int_{-\infty}^{t_0}T(t_0-s)\Theta(s-\tau)dW(s)\parallel^{p}.
\end{aligned}$$

Suppose that

$$\widetilde{W}(\alpha)=W(\alpha+t-t_0)-W(t-t_0).$$

We note that W and $\widetilde{W}$ are two Wiener process and have the same distribution. Using the Lemma 4.2, we obtain

$$\begin{aligned}
&\mathbb{E}\parallel(\mathcal{S}_1x)(t)-(\mathcal{S}_1x)(t_0)\parallel^{p}\\
&=\mathbb{E}\parallel\int_{-\infty}^{t_0}T(t_0-\alpha)\Theta(\alpha+t-t_0-\tau)d\widetilde{W}(\alpha)-\int_{-\infty}^{t_0}T(t_0-s)\Theta(s-\tau)dW(s)\parallel^{p}\\
&=\mathbb{E}\parallel\int_{-\infty}^{t_0}T(t_0-\alpha)\Theta(\alpha+t-t_0-\tau)d\widetilde{W}(\alpha)-\int_{-\infty}^{t_0}T(t_0-s)\Theta(s-\tau)d\widetilde{W}(s)\parallel^{p}\\
&=\mathbb{E}\parallel\int_{-\infty}^{t_0}T(t_0-s)\Big(\Theta(s+t-t_0-\tau)-\Theta(s-\tau)\Big)d\widetilde{W}(\alpha)\parallel^{p}\\
&\leqslant C_p\mathbb{E}\Big[\int_{-\infty}^{t_0}\parallel T(t_0-s)\Big(\Theta(s+t-t_0-\tau)-\Theta(s-\tau)\Big)\parallel^{2}ds\Big]^{\frac{p}{2}}.
\end{aligned}$$

From Holder's inequality, we get that

$$
\begin{aligned}
&\mathbb{E} \parallel (\mathcal{S}_1 x)(t) - (\mathcal{S}_1 x)(t_0) \parallel^p \\
&\leqslant C_p \mathbb{E}\Big[\int_{-\infty}^{t_0} \parallel T(t_0 - s)\Big(\Theta(s + t - t_0 - \tau) - \Theta(s - \tau)\Big) \parallel^2 ds\Big]^{\frac{p}{2}} \\
&\leqslant C_p \mathbb{E}\Big(\int_{-\infty}^{t_0} \parallel T(t_0 - s) \parallel^2 \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^2 ds\Big)^{\frac{p}{2}} \\
&\leqslant C_p \mathbb{E}\Big(\int_{-\infty}^{t_0} \parallel T(t_0 - s) \parallel^{2.\frac{p-2}{p}} . \parallel T(t_0 - s) \parallel^{\frac{4}{p}} \times \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^2 ds\Big)^{\frac{p}{2}} \\
&\leqslant C_p \mathbb{E}\Big[\Big(\int_{-\infty}^{t_0} (\parallel T(t_0 - s) \parallel^{2.\frac{p-2}{p}})^{\frac{p}{p-2}} ds\Big)^{\frac{p-2}{p}} \\
&\times\Big(\int_{-\infty}^{t_0} (\parallel T(t_0 - s) \parallel^{\frac{4}{p}} \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^2)^{\frac{p}{2}} ds\Big)^{\frac{2}{p}}\Big]^{\frac{p}{2}} \\
&\leqslant C_p K^p \Big(\int_{-\infty}^{t_0} e^{-2\omega(t_0 - s)} ds\Big)^{\frac{p-2}{2}} \times \int_{-\infty}^{t_0} e^{-2\omega(t_0 - s)} \mathbb{E} \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^p ds \\
&\leqslant C_p K^p (2\omega)^{\frac{2-p}{2}} \int_{-\infty}^{t_0} e^{-2\omega(t_0 - s)} \mathbb{E} \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^p ds.
\end{aligned}
$$

By the similar arguments as above, we obtain

$$\lim_{t \to t_0} \int_{-\infty}^{t_0} e^{-2\omega(t_0 - s)} \mathbb{E} \parallel \Theta(s + t - t_0 - \tau) - \Theta(s - \tau) \parallel^p ds = 0.$$

It implies that

$$\lim_{t \to t_0} \mathbb{E} \parallel \int_{-\infty}^{t} T(t - s)\Theta(s - \tau) dW(s) - \int_{-\infty}^{t_0} T(t_0 - s)\Theta(s - \tau) dW(s) \parallel^p = 0.$$

Thus,

$$\lim_{t \to t_0} \mathbb{E} \parallel \mathcal{S}_1 x(t) - \mathcal{S}_1 x(t_0) \parallel^p = 0.$$

So we demonstrate that $(\mathcal{S}_1 x)(t)$ is stochastically continuous. By an analogous argument, we verify that $(\mathcal{S}_2)x(t)$ is also continuous.

•• We know that $\Theta \in \mathbf{AP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$. So, $\forall \varepsilon > 0, \ \exists\, l > 0, \ \forall\, \alpha \in \mathbb{R}, \ \exists \delta \in [\alpha, \alpha + l]$ satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E} \parallel \Theta(t + \delta) - \Theta(t) \parallel^p < \varepsilon . \frac{1}{C_p} \Big(\frac{\sqrt{2\omega}}{K} \Big)^p.$$

Let $\sigma = s - \delta$, and $\widetilde{W}(\sigma) = W(\sigma + \delta) - W(\delta)$. From lemma 4.2, we obtain

$$
\begin{aligned}
&\mathbb{E} \parallel (\mathcal{S}_1 x)(t + \delta) - (\mathcal{S}_1 x)(t) \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t+\delta} T(t + \delta - s)\Theta(s - \tau) dW(s) - \int_{-\infty}^{t} T(t - s)\Theta(s - \tau) dW(s) \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t} T(t - \sigma)\Theta(\sigma + \delta - \tau) dW(\sigma + \delta) - \int_{-\infty}^{t} T(t - s)\Theta(s - \tau) dW(s) \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t} T(t - \sigma)\Theta(\sigma + \delta - \tau) d\widetilde{W}(\sigma) - \int_{-\infty}^{t} T(t - s)\Theta(s - \tau) d\widetilde{W}(s) \parallel^p \\
&= \mathbb{E} \parallel \int_{-\infty}^{t} T(t - s)\Big(\Theta(s + \delta - \tau) - \Theta(s - \tau)\Big) d\widetilde{W}(s) \parallel^p \\
&\leqslant C_p \mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t - s)\Theta(s + \delta - \tau) - \Theta(s - \tau) \parallel^2 ds\Big)^{\frac{p}{2}}.
\end{aligned}
$$

From Holder's inequality and the condition of exponential C_0−semi-group$(T(t))_{t\geq 0})$, we obtain

$$\begin{aligned}
&\mathbb{E} \parallel \mathcal{S}_1 x(t+\delta) - \mathcal{S}_1 x(t) \parallel^p \\
&\leqslant C_p \mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)\Theta(s+\delta-\tau) - \Theta(s-\tau) \parallel^2 ds \Big)^{\frac{p}{2}} \\
&\leqslant C_p \Big(\int_{-\infty}^{t} \parallel T(t-s) \parallel^2 ds \Big)^{\frac{p-2}{2}} \times \int_{-\infty}^{t} \parallel T(t-s) \parallel^2 .\mathbb{E} \parallel \Theta(s+\delta-\tau) - \Theta(s-\tau) \parallel^p ds \\
&\leqslant C_p K^{p-2} \Big(\int_{-\infty}^{t} e^{-2\omega(t-s)} ds \Big)^{\frac{p-2}{2}} \times K^2 \int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E} \parallel \Theta(s+\delta-\tau) - \Theta(s-\tau) \parallel^p ds \\
&\leqslant C_p K^p . (\frac{1}{2\omega})^{\frac{p-2}{2}} . \frac{1}{2\omega} . \varepsilon . \frac{1}{C_p} \Big(\frac{\sqrt{2\omega}}{K} \Big)^p \\
&\leqslant \varepsilon .
\end{aligned}$$

Consequently, $(\mathcal{S}_1 x)(t)$ is almost periodic.

•• In order to complete the proof we still have to show the ergodicity of $(\mathcal{S}_2 x)(t)$. By Lemma 4.2, we get that

$$\begin{aligned}
\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel \mathcal{S}_2 x(t) \parallel^p d\mu(t) &= \frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel \int_{-\infty}^{t} T(t-s)\varphi(s-\tau) dW(s) \parallel^p d\mu(t) \\
&\leqslant C_p \frac{1}{\nu([-r,r])} \int_{-r}^{r} \Big[\mathbb{E} \int_{-\infty}^{t} \parallel T(t-s)\varphi(s-\tau) \parallel^2 ds \Big]^{\frac{p}{2}} d\mu(t).
\end{aligned}$$

From Holder's inequality and Fubini's theorem, we infer that

$$\begin{aligned}
&\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel (\mathcal{S}_2 x)(t) \parallel^p d\mu(t) \\
&\leqslant C_p \frac{1}{\nu([-r,r])} \int_{-r}^{r} \Big(\int_{-\infty}^{t} \parallel T(t-s) \parallel^2 ds \Big)^{\frac{p-2}{2}} \times \int_{-\infty}^{t} \parallel T(t-s) \parallel^2 .\mathbb{E} \parallel \varphi(s-\tau) \parallel^p ds d\mu(t) \\
&\leqslant C_p \frac{K^p}{\nu([-r,r])} \int_{-r}^{r} \Big(\int_{-\infty}^{t} e^{-2\omega(t-s)} ds \Big)^{\frac{p-2}{2}} \times \Big(\int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E} \parallel \varphi(s-\tau) \parallel^p ds \Big) d\mu(t) \\
&\leqslant C_p \frac{K^p}{(2\omega)^{\frac{p-2}{2}}} \frac{1}{\nu([-r,r])} \int_{-r}^{r} \int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E} \parallel \varphi(s-\tau) \parallel^p ds d\mu(t) \\
&\leqslant C_p \frac{K^p}{(2\omega)^{\frac{p-2}{2}}} . \frac{1}{\nu([-r,r])} \int_{-r}^{r} \int_{\mathbb{R}} \mathbf{1}_{]-\infty,t]}(s) e^{-2\omega(t-s)} \mathbb{E} \parallel \varphi(s-\tau) \parallel^p ds d\mu(t) \\
&\leqslant C_p \frac{K^p}{(2\omega)^{\frac{p-2}{2}}} . \frac{1}{\nu([-r,r])} \int_{\mathbb{R}} \int_{-r}^{r} \mathbf{1}_{]-\infty,t]}(s) e^{-2\omega(t-s)} \mathbb{E} \parallel \varphi(s-\tau) \parallel^p d\mu(t) ds.
\end{aligned}$$

Let $v = t - s$, then we obtain

$$\frac{1}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel (\mathcal{S}_2 x)(t) \parallel^p d\mu(t) \leqslant C_p \frac{K^p}{(2\omega)^{\frac{p-2}{2}}} \int_{0}^{+\infty} \frac{e^{-2\omega v}}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel \varphi(t-v-\tau) \parallel^p d\mu(t) dv.$$

Add to that, we have

$$\Big| \frac{e^{-2\omega v}}{\nu([-r,r])} \int_{-r}^{r} \mathbb{E} \parallel \varphi(t-v-\tau) \parallel^p d\mu(t) \Big| \leqslant e^{-2\omega v} \parallel \varphi \parallel_{\infty}^p .$$

Since, $\varphi \in \xi_p(\mathbb{R}, \mathbb{H}, \mu, \nu)$ which is translation invariant, and from the Lebesgue dominated convergence theorem, we deduce that

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel(\mathcal{S}_2x)(t)\parallel^p d\mu(t)\\
&\leqslant C_p\frac{K^p}{(2\omega)^{\frac{p-2}{2}}}\int_0^{+\infty}e^{-2\omega v}\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}\mathbb{E}\parallel\varphi(t-v,x(t-v-\tau)\parallel^p d\mu(t)dv\\
&=0,
\end{aligned}$$

which complete the proof. □

Theorem 4.4. *Let $\mu,\ \nu\in M$ satisfy $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_2)$. Assume that $f,\ \psi\in PAP(\mathbb{R}\times\mathcal{L}^p(\Omega,\mathbb{H}),\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ which satisfy the Lipschitz condition ,that is there exist two positive constants L and L' such that*

$$\mathbb{E}\parallel f(t,x)-f(t,y)\parallel^p\leqslant L\mathbb{E}\parallel x-y\parallel^p, \tag{10}$$

$$\mathbb{E}\parallel \psi(t,x)-\psi(t,y)\parallel^p\leqslant L'\mathbb{E}\parallel x-y\parallel^p, \tag{11}$$

$\forall t\in\mathbb{R}$ and $x,y\in\mathcal{L}^p(\Omega,\mathbb{H})$. If

$$2^{p-1}K^p\Big(\frac{L}{\omega^p}+C_pL'\Big(\frac{1}{2\omega}\Big)^{p/2}\Big)<1,\quad p>2,$$

and

$$K^2\Big(2\frac{L}{\omega^2}+\frac{L'}{\omega}\Big)<1,$$

then Equation (1) has a unique (μ,ν)-pseudo almost periodic mild solution in p-th mean sense on $\mathbb{R}$.

Proof. It is clear that $x:\mathbb{R}\to\mathcal{L}^p(\Omega,\mathbb{H})$ is a solution of (1) if and only if it satisfies the stochastic integral equation :

$$x(t)=\int_{-\infty}^{t}T(t-s)f(s,x(s-\tau))ds+\int_{-\infty}^{t}T(t-s)\psi(s,x(s-\tau))dW(s). \tag{12}$$

Let

$$(\gamma x)(t)=\int_{-\infty}^{t}T(t-s)f(s,x(s-\tau))ds+\int_{-\infty}^{t}T(t-s)\psi(s,x(s-\tau))dW(s).$$

From Theorem 3.5 and Theorem 4.3, we infer that γ is a self-mapping from $\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ to itself. In order to demonstrate this theorem, it is sufficient to show that γ is a contraction mapping. Let $x,y\in\mathbf{PAP}(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}),\mu,\nu)$ and $t\in\mathbb{R}$

$$\begin{aligned}
\mathbb{E}\parallel(\gamma x)(t)-(\gamma y)(t)\parallel^p &= \mathbb{E}\parallel\int_{-\infty}^{t}T(t-s)f(s,x(s-\tau))ds+\int_{-\infty}^{t}T(t-s)\psi(s,x(s-\tau))dW(s)\\
&- \int_{-\infty}^{t}T(t-s)f(s,y(s-\tau))ds-\int_{-\infty}^{t}T(t-s)\psi(s,y(s-\tau))dW(s)\parallel^p\\
&= \mathbb{E}\parallel\int_{-\infty}^{t}T(t-s)[f(s,x(s-\tau))-f(s,y(s-\tau))]ds\\
&+ \int_{-\infty}^{t}T(t-s)[\psi(s,x(s-\tau))-\psi(s,y(s-\tau))]dW(s)\parallel^p\\
&\leqslant 2^{p-1}\mathbb{E}\Big(\parallel\int_{-\infty}^{t}T(t-s)[f(s,x(s-\tau))-f(s,y(s-\tau))]ds\parallel^p\Big)\\
&+ 2^{p-1}\mathbb{E}\Big(\parallel\int_{-\infty}^{t}T(t-s)[\psi(s,x(s-\tau))-\psi(s,y(s-\tau))]dW(s)\parallel^p\Big)\\
&= 2^{p-1}(\gamma_1+\gamma_2).
\end{aligned}$$

Firstly, we evaluate γ_1. From Holder's inequality and the Lipschitz conditions, we have

$$\begin{aligned}
\gamma_1 &= \mathbb{E} \parallel \int_{-\infty}^{t} T(t-s)[f(s,x(s-\tau)) - f(s,y(s-\tau))]ds \parallel^p \\
&\leqslant \Big(\int_{-\infty}^{t} \parallel T(t-s) \parallel ds \Big)^{p-1} \int_{-\infty} \parallel T(t-s) \parallel .\mathbb{E} \parallel f(s,x(s-\tau)) - f(s,y(s-\tau)) \parallel^p ds \\
&\leqslant \frac{K^p}{\omega^{p-1}} \int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E} \parallel f(s,x(s-\tau)) - f(s,y(s-\tau)) \parallel^p ds \\
&\leqslant \frac{K^p}{\omega^{p-1}}.L. \int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E} \parallel x(s-\tau) - y(s-\tau) \parallel^p ds \\
&\leqslant \frac{K^p}{\omega^{p-1}}.L. \sup_{t\in\mathbb{R}} \mathbb{E} \parallel x(t-\tau) - y(t-\tau) \parallel^p \int_{-\infty}^{t} e^{-\omega(t-s)} ds \\
&\leqslant \frac{K^p}{\omega^{p}}.L \sup_{t\in\mathbb{R}} \mathbb{E} \parallel x(t-\tau) - y(t-\tau) \parallel^p .
\end{aligned}$$

Secondly, using Lemma 4.2, Holder theorem and the Lipschitz condition, we get that

$$\begin{aligned}
\gamma_2 &= \mathbb{E}\Big(\parallel \int_{-\infty}^{t} T(t-s)[\psi(s,x(s-\tau)) - \psi(s,y(s-\tau))]dW(s) \parallel^p \Big) \\
&\leqslant C_p \mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)[\psi(s,x(s-\tau)) - \psi(s,y(s-\tau))] \parallel^2 ds \Big)^{\frac{p}{2}} \\
&\leqslant C_p \Big(\int_{-\infty}^{t} \parallel T(t-s) \parallel^2 ds \Big)^{\frac{p-2}{2}} \times \int_{-\infty}^{t} \parallel T(t-s) \parallel^2 .\mathbb{E} \parallel \psi(s,x(s-\tau)) - \psi(s,y(s-\tau)) \parallel^p ds \\
&\leqslant C_p K^{p-2} \Big(\int_{-\infty}^{t} e^{-2\omega(t-s)} ds \Big)^{\frac{p-2}{2}} \times K^2 \int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E} \parallel \psi(s,x(s-\tau)) - \psi(s,y(s-\tau)) \parallel^p ds \\
&\leqslant C_p. \frac{K^p}{(2\omega)^{p/2}}.L' \sup_{t\in\mathbb{R}} \mathbb{E} \parallel x(t-\tau) - y(t-\tau) \parallel^p .
\end{aligned}$$

Therefore,

$$\mathbb{E} \parallel (\gamma x)(t) - (\gamma y)(t) \parallel^p \leqslant \Big(\frac{2^{p-1}K^p L}{\omega^p} + \frac{2^{p-1}K^p C_p L'}{(2\omega)^{p/2}} \Big) \parallel x - y \parallel_\infty^p .$$

Actually in the case $p = 2$, by the same method as above and from Ito's isometry identity, we get that

$$\begin{aligned}
\mathbb{E} \parallel (\gamma x)(t) - (\gamma y)(t) \parallel^2 &\leqslant 2\mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)[f(s,x(s-\tau)) - f(s,y(s-\tau))] \parallel ds \Big)^2 \\
&+ 2\mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)[\psi(s,x(s-\tau)) - \psi(s,y(s-\tau))]dW(s) \parallel \Big)^2 \\
&\leqslant 2\mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)[f(s,x(s-\tau)) - f(s,y(s-\tau))] \parallel ds \Big)^2 \\
&+ 2\mathbb{E}\Big(\int_{-\infty}^{t} \parallel T(t-s)[\psi(s,x(s-\tau)) - \psi(s,y(s-\tau))] \parallel^2 ds \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\mathbb{E}\Big(\int_{-\infty}^{t} \| T(t-s) \|\| f(s,x(s-\tau)) - f(s,y(s-\tau)) \| ds\Big)^2 \\
&+ 2\mathbb{E}\Big(\int_{-\infty}^{t} \| T(t-s) \|^2 \| \psi(s,x(s-\tau)) - \psi(s,y(s-\tau)) \|^2 ds \\
&\leqslant 2K^2\Big(\int_{-\infty}^{t} e^{-\omega(t-s)} ds\Big)\Big(\int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E} \| f(s,x(s-\tau)) - f(s,y(s-\tau)) \|^2 ds\Big) \\
&+ 2K^2\Big(\int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E} \| \psi(s,x(s-\tau)) - \psi(s,y(s-\tau)) \|^2 ds\Big) \\
&\leqslant \frac{2K^2L}{\omega^2} \sup_{t\in\mathbb{R}} \mathbb{E} \| x(t-\tau) - y(t-\tau) \|^2 + \frac{K^2L'}{\omega} \sup_{t\in\mathbb{R}} \mathbb{E} \| x(t-\tau) - y(t-\tau) \|^2 \\
&\leqslant \Big(\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega}\Big) \| x - y \|_\infty^2 .
\end{aligned}
$$

If

$$\Big(\frac{2^{p-1}K^pL}{\omega^p} + \frac{2^{p-1}K^pC_pL'}{(2\omega)^{p/2}}\Big) < 1,$$

and

$$\Big(\frac{2K^2L}{\omega^2} + \frac{K^2L'}{\omega}\Big) < 1,$$

then, γ is a contraction mapping in the Banach space $\mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}), \mu, \nu)$. Therefore, by the Banach fixed point theorem, we deduce that Equation (1) has a unique (μ,ν)-pseudo almost periodic mild solution in p-th mean. □

5. Stability of (μ, ν)-pseudo almost periodic solutions

In this section, we will establish the stability of the solution for the stochastic evolution equation (1).

Theorem 5.1. *Suppose that all the conditions of theorem 4.4 hold. If*

$$\frac{3^{p-1}K^pL}{\omega^p} + \frac{3^{p-1}K^pC_pL'}{(2\omega)^{p/2}} < 1.$$

Then the (μ,ν)-pseudo almost periodic mild solution $x^*(t)$ on $\mathbb{R}$ of equation (1) *is globally exponentially stable in the p-th mean sense.*

To prove this theorem, we use same steps in the proof of Theorem 6.2 [11].

Corollary 5.2. *Suppose that $p = 2$ and all the conditions of Theorem 4.4 hold. If*

$$\frac{3K^2L}{\omega^2} + \frac{3K^2L'}{2\omega} < 1,$$

then the square-mean (μ,ν)-pseudo almost periodic mild solution $x^*(t)$ on $\mathbb{R}$ of Equation (1) *is globally exponentially stable.*

6. Example

We consider the following one-dimensional heat equation

$$
\begin{cases}
dv(t,x) = \frac{\partial^2}{\partial x^2} v(t,x)dt + f(t,v(t,x))dt + \psi(t,v(t,x))dW(t), \\
(t,x) \in \mathbb{R} \times (0,1), \\
u(t,0) = u(t,1) = 0 \ for \ t \in \mathbb{R}.
\end{cases} \tag{13}
$$

Suppose that μ is a positive measure, where its Radon-Nikodym derivative is

$$\rho(t) = \begin{cases} exp(t) & if \;\; t \leqslant 0 \\ 1 & if \;\; t > 0, \end{cases} \tag{14}$$

and ν is the Lebesgue measure. Then from [3] μ and ν satisfy $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$.
In order to write Eq.13 on the abstract from (1), we consider the linear operator

$$A \;:\; D(A) \subset \mathcal{L}^2(0,1) \to \mathcal{L}^2(0,1),$$

given by

$$D(A) = \mathcal{H}^2(0,1) \cap \mathcal{H}^1_0(0,1), \; and \; Av = v' \; for \; v \in D(A).$$

It is well known that A generates a C_0 semi-group $(T(t))_{t\geq 0}$ such that $\| T(t) \| \leqslant e^{-\omega t}$ for $t, \omega \geq 0$. Let

$$f(t,v) = (\sin t + \sin 2\pi \sqrt{2}t)v + e^{-t^2+a}(cosv + sinv), \;\; for \; a > 0$$

and

$$\theta(t,v) = (\sin 2t + \sin t)v + \sqrt{2}e^{-|t|}cosv.$$

We have

$$\left[(\sin t + \sin 2\pi \sqrt{2}t)v + e^{-t^2+a}(\cos v + \sin v)\right] \in \mathbf{PAP}(\mathbb{R} \times \mathcal{L}^p(\Omega, \mathcal{L}^2(0,1)), \mathcal{L}^p(\Omega, \mathcal{L}^2(0,1)), \mu, \nu)$$

where $(\sin t + \sin 2\pi \sqrt{2}t)v$ is almost periodic and $e^{-t^2+a}(\cos v + \sin v)$ is (μ,ν)-ergodic, since

$$\begin{aligned} \frac{1}{\nu([-r,r])} \int_{-r}^{0} \mathbb{E} \| e^{-t^2+a}(\cos v + \sin v) \|^p \, d\mu(t) &\leqslant 2^{p-1} \frac{e^{ap}}{r} \int_{-r}^{0} e^{-pt^2}.e^t dt \\ &= 2^{p-1} \frac{e^{ap+\frac{1}{4p}}}{r} \int_{-r}^{0} e^{-p(t-\frac{1}{2p})^2} dt \to 0 \; as \; r \to +\infty. \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\nu([-r,r])} \int_{0}^{r} \mathbb{E} \| e^{-t^2+a}(\cos v + \sin v) \|^p \, d\mu(t) &\leqslant 2^{p-1} \frac{e^{ap}}{r} \int_{0}^{r} e^{-pt^2} dt \\ &= 2^{p-2} \frac{e^{ap}}{r} \int_{-r}^{r} e^{-pt^2} dt \to 0 \; as \; r \to +\infty. \end{aligned}$$

Consequently $(t,v) \longmapsto f(t,v) \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathcal{L}^2(0,1))\mu, \nu)$. By the same arguments performed above and from [11], we deduce that

$$(t,v) \longmapsto \theta(t,v) = (\sin 2t + \sin t)v + \sqrt{2}e^{-|t|}cosv \in \mathbf{PAP}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathcal{L}^2(0,1))\mu, \nu).$$

It is easy to verify that f and θ satisfy the Lipschitz conditions in Theorem 4.4, with $K = 1$, $L = 2^p(1 + e^a)^p$ and $L' = (2 + \sqrt{2})^p$.
We conclude by Theorem 4.4, that the equation (13) has a unique (μ,ν)-pseudo almost periodic mild solution in p-th mean sense. □

References

[1] S. Abbas, Pseudo almost periodic solution of stochastic functional differential equations, Int. J. Evol. Equat. 5(2011)1–13.
[2] PH. Bezandry and T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, Appl. Anal. 86(7)(2007)819–827.
[3] J. Blot, P. Cieutat and K. Ezzinbi, New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications. Applicable Analysis. 92(3) (2013)493–526.

[4] PH. Bezandry, Existence of almost periodic solutions for semilinear stochastic evolution equations driven by fractional Brownian motion, Electron. J Differential Equations. 156(2012)1-21.
[5] PH. Bezandry and T. Diagana, Almost Periodic Stochastic Processes. Springer, NewYork (2011).
[6] JF. Cao, QG. Yang and ZT. Huang, On almost periodic mild solutions for stochastic functional differential equations, Nonlinear Anal. RWA. 13(1)(2012)275–286.
[7] F. Chérif, Quadratic-mean pseudo almost periodic solutions to some stochastic differential equations in a Hilbert space, J Appl Math Comput. 40(2012)427–443.
[8] F. Chérif and M. Miraoui, New results for a Lasota–Wazewska model. International Journal of Biomathematics. 12(2)(2019) 1950019.
[9] T. Diagana, Weighted pseudo almost periodic solutions to some differential equations, Nonlinear Analysis: Theory, Methods and Applications, 68(2008)2250–2260.
[10] T. Diagana, K. Ezzinbi, K and M. Miraoui, Pseudo almost periodic and pseudo almost automorphic solutions to some evolution equations involving theoretical measure theory, CUBO A Mathematical Journal, 16(2014)01–31.
[11] MA. Diop, K. Ezzinbi and MM. Mbaye, Existence and global attractiveness of a pseudo almost periodic solution in p-th mean sense for stochastic evolution equation driven by a fractional Brownian motion, Stochastics. 87(2015)1061–1093.
[12] C. Jing and R. Wenping, Existence and stability of $\mu-$pseudo almost automorphic solutions for stochastic evolution equations. Front. Math. China. doi.org/10.1007/s11464-019-0754-z, (2019).
[13] M. Miraoui, Existence of μ-pseudo almost periodic solutions to some evolution equations, Mathematical Methods in the Applied Sciences. 40(13)(2017)4716–4726.
[14] M. Miraoui, *μ-pseudo almost automorphic solutions for some differential equations with reflection of the argument*, Numerical Functional Analysis and Optimization. 38(2017) 371-394.
[15] M. Miraoui and N. Yaakobi, *Measure pseudo almost periodic solutions of shunting inhibitory cellular neural networks with mixed delays*. Numerical Functional Analysis and Optimization, 40(5)(2019)571–585.
[16] GM. N'Guérékata, *Almost automorphic and almost periodic functions in abstract spaces*. Kluwer Academic Publishers, New-York, 2001.
[17] J. Seidler and Da Prato-Zabczyk's, Maximal inequality revisited I, Math Bohem. 118(1993)67–106.
[18] CY. Zhang, Pseudo almost periodic solutions of some differential equations I. Journal of Mathematical Analysis and Applications, 151(1994)62–76.
[19] CY. Zhang, Pseudo almost periodic solutions of some differential equations II. Journal of Mathematical Analysis and Applications, 192(1995)543–561.