



Dynamics of a Single Predator Multiple Prey Model With Stochastic Perturbation and Seasonal Variation

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Abstract. Considering various factors are stochastic rather than deterministic in the evolution of populations growth, in this paper, we propose a single predator multiple prey stochastic model with seasonal variation. By using the method of solving an explicit solution, the existence of global positive solution of this model are obtained. The method is more convenient than Lyapunov analysis method for some population models. Moreover, the stochastically ultimate boundedness are considered by using the comparison theorem of stochastic differential equation. Further, some sufficient conditions for the extinction and strong persistence in the mean of populations are discussed, respectively. In addition, by constructing some suitable Lyapunov functions, we show that this model admits at least one periodic solution. Finally, numerical simulations clearly illustrate the main theoretical results and the effects of white noise and seasonal variation for the persistence and extinction of populations.

1. Introduction

Relationship between predator and prey is one of the most important relationships between biology in nature due to its universal existence. Qualitative description of this biology phenomenon has the significant practical meaning. Among them, mathematicians made a notable contribution in this neighborhood primarily through used mathematical modelling. Since the initial and simplest predator-prey mathematical model was proposed by Lotka and Volterra [1, 2], many scholars established various mathematical models to analyze the relationship and evolution of populations (see [3–5] and the references therein). More research results also can be found in the monographs of Chen [6] and Murray [7].

Consider that the complexity of predator-prey relationships between species in the real world, several researchers have showed clearly that two-species predator-prey models can't describe the real world accurately, the relationship between predators and preys can only be shown by models with three or more species. Therefore, there are some scholars have discussed the dynamics of multi-species of predator-prey model (for example, see [8, 9] and the references therein). Particularly, Walid et al. [10] introduced a three species predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response,

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and established the sufficient conditions of global stability of the positive steady states. Blé et al. [11] proposed a tritrophic food chain model with Holling III and Holling II functional response for the predator and the top-predator, respectively, and proved that this model has stable periodic orbit for adequate values of its parameters. Wang [12] introduced a n -species competitive predator-prey model with Holling-type II functional response, and established some easily verifiable criteria on the global existence of multiple positive periodic solutions by using the Gaines and Mawhin's coincidence degree theory.

Throughout the previous literatures, it is not hard to find that most of the multi-species models are mainly focussed on the food chain relationship, while less work has been undertaken looking at a single predator multiple prey model. In fact, this phenomenon exists widely in the nature, for example, most of the predators (senior carnivore, such as, tiger, lion) feed on variety preys in different types of consumption ways. With this in mind, Freedman [13] introduced a single predator two preys model which takes the form of

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[b_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}y(t)], \\ \frac{dx_2(t)}{dt} = x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2(t) - a_{23}y(t)], \\ \frac{dy(t)}{dt} = y(t)[b_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}y(t)], \end{cases} \quad (1)$$

where $x_1(t)$, $x_2(t)$ and $y(t)$ are the populations sizes of prey-1, prey-2 and predator at time t , respectively; $b_1 > 0$ and $b_2 > 0$ denote the intrinsic growth rate of prey-1 and prey-2, respectively; $b_3 > 0$ is the intrinsic growth rate of predator; $a_{11} > 0$, $a_{22} > 0$ and $a_{33} > 0$ are the interspecific competition coefficients of prey-1, prey-2 and predator, respectively; $a_{13} > 0$ and $a_{23} > 0$ are capture rates, $a_{31} > 0$ and $a_{32} > 0$ measure the efficiencies of food conversion, a_{12} and a_{21} are the competition coefficients between the prey-1 and prey-2.

In the recent, biologists noted that biological populations are always subject to white noises. Therefore, deterministic models cannot be described well with random fluctuations factors of natural phenomena, and it is important to study how the white noises affect the rule of evolution of populations. For all this, stochastic population models have been studied by some scholars recently (see [14–18] and the references therein). For instance, Rudnicki et al. [19] introduced a stochastic prey-predator model, and pointed out the differences between the deterministic and stochastic models. Costa et al. [20] put forward an individual-based model of the community that taken into account both prey and predator phenotypes, and proved the existence of a unique globally asymptotically stable equilibrium under specific conditions on the interaction among prey individuals. Zhao et al. [21] proposed a stochastic Leslie-Gower predator-prey model with randomized intrinsic growth rate, and obtained some sufficient conditions for the permanence in mean and almost sure extinction of this model.

In addition, consider the effect of seasonal variation on populations, Zu et al. [22] introduced a stochastic Lotka-Volterra prey-predator model with seasonal variation, and studied the extinction, persistence and existence of positive periodic solution. Zuo et al. [23] investigated a stochastic periodic Holling-Tanner predator-prey model with impulsive effects, and given the sufficient condition for the existence of positive T -periodic solution by choosing a suitable Lyapunov function. Meng et al. [24] studied a non-autonomous Lotka-Volterra almost periodic predator-prey dispersal model, and proved the uniformly persistent of population by using the comparison theorem and fundamental theory of delay differential equation. More related stochastic non-autonomous or periodic population models can be found in [25, 26] and the references therein.

For all the reasons that have previously been discussed, in this paper, we propose a stochastic predator-prey model with one predator and two preys. The main motivation is to discuss how white noise and the seasonal variation influence on the extinction and persistence of populations. The organization of this paper is as follows. In the next section, we present some lemmas and definitions, which are necessary for the future discussion. The global, nonnegative and stochastic boundedness of solution of this model are considered in Section 3, and the sufficient conditions for the extinction and strong persistence in the mean of populations are obtained in Section 4, respectively. In Section 5, we focus our attention on the existence of positive periodic solution. The numerical simulations are carried out in Section 6 to verify and extend

our theoretical result, and a brief summary of the main results are provided in the last section.

2. Model formulation and preliminaries

As an extension of model (1) and to better depict the actual phenomenon, we assume that all the parameters of model (1) and $\sigma_i(t)$ are positive θ -periodic continuous functions, where, $\sigma_i(t) > 0$ ($i = 1, 2, 3$) represents the intensities of the white noise, θ is a positive constant. In addition, we also suppose that the growth rates of populations is stochastically perturbed, that is

$$b_i(t) \rightarrow b_i(t) + \sigma_i(t)dB_i(t), \quad i = 1, 2, 3,$$

where $B_i(t)$ are independent standard Brownian motions with $B_i(0) = 0$. Hence the stochastic version corresponding to model (1) takes the following form

$$\begin{cases} dx_1(t) = x_1(t)[b_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - a_{13}(t)y(t)]dt + \sigma_1(t)x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)[b_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - a_{23}(t)y(t)]dt + \sigma_2(t)x_2(t)dB_2(t), \\ dy(t) = y(t)[b_3(t) + a_{31}(t)x_1(t) + a_{32}(t)x_2(t) - a_{33}(t)y(t)]dt + \sigma_3(t)y(t)dB_3(t). \end{cases} \tag{2}$$

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and \mathcal{F}_0 contains all \mathbb{P} -null sets). Set $\mathbb{R}_+^n := \{(x_1, x_2, \dots, x_n) : x_i \geq 0, i = 1, 2, \dots, n\}$.

For convenience sake, we introduce, firstly, the following definitions and lemmas.

Definition 2.1 ([27]). The population $x_1(t)$ is said to be extinct if $\lim_{t \rightarrow \infty} x_1(t) = 0$ a.s.; the population $x_1(t)$ is said to be strongly persistent in the mean if $\liminf_{t \rightarrow \infty} t^{-1} \int_0^t x_1(s) ds > 0$ a.s.

Definition 2.2 ([28]). A stochastic process $\xi(t) = \xi(t, \theta)$ ($-\infty < t < +\infty$) is said to be periodic with period θ if for every finite sequence of numbers t_1, t_2, \dots, t_n , the joint distribution of random variables $\xi(t_1 + k\theta), \dots, \xi(t_n + k\theta)$ is independent of k ($k = 1, 2, \dots$).

Remark 2.3. In Ref. [28], it is verified that a Markov process $X_1(t)$ is θ -periodic if and only if its transition probability function is θ -periodic and the function $P_0(t, A) = P\{X_1(t) \in A\}$ satisfies the equation $P_0(s, A) = \int_{\mathbb{R}^d} P_0(s, dx)P(s, x, s + \theta, A) \equiv P_0(s + \theta, A)$, where $A \in \mathcal{B}$ and \mathcal{B} is σ -algebra.

Nextly, consider the following equation

$$X_1(t) = X(t_0) + \int_{t_0}^t b(s, X(s)) ds + \sum_{i=1}^k \int_{t_0}^t \sigma_i(s, X(s))dB_i(s) \quad X \in \mathbb{R}^d, \tag{3}$$

where vectors $b(s, z)$, $\sigma_1(s, z)$ and $\sigma_k(s, z)$ ($s \in (t_0, T)$, $z \in \mathbb{R}^d$) are continuous functions of (s, z) . We need the following lemma on the existence of θ -periodic, which was proposed in Ref. [28].

Lemma 2.4 ([28]). Suppose that the coefficients of equation (3) are θ -periodic in t and satisfy conditions

$$\begin{aligned} |b(s, x) - b(s, y)| + \sum_{i=1}^k |\sigma_i(s, x) - \sigma_i(s, y)| &\leq B|x - y|, \\ |b(s, x)| + \sum_{i=1}^k |\sigma_i(s, x)| &\leq B(1 + |x|), \end{aligned} \tag{4}$$

in every cylinder $I \times U$, where B is a constant; and suppose further that there exists a C^2 -function $V(t, x)$ which is θ -periodic in t and satisfies the following conditions

$$\inf_{|x|>\rho} V(t, x) \rightarrow \infty \quad \text{as } \rho \rightarrow \infty \tag{5}$$

and

$$\mathcal{L}V(t, x) \leq -1, \quad \text{outside of some compact set,} \tag{6}$$

where the operator \mathcal{L} is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^l b_i(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^l a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij} = \sum_{r=1}^l \sigma_r^i(t, x) \sigma_r^j(t, x).$$

Then there exists a solution of equation (3) which is a θ -periodic Markov process.

Suppose that $f(t)$ is an integrable function on $[0, \infty)$, and define $\langle f \rangle_t = t^{-1} \int_0^t f(s) ds$. If $f(t)$ is a bounded function on $[0, \infty)$, define $f^u = \sup_{t \in [0, \infty)} f(t)$ and $f^l = \inf_{t \in [0, \infty)} f(t)$. Since all the parameters of model (2) are positive θ -periodic continuous functions, the following results holds

$$0 < a_{ij}^l \leq a_{ij}^u < \infty, \quad 0 < b_i^l \leq b_i^u < \infty \quad \text{and} \quad 0 < \sigma_i^l \leq \sigma_i^u < \infty, \quad i, j = 1, 2, 3.$$

Lemma 2.5 (Lemma in Liu et al. [29]). Suppose that $x(t) \in C(\Omega \times [0, \infty), \mathbb{R}_+)$,

(i) if there exist three constants $T > 0, \lambda_0 > 0$ and $\lambda \geq 0$ such that for all $t \geq T$

$$\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t),$$

where β_i are constants ($1 \leq i \leq n$), then $\limsup_{t \rightarrow \infty} \langle x(s) \rangle_t \leq \lambda / \lambda_0$ a.s.;

(ii) if there exist three constants $T > 0, \lambda \geq 0$ and $\lambda_0 > 0$ such that for all $t \geq T$

$$\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t),$$

where β_i are constants ($1 \leq i \leq n$), then $\liminf_{t \rightarrow \infty} \langle x(s) \rangle_t \geq \lambda / \lambda_0$ a.s.

3. Global positive solutions and stochastic boundedness

Let $X(t) = (x_1(t), x_2(t), y(t))$, $h_i(t) = b_i(t) - \sigma_i^2(t)/2$ and $g_i(t) = \int_0^t \sigma_i(s) dB_i(s)$ ($i = 1, 2, 3$). On the global, nonnegative and stochastic ultimately bounded of model (2), we have the following result.

Theorem 3.1. For any given initial value $X(0) \in \mathbb{R}_+^3$, there is a unique positive solution $X(t)$ of model (2) for all $t \geq 0$ and the solution will remain in \mathbb{R}_+^3 with probability one.

Proof. Since coefficients of model (2) are locally Lipschitz continuous, then for any given the initial value $X(0) \in \mathbb{R}_+^3$, there is a unique local solution $X(t)$ on $t \in [0, \tau_e)$, where τ_e is t the explosion time (see [28]). To show this solution is global and positive, we only need to show that $\tau_e = \infty$. For $t \in [0, \tau_e)$, we can directly calculate the explicit solution of model (2) with the initial value $X(0)$ as follows

$$\begin{aligned} x_1(t) &= \frac{\exp \left\{ \int_0^t h_1(s) ds - \int_0^t a_{12}(s)x_2(s) ds - \int_0^t a_{13}(s)y(s) ds + g_1(t) \right\}}{\frac{1}{x(0)} + \int_0^t a_{11}(s) \exp \left\{ \int_0^s h_1(\tau) d\tau - \int_0^s a_{12}(\tau)x_2(\tau) d\tau - \int_0^s a_{13}(\tau)y(\tau) d\tau + g_1(s) \right\} ds}, \\ x_2(t) &= \frac{\exp \left\{ \int_0^t h_2(s) ds - \int_0^t a_{21}(s)x_1(s) ds - \int_0^t a_{23}(s)y(s) ds + g_2(t) \right\}}{\frac{1}{y(0)} + \int_0^t a_{22}(s) \exp \left\{ \int_0^s h_2(\tau) d\tau - \int_0^s a_{21}(\tau)x_1(\tau) d\tau - \int_0^s a_{23}(\tau)y(\tau) d\tau + g_2(s) \right\} ds}, \\ y(t) &= \frac{\exp \left\{ \int_0^t h_3(s) ds + \int_0^t a_{31}(s)x_1(s) ds + \int_0^t a_{32}(s)x_2(s) ds + g_3(t) \right\}}{\frac{1}{z(0)} + \int_0^t a_{22}(s) \exp \left\{ \int_0^s h_3(\tau) d\tau + \int_0^s a_{31}(\tau)x_1(\tau) d\tau + \int_0^s a_{32}(\tau)x_2(\tau) d\tau + g_3(s) \right\} ds}. \end{aligned}$$

Note that $x_1(t) > 0, x_2(t) > 0$ and $y(t) > 0$ for $t \in [0, \tau_e)$, therefore, we only need to show that $\tau_e = \infty$. For all this, consider the following comparison equation

$$\begin{cases} d\bar{x}_1(t) = \bar{x}_1(t)[b_1(t) - a_{11}(t)\bar{x}_1(t)]dt + \sigma_1(t)\bar{x}_1(t)dB_1(t), \\ d\bar{x}_2(t) = \bar{x}_2(t)[b_2(t) - a_{22}(t)\bar{x}_2(t)]dt + \sigma_2(t)\bar{x}_2(t)dB_2(t), \\ d\bar{y}(t) = \bar{y}(t)[b_3(t) + a_{31}(t)\bar{x}_1(t) + a_{32}(t)\bar{x}_2(t) - a_{33}\bar{y}(t)]dt + \sigma_3(t)\bar{y}(t)dB_3(t), \end{cases} \tag{7}$$

with the initial value $(\bar{x}_1(0), \bar{x}_2(0), \bar{y}(0)) = X(0)$. Obviously, the explicit solution of model (7) can be given

$$\begin{aligned} \bar{x}_1(t) &= \frac{\exp\left\{\int_0^t h_1(s) ds + g_1(t)\right\}}{\frac{1}{x(0)} + \int_0^t a_{11}(s) \exp\left\{\int_0^s h_1(\tau) d\tau + g_1(s)\right\} ds}, \\ \bar{x}_2(t) &= \frac{\exp\left\{\int_0^t h_2(s) ds + g_2(t)\right\}}{\frac{1}{y(0)} + \int_0^t a_{22}(s) \exp\left\{\int_0^s h_2(\tau) d\tau + g_2(s)\right\} ds}, \\ \bar{y}(t) &= \frac{\exp\left\{\int_0^t h_3(s) ds + \int_0^t a_{31}(s)\bar{x}_1(s) ds + \int_0^t a_{23}(s)\bar{x}_2(s) ds + g_3(t)\right\}}{\frac{1}{y(0)} + \int_0^t a_{22}(s) \exp\left\{\int_0^s h_3(\tau) d\tau + \int_0^\tau a_{31}(\tau)\bar{x}_1(\tau) d\tau + \int_0^s a_{23}(\tau)\bar{x}_2(\tau) d\tau + g_3(s)\right\} ds}. \end{aligned}$$

By the comparison theorem of stochastic differential equation, we have $x_1(t) \leq \bar{x}_1(t), x_2(t) \leq \bar{x}_2(t)$ and $y(t) \leq \bar{y}(t)$. Since $b_i(t), a_{ij}(t),$ and $\sigma_i(t)$ ($i, j = 1, 2, 3$) are positive bounded θ -periodic functions. Therefore, $x_1(t), x_2(t)$ and $y(t)$ will not explode in finite time, then $\tau_e = \infty$. This completes the proof. \square

The following Theorem 3.2 is on the stochastically ultimate boundedness of solution of model (2).

Theorem 3.2. *For any $X(0) \in \mathbb{R}_+^3$, then solution $X(t)$ of model (2) with the initial value $X(0)$, is stochastic ultimately bounded. That is, for any $\varepsilon \in (0, 1)$, there is a positive constant $M = M(\varepsilon)$ such that solution $X(t)$ of model (2) has the property*

$$\limsup_{t \rightarrow \infty} \mathbb{P}\{|X(t)| > M\} < \varepsilon.$$

Proof. Firstly, we prove that the follows inequalities are true.

$$E[x_1^p(t)] \leq K_1(p), \quad E[x_2^p(t)] \leq K_2(p), \quad E[y^p(t)] \leq K_3(p), \quad t \in [0, +\infty),$$

where $p > 1$. Applying Itô's formula, one can derive

$$\begin{aligned} dx_1^p &= px_1^{p-1} \left[b_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2 - a_{13}y + \frac{1}{2}(p-1)\sigma_1^2(t) \right] dt + px_1^{p-1}\sigma_1(t)dB_1(t) \\ &\leq px_1^{p-1} \left[b_1^u - a_{11}^l x_1 + \frac{1}{2}p(\sigma_1^u)^2 \right] dt + px_1^{p-1}\sigma_1(t)dB_1(t), \\ dx_2^p &= px_2^{p-1} \left[b_2(t) - a_{21}(t)x_1 - a_{22}(t)x_2 - a_{23}y + \frac{1}{2}(p-1)\sigma_2^2(t) \right] dt + px_2^{p-1}\sigma_2(t)dB_2(t) \\ &\leq px_2^{p-1} \left[b_2^u - a_{22}^l x_2 + \frac{1}{2}p(\sigma_2^u)^2 \right] dt + px_2^{p-1}\sigma_2(t)dB_2(t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} dy^p &= py^{p-1} \left[b_3(t) + a_{31}(t)x_1 + a_{32}(t)x_2 - a_{33}y + \frac{1}{2}(p-1)\sigma_3^2(t) \right] dt + py^{p-1}\sigma_3(t)dB_3(t) \\ &\leq py^{p-1} \left[b_3^u + a_{31}^u x_1 + a_{32}^u x_2 - a_{33}^l y + \frac{1}{2}p(\sigma_3^u)^2 \right] dt + py^{p-1}\sigma_3(t)dB_3(t). \end{aligned} \tag{9}$$

Taking the exception of both sides on the first inequality of (8) yields that

$$\frac{dE[x_1^p(t)]}{dt} \leq pE[x_1^{p-1}(t)] \left\{ \left[b_1^u + \frac{1}{2}p(\sigma_1^u)^2 \right] - a_{11}^l (E[x_1^p(t)])^{\frac{1}{p}} \right\}.$$

Consider the following comparison equation

$$\frac{dW_1(t)}{dt} = pW_1(t) \left\{ \left[b_1^u + \frac{1}{2}p(\sigma_1^u)^2 \right] - a_{11}^l W_1(t)^{\frac{1}{p}} \right\}.$$

It is easy to see that the above equation is a Bernoulli equation, then one can get

$$[W_1(t)]^{-\frac{1}{p}} = \left(\int_0^t a_{11}^l \exp \left\{ \left[b_1^u + \frac{1}{2}p(\sigma_1^u)^2 \right] s \right\} ds + W_1(0) \right) \exp \left\{ \left[-b_1^u - \frac{1}{2}p(\sigma_1^u)^2 \right] t \right\}.$$

In view of L'Hospital's rule leads to

$$\lim_{t \rightarrow \infty} W_1(t) = \left(\frac{b_1^u + \frac{1}{2}p(\sigma_1^u)^2}{a_{11}^l} \right)^p.$$

Therefore, by the comparison theorem, we have

$$\limsup_{t \rightarrow \infty} E[x_1^p(t)] \leq \lim_{t \rightarrow \infty} W_1(t) = \left(\frac{b_1^u + \frac{1}{2}p(\sigma_1^u)^2}{a_{11}^l} \right)^p := \bar{K}_1(p).$$

Further, for any given $\varepsilon > 0$, there exists a constant $T > 0$ such that for all $t > T$,

$$E[x_1^p(t)] \leq \bar{K}_1(p) + \varepsilon.$$

Together with the continuity of $E[x_1(t)^p]$, there exists $\widehat{K}_1(p) > 0$ such that $E[x_1^p(t)] < \widehat{K}_2(p)$ for $t \leq T$. Let

$$K_1(p) = \max \{ \bar{K}_1(p), \widehat{K}_1(p) \},$$

then, for all $t \in \mathbb{R}_+$, $E[x_1(t)^p] \leq K_1(p)$. Similarly, we have $E[y^p(t)] \leq K_2(p)$ for all $t \in \mathbb{R}_+$.

Next, we claim that $E[y^p(t)] \leq K_3(p)$ for all $t \in \mathbb{R}_+$. Taking exception on (9), we get

$$\begin{aligned} \frac{dE[y^p(t)]}{dt} &\leq pE[y^p(t)] \left\{ \left[b_3^u + a_{31}^u (E[x^p(t)])^{\frac{1}{p}} + a_{32}^u (E[y^p(t)])^{\frac{1}{p}} + \frac{1}{2}p(\sigma_3^u)^2 \right] - a_{33}^l (E[y^p(t)])^{\frac{1}{p}} \right\} \\ &\leq pE[y^p(t)] \left\{ \left[b_3^u + a_{31}^u [K_1(p)]^{\frac{1}{p}} + a_{32}^u [K_2(p)]^{\frac{1}{p}} + \frac{1}{2}p(\sigma_3^u)^2 \right] - a_{33}^l (E[y^p(t)])^{\frac{1}{p}} \right\}. \end{aligned}$$

By the same method as above, we can show that $E[y^p(t)] \leq K_3(p)$ for all $t \in \mathbb{R}_+$.

According to above discussion and analysis, one can see that $E[|X(t)|^p] \leq K(p)$, where $K(p) = 3^{p/2}[K_1(p) + K_2(p) + K_3(p)]$. Further, for arbitrary $\varepsilon > 0$, set $M = (K(p)/\varepsilon)^{1/p}$, then by virtue of Chebyshev's inequality, one can see that

$$\mathbb{P}\{|X(t)| > M\} < E[|X(t)|^p]M^{-p} \leq K(p) \left(\frac{K(p)}{\varepsilon} \right)^{-1} = \varepsilon.$$

The proof is complete. \square

4. Persistence and extinction

In this section, we focus on the extinction and persistence of model (2).

Theorem 4.1. For any $X(0) \in \mathbb{R}_+^3$, then the solution $X_1(t)$ of model (2) with the initial value $X(0)$ obeys

$$\limsup_{t \rightarrow \infty} \frac{\ln x_1(t)}{t} \leq 0, \quad \limsup_{t \rightarrow \infty} \frac{\ln x_2(t)}{t} \leq 0, \quad \limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq 0, \quad a.s.$$

Proof. Consider the following comparison theorem

$$\begin{cases} d\bar{x}_1(t) = \bar{x}_1(t)[b_1(t) - a_{11}(t)\bar{x}_1(t)]dt + \sigma_1(t)\bar{x}_1(t)dB_1(t), \\ d\bar{x}_2(t) = \bar{x}_2(t)[b_2(t) - a_{22}(t)\bar{x}_2(t)]dt + \sigma_2(t)\bar{x}_2(t)dB_2(t). \end{cases}$$

In view of the comparison theorem, we can easy to get

$$x_1(t) \leq \bar{x}_1(t), \quad x_2(t) \leq \bar{x}_2(t), \quad a.s., \quad t \in [0, \infty).$$

By the same method as Lemma 3.4 in Ref. [30], it follows that

$$\limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_1(t)}{\ln t} \leq 1, \quad \limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_2(t)}{\ln t} \leq 1.$$

Moreover,

$$\limsup_{t \rightarrow \infty} \frac{\ln x_1(t)}{\ln t} \leq \limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_1(t)}{\ln t} \leq 1, \quad \limsup_{t \rightarrow \infty} \frac{\ln x_2(t)}{\ln t} \leq \limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_2(t)}{\ln t} \leq 1.$$

In addition,

$$\limsup_{t \rightarrow \infty} \frac{\ln x_1(t)(t)}{t} = \limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_1(t)}{\ln t} \frac{\ln t}{t} \leq \limsup_{t \rightarrow \infty} \frac{\ln t}{t} = 0$$

and

$$\limsup_{t \rightarrow \infty} \frac{\ln x_2(t)}{t} = \limsup_{t \rightarrow \infty} \frac{\ln \bar{x}_2(t)}{\ln t} \frac{\ln t}{t} \leq \limsup_{t \rightarrow \infty} \frac{\ln t}{t} = 0.$$

Now, we claim that $\limsup_{t \rightarrow \infty} \ln y(t)/t \leq 0$. In fact, the proof is a slight modification of that for Lemma 3.4 in Li and Mao [30] by replacing the function $V(x) = \sum_{i=1}^n x_i$ in the proof of this Lemma with $V(x) = x_1 + x_2 + \eta y$ in this paper, where, η is a positive constant satisfied

$$\eta \max \{a_{31}^u, a_{32}^u\} \leq \min \{a_{13}^l, a_{23}^l\}. \tag{10}$$

This completes the proof. \square

Now, we turn our attention to the persistence of model (2).

Theorem 4.2. *Let $(x_1(t), x_2(t), y(t))$ be solution of model (2) with the initial value $(x(0), y(0), y(0)) \in \mathbb{R}_+^3$, if $\lambda_1 > 0$, $\lambda_2 > 0$ and $\langle h_i(t) \rangle_\theta > 0$ ($i = 1, 2, 3$), then the solution of model (2) are obeys*

$$\frac{\lambda_1}{a_{11}^u} \leq \liminf_{t \rightarrow \infty} \langle x_1(s) \rangle_t \leq \limsup_{t \rightarrow \infty} \langle x_1(s) \rangle_t \leq \frac{\langle h_1(t) \rangle_\theta}{a_{11}^l},$$

$$\frac{\lambda_2}{a_{22}^u} \leq \liminf_{t \rightarrow \infty} \langle x_2(s) \rangle_t \leq \limsup_{t \rightarrow \infty} \langle x_2(s) \rangle_t \leq \frac{\langle h_2(t) \rangle_\theta}{a_{22}^l}$$

and

$$\frac{\langle h_3(t) \rangle_\theta}{a_{33}^u} \leq \liminf_{t \rightarrow \infty} \langle y(s) \rangle_t \leq \limsup_{t \rightarrow \infty} \langle y(s) \rangle_t \leq \frac{\beta}{a_{33}^l},$$

where

$$\lambda_1 = \langle h_1(t) \rangle_\theta - \frac{a_{12}^u \langle h_2(t) \rangle_\theta}{a_{22}^l} - \frac{a_{13}^u \beta}{a_{33}^l}, \quad \lambda_2 = \langle h_2(t) \rangle_\theta - \frac{a_{21}^u \langle h_1(t) \rangle_\theta}{a_{11}^l} - \frac{a_{23}^u \beta}{a_{33}^l}, \quad \beta = \langle h_3(t) \rangle_\theta + \frac{a_{31}^u \langle h_1(t) \rangle_\theta}{a_{11}^l} + \frac{a_{32}^u \langle h_2(t) \rangle_\theta}{a_{22}^l}.$$

Proof. By Itô's formula, we drive from model (2) that from

$$\begin{aligned} \ln \frac{x_1(t)}{x_1(0)} &= t \langle h_1(s) \rangle_t - t \langle a_{11}(s)x_1(s) \rangle_t - t \langle a_{12}(s)x_2(s) \rangle_t - t \langle a_{13}(s)y(s) \rangle_t + \int_0^t \sigma_1(s) dB_1(s), \\ \ln \frac{x_2(t)}{x_2(0)} &= t \langle h_2(s) \rangle_t - t \langle a_{21}(s)x_1(s) \rangle_t - t \langle a_{22}(s)x_2(s) \rangle_t - t \langle a_{23}(s)y(s) \rangle_t + \int_0^t \sigma_2(s) dB_2(s), \\ \ln \frac{y(t)}{y(0)} &= t \langle h_3(s) \rangle_t + t \langle a_{31}(s)x_1(s) \rangle_t + t \langle a_{32}(s)x_2(s) \rangle_t - t \langle a_{33}(s)y(s) \rangle_t + \int_0^t \sigma_3(s) dB_3(s). \end{aligned} \tag{11}$$

Moreover, we can get

$$\begin{aligned} \ln x_1(t) &\leq \ln x_1(0) + t\langle h_1(s) \rangle_t - a_{11}^l t\langle x_1(s) \rangle_t + \int_0^t \sigma_1(s) dB_1(s), \\ \ln x_2(t) &\leq \ln x_2(0) + t\langle h_2(s) \rangle_t - a_{22}^l t\langle x_2(s) \rangle_t + \int_0^t \sigma_2(s) dB_2(s), \\ \ln y(t) &\geq \ln y(0) + t\langle h_3(s) \rangle_t - a_{33}^u t\langle y(s) \rangle_t + \int_0^t \sigma_3(s) dB_3(s). \end{aligned}$$

From the strong law of large number for local martingales (see, Ref. [27]), one have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma_i(s) dB_i(s) \leq \lim_{t \rightarrow \infty} \frac{\sigma_i^u B_i}{t} = 0, \quad a.s. \quad i = 1, 2, 3.$$

Further, by the periodicity of $b_i(t)$ and $\sigma_i(t)$, it can be easily proved that $\lim_{t \rightarrow \infty} \langle h_i(s) \rangle_t = \langle h_i(t) \rangle_\theta$, ($i = 1, 2, 3$). Note that $\langle h_i(t) \rangle_\theta > 0$, then Lemma 2.5 yields

$$\limsup_{t \rightarrow \infty} \langle x_1(s) \rangle_t \leq \frac{\langle h_1(t) \rangle_\theta}{a_{11}^l}, \quad \limsup_{t \rightarrow \infty} \langle x_2(s) \rangle_t \leq \frac{\langle h_2(t) \rangle_\theta}{a_{22}^l}, \quad \liminf_{t \rightarrow \infty} \langle y(s) \rangle_t \geq \frac{\langle h_3(t) \rangle_\theta}{a_{33}^u}. \tag{12}$$

On the other hand, it follows from (11) that

$$\begin{aligned} \ln x_1(t) &\geq \ln x_1(0) + t\langle h_1(s) \rangle_t - a_{11}^u t\langle x_1(s) \rangle_t - a_{12}^u t\langle x_2(s) \rangle_t - a_{13}^u t\langle y(s) \rangle_t + \int_0^t \sigma_1(s) dB_1(s), \\ \ln x_2(t) &\geq \ln x_2(0) + t\langle h_2(s) \rangle_t - a_{21}^u t\langle x_1(s) \rangle_t - a_{22}^u t\langle x_2(s) \rangle_t - a_{23}^u t\langle y(s) \rangle_t + \int_0^t \sigma_2(s) dB_2(s), \\ \ln y(t) &\leq \ln y(0) + t\langle h_3(s) \rangle_t + a_{31}^u t\langle x_1(s) \rangle_t + a_{32}^u t\langle x_2(s) \rangle_t - a_{33}^l t\langle y(s) \rangle_t + \int_0^t \sigma_3(s) dB_3(s). \end{aligned}$$

In view of $\lim_{t \rightarrow \infty} \langle h_3(s) \rangle_t = \langle h_3(t) \rangle_\theta$ and (12), then for arbitrary $\varepsilon > 0$, there is a positive constant $T = T(\varepsilon)$ such that for all $t > T$

$$\langle x_1(s) \rangle_t \leq \frac{\langle h_1(t) \rangle_\theta}{a_{11}^l} + \frac{\varepsilon}{3a_{31}^u}, \quad \langle x_2(s) \rangle_t \leq \frac{\langle h_2(t) \rangle_\theta}{a_{22}^l} + \frac{\varepsilon}{3a_{32}^u}, \quad \langle h_3(s) \rangle_t \leq \langle h_3(t) \rangle_\theta + \frac{\varepsilon}{3}.$$

Then we can see that for all $t > T$

$$\ln \frac{y(t)}{y(0)} \leq t \left(\langle h_3(t) \rangle_\theta + \frac{a_{31}^u \langle h_1(t) \rangle_\theta}{a_{11}^l} + \frac{a_{32}^u \langle h_2(t) \rangle_\theta}{a_{22}^l} + \varepsilon \right) - ta_{33}^l \langle y(s) \rangle_t + \int_0^t \sigma_3(s) dB_3(s).$$

By Lemma 2.5 and according to the arbitrariness of ε , we get

$$\limsup_{t \rightarrow \infty} \langle y(s) \rangle_t \leq \frac{\langle h_3(t) \rangle_\theta + \frac{a_{31}^u \langle h_1(t) \rangle_\theta}{a_{11}^l} + \frac{a_{32}^u \langle h_2(t) \rangle_\theta}{a_{22}^l}}{a_{33}^l} = \frac{\beta}{a_{33}^l}. \tag{13}$$

On the other hand, from $\limsup_{t \rightarrow \infty} \langle x_2(s) \rangle_t \leq \langle h_2(t) \rangle_\theta / a_{22}^l$, $\lim_{t \rightarrow \infty} \langle h_1(s) \rangle_t = \langle h_1(t) \rangle_\theta$ and (13), for arbitrary $\delta > 0$, there is a positive constant $\bar{T} = \bar{T}(\delta)$ such that for all $t > \bar{T}$

$$\langle h_1(s) \rangle_t \leq \langle h_1(t) \rangle_\theta + \delta, \quad \langle x_2(s) \rangle_t \leq \frac{\langle h_2(t) \rangle_\theta}{a_{22}^l} + \frac{\delta}{2a_{12}^u}, \quad \langle y(s) \rangle_t \leq \frac{\beta}{a_{33}^l} + \frac{\delta}{2a_{13}^u}.$$

Then, for sufficiently large t , we can get

$$\ln \frac{x_1(t)}{x_1(0)} \geq t \left(\langle h_1(t) \rangle_\theta - \frac{a_{12}^u \langle h_2(t) \rangle_\theta}{a_{22}^l} - \frac{a_{13}^u \beta}{a_{33}^l} \right) - a_{11}^u t\langle x_1(s) \rangle_t + \int_0^t \sigma_1(s) dB_1(s).$$

By the Lemma 2.5 again, it yields that

$$\liminf_{t \rightarrow \infty} \langle x_1(s) \rangle_t \geq \frac{\langle h_1(t) \rangle_\theta - \frac{a_{12}^u \langle h_2(t) \rangle_\theta}{a_{22}^u} - \frac{a_{13}^u \beta}{a_{33}^u}}{a_{11}^u} = \frac{\lambda_1}{a_{11}^u}.$$

By the same method as above, it can be easily shown that

$$\liminf_{t \rightarrow \infty} \langle x_2(s) \rangle_t \geq \frac{\langle h_2(t) \rangle_\theta - \frac{a_{21}^u \langle h_1(t) \rangle_\theta}{a_{11}^u} - \frac{a_{23}^u \beta}{a_{33}^u}}{a_{22}^u} = \frac{\lambda_2}{a_{22}^u}.$$

The proof is complete. \square

On the persistence and extinction of a single species of model (2), direct application Theorem 4.2, we have the following corollary.

- Corollary 4.3.** (i) If $\langle h_1(t) \rangle_\theta < 0$ (or $\langle h_2(t) \rangle_\theta < 0$), then the population $x_1(t)$ (or $x_2(t)$) will go to extinction, a.s.;
- (ii) if $\lambda_1 > 0$ (or $\lambda_2 > 0$), then the population $x_1(t)$ (or $x_2(t)$) will be strongly persistent in the mean, a.s.;
- (iii) if $\beta < 0$, then the population $y(t)$ will go to extinction, a.s.;
- (iv) if $\langle h_3(t) \rangle_\theta > 0$, then the population $y(t)$ will be strongly persistent in the mean, a.s.

5. Existence of periodic solution

On the existence of a positive θ -periodic solution of model (2), we have the following Theorem 5.1.

Theorem 5.1. If $b_1^l > (\sigma_1^u)^2/2$, $b_2^l > (\sigma_2^u)^2/2$ and $b_3^l > (\sigma_3^u)^2$, then model (2) exists at least one positive θ -periodic solution.

Proof. Since all coefficients of model (2) are continuous bounded positive periodic functions, it is clear that the condition (4) of Lemma 2.4 holds. Now, we show the conditions (5) and (6) hold. Define function $V(t, x_1, x_2, y)$ is given as follows

$$V(t, x_1, y) = V_1(x_1, x_2, y) + \widehat{c}V_2(x_1, x_2) + V_3(y) + V_4(t),$$

here

$$V_1(x_1, x_2, y) = x_1 + x_2 + \eta y, \quad V_2(x_1) = -(\ln x_1 + \ln x_2), \quad V_3(y) = \frac{1}{y}, \quad \text{and} \quad V_4(t) = \omega(t),$$

η is a positive constant and satisfies condition (10), \widehat{c} is a positive constant to be chosen later, and

$$\frac{d\omega(t)}{dt} = \frac{1}{\theta} \int_0^\theta b_1(s) ds - b_1(t) = \langle b_1(t) \rangle_\theta - b_1(t).$$

First, we can claim that $\omega(t)$ is a θ -periodic function on $[0, \infty)$. In fact integrating above equality from t to $t + \theta$, one get

$$\omega(t + \theta) - \omega(t) = \int_t^{t+\theta} \omega(s) ds = \int_0^\theta b_1(s) ds - \int_t^{t+\theta} b_1(s) ds = 0.$$

By the periodicity of $\omega(t)$, $V(t, x_1, x_2, y)$ is a θ -periodic function on t and satisfies condition (5), where we use the fact that the coefficients of quadratic in $V(t, x_1, x_2, y)$ are all positive.

Next, we verify the condition (6) of Lemma 2.4. By using Itô's formula, it follows that

$$\begin{aligned} \mathcal{L}V_1 = & b_1(t)x_1 - a_{11}(t)x_1^2 - a_{12}(t)x_1x_2 - a_{13}x_1 + b_2(t)x_2 - a_{21}(t)x_1y - a_{22}(t)x_2^2 - a_{23}(t)x_2y \\ & + \eta b_3(t)y + \eta a_{31}(t)x_1y + \eta a_{32}(t)x_2y - \eta a_{33}(t)y^2 \end{aligned}$$

$$\begin{aligned} &\leq -a_{11}^l x_1^2 - a_{22}^l x_2^2 - \eta a_{33}^l y^2 + b_1^u x_1 + b_2^u x_2 + \eta b_3^u y, \\ \mathcal{L}V_2 &= -[(b_1(t) - a_{11}(t)x_1 - a_{12}(t)x_2 - a_{13}(t)y)] + \frac{1}{2}\sigma_1^2(t) \\ &\quad - [(b_2(t) - a_{21}(t)x_1 - a_{22}(t)y - a_{23}(t)y)] + \frac{1}{2}\sigma_2^2(t) \\ &\leq a_{11}^u x_1 + a_{12}^u x_2 + a_{13}^u y - b_1^l + \frac{1}{2}(\sigma_1^u)^2 + a_{21}^u x_1 + a_{22}^u x_2 + a_{23}^u y - b_2^l + \frac{1}{2}(\sigma_2^u)^2, \\ \mathcal{L}V_3 &= -\frac{1}{y}[b_3(t) + a_{31}(t)x_1 + a_{32}(t)x_2 - a_{33}(t)y] + \frac{\sigma_3^2(t)}{y} \leq -\frac{b_3^l - (\sigma_3^u)^2}{y} + a_{33}^u \end{aligned}$$

and $\mathcal{L}V_4 = \langle b_1(t) \rangle_\theta - b_1(t) \leq \langle b_1(t) \rangle_\theta - b_1^l$. So, it is easy to calculate that

$$\mathcal{L}V \leq f(x_1) + g(x_2) + h(y),$$

where

$$\begin{aligned} f(x_1) &= -a_{11}^l x_1^2 + [b_1^u + \widehat{c}(a_{11}^u + a_{21}^u)]x_1 - \widehat{c}\left(b_1^l - \frac{1}{2}(\sigma_1^u)^2\right), \\ g(x_2) &= -a_{22}^l x_2^2 + [b_2^u + \widehat{c}(a_{12}^u + a_{22}^u)]x_2 - \widehat{c}\left(b_2^l - \frac{1}{2}(\sigma_2^u)^2\right), \\ h(y) &= -\eta a_{33}^l y^2 + [\eta b_3^u + \widehat{c}(a_{13}^u + a_{23}^u)]y - \frac{b_3^l - (\sigma_3^u)^2}{y} + a_{33}^u + \langle b_1(t) \rangle_\theta - b_1^l. \end{aligned}$$

In view of $b_3^l > (\sigma_2^u)^2$, we get

$$\begin{aligned} f(x_1) + g^u + h^u &\rightarrow -\infty, & \text{as } x_1 &\rightarrow +\infty, \\ f^u + g(x_2) + h^u &\rightarrow -\infty, & \text{as } x_2 &\rightarrow +\infty, \\ f^u + g^u + h(y) &\rightarrow -\infty, & \text{as } y &\rightarrow +\infty \text{ or } y \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} f(x_1) + g^u + h^u &\rightarrow g^u + h^u - \widehat{c}\left(b_1^l - \frac{1}{2}(\sigma_1^u)^2\right), & \text{as } x_1 &\rightarrow 0, \\ f^u + g(x_2) + h^u &\rightarrow f^u + h^u - \widehat{c}\left(b_2^l - \frac{1}{2}(\sigma_2^u)^2\right), & \text{as } y &\rightarrow 0. \end{aligned}$$

By conditions $b_1^l > (\sigma_1^u)^2/2$ and $b_2^l > (\sigma_2^u)^2/2$, there exists a constant \widehat{c} such that

$$g^u + h^u - \widehat{c}\left(b_1^l - \frac{1}{2}(\sigma_1^u)^2\right) \leq -2, \quad f^u + h^u - \widehat{c}\left(b_2^l - \frac{1}{2}(\sigma_2^u)^2\right) \leq -2. \tag{14}$$

Take κ sufficiently large and

$$U = \left\{ (x_1, x_2, y) : \text{and insert space } \frac{1}{\kappa} \leq x_1 \leq \kappa, \frac{1}{\kappa} \leq x_2 \leq \kappa, \frac{1}{\kappa} \leq y \leq \kappa \right\}.$$

Obviously,

$$\mathcal{L}V \leq -1, \quad (x_1, x_2, y) \in \mathbb{R}_+^3/U.$$

Thus, condition (6) is satisfied. This completes the proof. \square

6. Numerical simulations

To verify the mathematical results obtained above, in this section, we perform numerical simulations with the help of software from MATLAB soft. By using Milsten method mentioned in Higham [31], the corresponding discretization equation of model (2) takes the form

$$x_{1_{k+1}} = x_{1_k} + x_{1_k} [b_1(k\Delta_t) - a_{11}(k\Delta_t)x_{1_k} - a_{12}(k\Delta_t)x_{2_k} - a_{13}(k\Delta_t)y_k] \Delta_t$$

$$\begin{aligned}
 & + \sigma_1(k\Delta_t)x_{1k}\gamma_{1k}\sqrt{\Delta_t} + \sigma_1^2(k\Delta_t)x_{1k}(\gamma_{1k}^2 - 1)\Delta_t/2, \\
 x_{2_{k+1}} = & x_{2k} + x_{2k} [b_2(k\Delta_t) - a_{21}(k\Delta_t)x_{1k} - a_{22}(k\Delta_t)x_{2k} - a_{23}(k\Delta_t)y_k] \Delta_t \\
 & + \sigma_2(k\Delta_t)x_{2k}\gamma_{2k}\sqrt{\Delta_t} + \sigma_2^2(k\Delta_t)x_{2k}(\gamma_{2k}^2 - 1)\Delta_t/2, \\
 y_{k+1} = & y_k + y_k [b_3(k\Delta_t) - a_{31}(k\Delta_t)x_{1k} - a_{32}(k\Delta_t)x_{2k} - a_{33}(k\Delta_t)y_k] \Delta_t \\
 & + \sigma_3(k\Delta_t)y_k\gamma_{3k}\sqrt{\Delta_t} + \sigma_3^2(k\Delta_t)y_k(\gamma_{3k}^2 - 1)\Delta_t/2,
 \end{aligned}$$

where γ_{1k} , γ_{2k} and γ_{3k} ($k = 1, 2, \dots, n$) are the independent Gaussian random variables $N(0, 1)$. Given that the effect of seasonal variation on populations, we let the basic model parameters as

$$\begin{aligned}
 a_{11}(t) &= 1 + \rho \sin(\pi t/15), \quad a_{12}(t) = 0.3 + \rho \sin(\pi t/15), \quad a_{13}(t) = 0.2 + \rho \sin(\pi t/15), \\
 a_{21}(t) &= 0.3 + \rho \sin(\pi t/15), \quad a_{22}(t) = 1 + \rho \sin(\pi t/15), \quad a_{23}(t) = 0.2 + \rho \sin(\pi t/15), \\
 a_{31}(t) &= 0.1 + \rho \sin(\pi t/15), \quad a_{32}(t) = 0.1 + \rho \sin(\pi t/15), \quad a_{33}(t) = 1 + \rho \sin(\pi t/15).
 \end{aligned} \tag{15}$$

We assume that all of parameters of model (2) are 30-periodic functions which are for the convenience of numerical simulations, and ρ is a positive constant to be chosen later. In fact, we can appropriately adjust periodic length, making it more meaningful, such as a quarter, or a year.

The plots in Figure 1(a) show that deterministic model (1) without or with seasonal variation have a stable positive equilibrium and a stable positive periodic solution, respectively. However, under the effect of white noise, stochastic model (2) without or with seasonal variation show more complex dynamic behaviors, which are shown in blue and green lines in Figure 1(a). Further, if we choose other model parameters, there is a significantly different between the stochastic model and the corresponding deterministic model, white noise can make the population to become extinct, as showed in Figure 1(b). These suggest that the effects of white noise and seasonal variation have significant influence on the development trend of populations. Therefore, we do such consideration that white noises and seasonal variation for a predator-prey model with single predator and two preys is completely necessary.

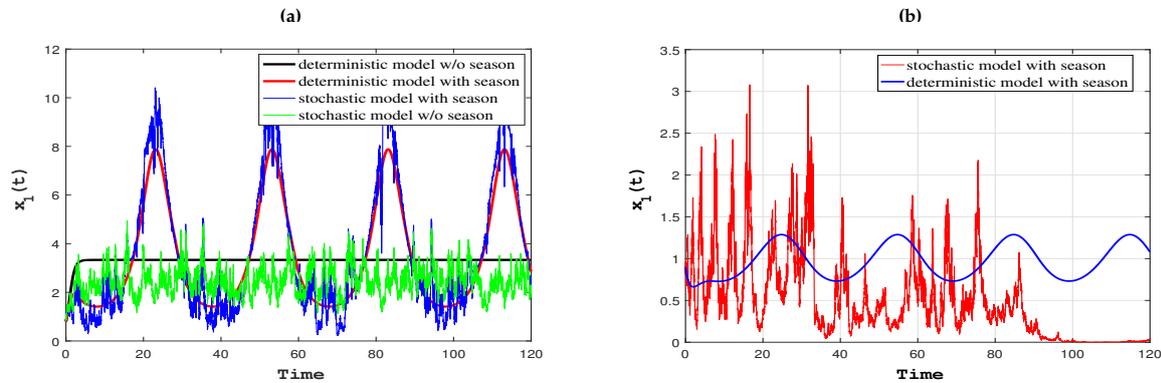


Figure 1: Comparison diagrams: (a) the dynamics of four models, where white notices and seasonal variation are introduced; (b) the comparison between stochastic model and the corresponding deterministic model under the effect of seasonal variation.

Next we will be more qualitative analysis of these problems in detail.

Example 6.1. Extinction and persistence of populations.

Draw on the selection of parameters in References [22, 24–26], for model (2), we choose the parameters as

$$\begin{aligned}
 b_1(t) &= 0.9 + \rho \sin(\pi t/15), \quad b_2(t) = 0.9 + \rho \sin(\pi t/15), \quad b_3(t) = 0.3 + \rho \sin(\pi t/15), \\
 \frac{\sigma_1^2(t)}{2} &= 0.1 + \rho \sin(\pi t/15), \quad \frac{\sigma_2^2(t)}{2} = 0.1 + \rho \sin(\pi t/15), \quad \frac{\sigma_3^2(t)}{2} = 0.1 + \rho \sin(\pi t/15)
 \end{aligned} \tag{16}$$

and the other parameters are taken as (15), here, we always assume that the intensity of reflect seasonal variation is $\rho = 0.1$ except for the other specification. It is easy to calculate $\langle h_1(t) \rangle_\theta = \langle h_2(t) \rangle_\theta = 0.8 > 0$,

$\langle h_3(t) \rangle_\theta = 0.2 > 0$, $\beta = 0.556 > 0$ and $\lambda_1 = \langle h_1(t) \rangle_\theta - a_{12}^u \langle h_1(t) \rangle_\theta / a_{22}^l - a_{13}^u \beta / a_{33}^l = \lambda_2 = 0.259 > 0$. In view of Theorem 4.2 and Corollary 4.3, populations $x_1(t)$, $x_2(t)$ and $y(t)$ are strongly persistent in the mean. This can be seen more obviously from in Figure 2(a). However, when we adjust the parameters and makes $\langle h_1(t) \rangle_\theta < 0$, for example, $b_1(t) = 0.4 + \rho \sin(\pi t/15)$, $\sigma_1^2(t)/2 = 0.5 + \rho \sin(\pi t/15)$, other parameters are taken as (15) and (16). By virtue of Corollary 4.3, the prey-1 population $x_1(t)$ is extinct and populations $x_2(t)$ and $y(t)$ are strongly persistent in the mean, which is shown in Figure 2(b). Further, let $b_2(t) = 0.4 + \rho \sin(\pi t/15)$, $\sigma_2^2(t)/2 = 0.5 + \rho \sin(\pi t/15)$ and the other values of model parameters are invariant. It is not hard to compute $\langle h_1(t) \rangle_\theta = \langle h_2(t) \rangle_\theta = -0.1 < 0$ and $\langle h_3(t) \rangle_\theta = 0.2 > 0$. Then, prey populations $x_1(t)$ and $x_2(t)$ are extinct and predator population is strongly persistent in the mean by Corollary 4.3. Which be verified by Figure 2(c). In fact, numerical simulations and theoretical results indicate that the predator population is extinct or persistent not directly related to the extinction of prey populations. In addition, if we choose $b_i(t) = 0.4 + \rho \sin(\pi t/15)$, $\sigma_i^2(t)/2 = 0.5 + \rho \sin(\pi t/15)$ ($i = 1, 2, 3$) and the other parameters are taken as (15) and (16), it can be easily shown that $\langle h_1(t) \rangle_\theta = \langle h_2(t) \rangle_\theta = -0.1 < 0$ and $\beta = -0.2 < 0$. In view of Corollary 4.3, all populations of model (2) are extinct, as can be clearly seen from Figure 2(d).

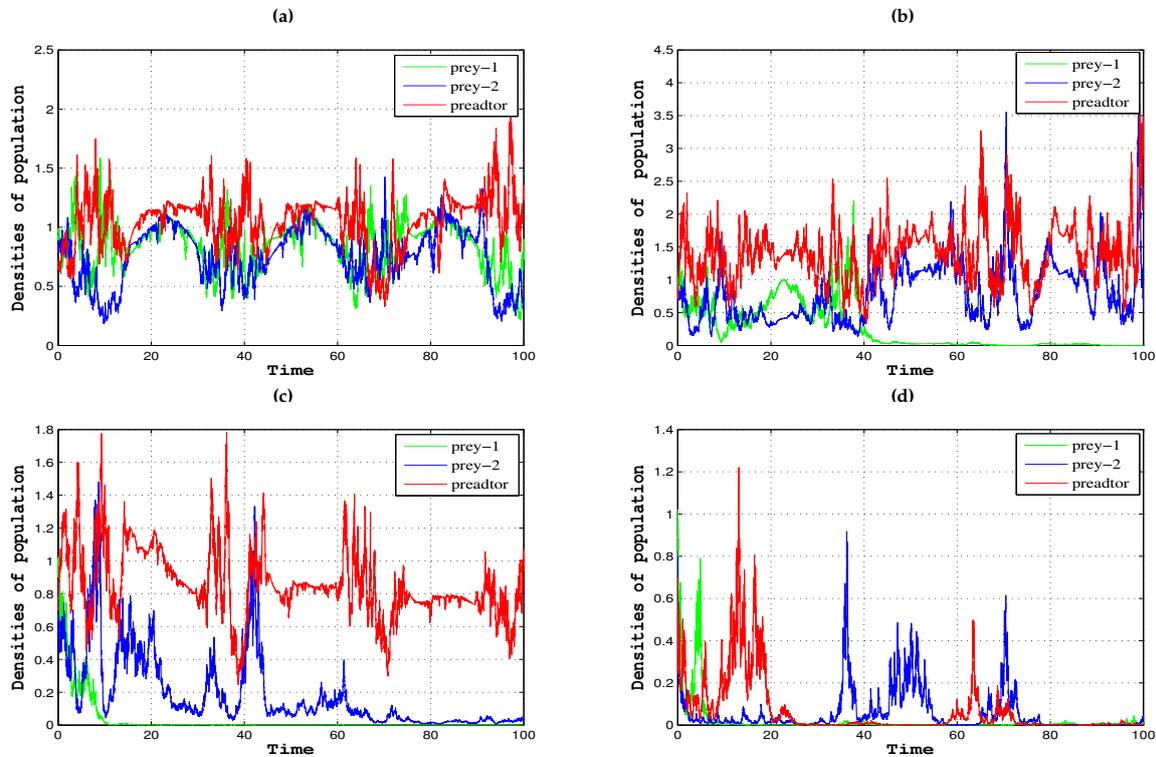


Figure 2: The extinction and persistence: (a) all populations are persistent; (b) population $x_1(t)$ is extinct and the other populations are persistent; (c) population $y(t)$ is persistent and all prey populations are extinct; (d) all populations are extinct.

Example 6.2. Existence of periodic solution.

In order to better reflect the effect of seasonal variation on populations, we increase value of ρ to 0.18 and the other parameters are the same as (15) and (16). In this case, we can verify that $b_i^l = 0.72 > 0.0392 = (\sigma_i^u)^2/2$ ($i = 1, 2$) and $b_3^l = 0.12 > 0.0784 = (\sigma_3^u)^2$. That is, all the conditions of Theorem 5.1 are hold. Hence, model (2) admits at least one positive periodic solution. The theoretical result can be seen more obviously from in Figure 3(a) and Figure 3(b). Further, the plots in Figure 3(c) show that the positive periodic solutions of model (2) is global attractive. Therefore, we present an interesting opening question: model (2) admits a global attractive positive periodic solution under some conditions.

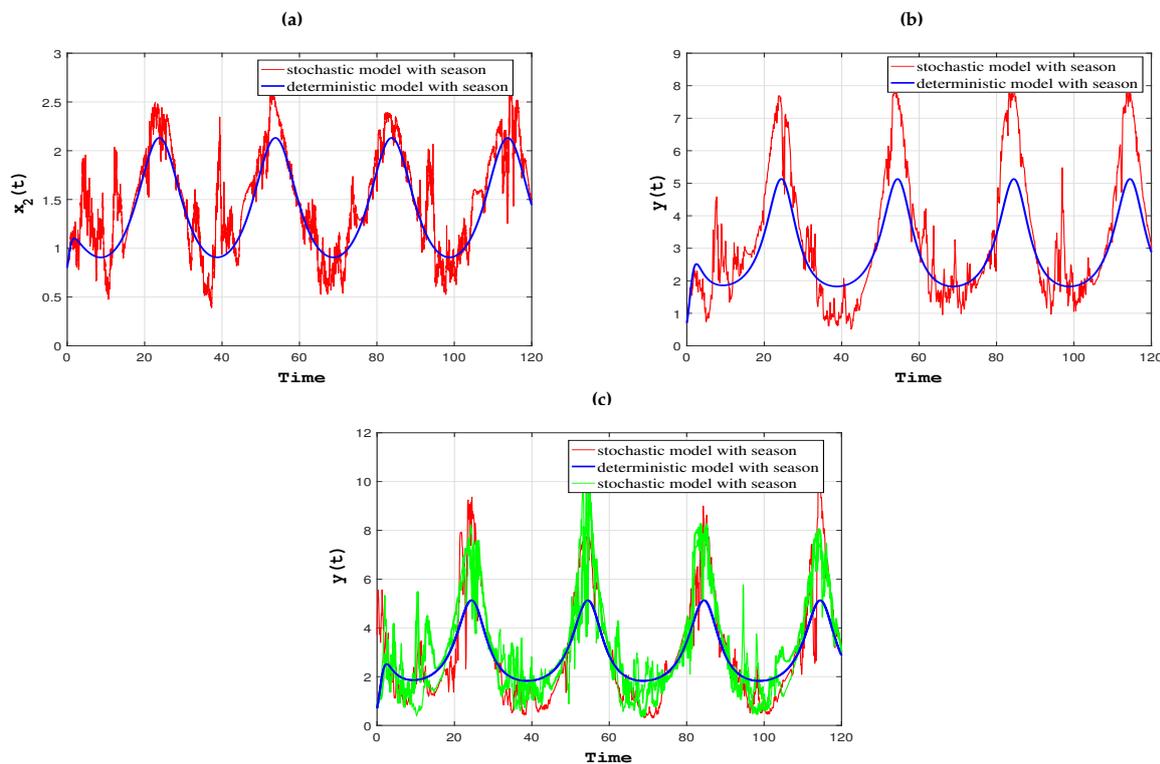


Figure 3: The existence of periodic solution of stochastic model (2).

Example 6.3. *The effect of white noises on the dynamics of population.*

In this example, we choose the different intensities of white noise $\sigma_3^2(t)/2 = 0.5 + \rho \sin(\pi t/15)$, $\sigma_3^2(t)/2 = 0.3 + \rho \sin(\pi t/15)$, and $\sigma_3^2(t)/2 = 0.1 + \rho \sin(\pi t/15)$, respectively, and the other parameters are fixed as (15) and (16) (Specially, we still let $\rho = 0.18$). Numerical simulations show that the deterministic model with seasonal variation has a stable periodic solution, which describes by the black line in Figure 4. However, when white noises are introduced, the densities of populations are oscillating and the amplitude are directly proportional to the intensities of white noises, and if the intensity value exceeds a certain threshold, the population loses its persistence and becomes extinct, as depicted by blue lines and green lines in Figure 4.

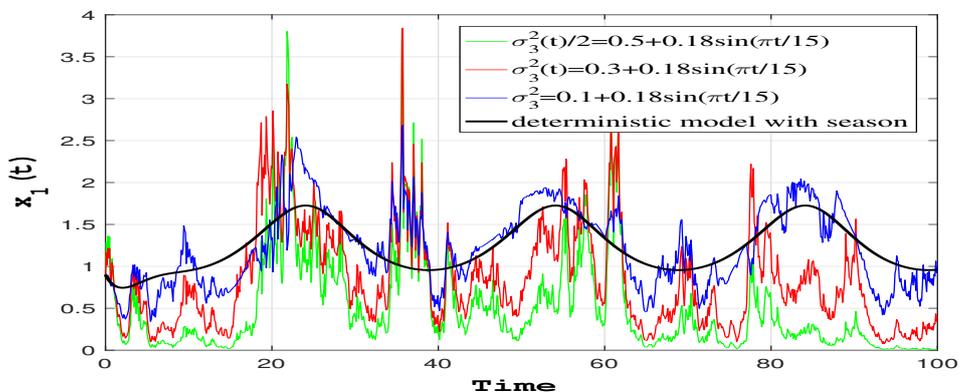


Figure 4: The effects of white noises on the dynamics of model (2).

7. Conclusions

Multi-species predator-prey models should be considered due to the diversity of species and the complexity of predator-prey relationships in nature. In addition, for the population, always inevitably subjected to the influence of white noise and seasonal variation. For that reason, in this paper, the dynamic complexity of a single predator multiple prey stochastic model with seasonal variation is studied. Firstly, we show that model is a worldwide positive solution and stochastically ultimate boundedness by solving its explicit expression method. In fact, this method is applicable to the general Lotka-Volterra models, which is a simple and easy way to determine the global positive of solutions for stochastic model. Further, we obtain some sufficient conditions on the persistence and extinction of populations and partial populations. From the expressions of the threshold condition of Theorem 4.2 and Corollary 4.3, it is clearly show that white noises directly affect the persistence and extinction of populations. Moreover, by using the theory of Khasminskii [28], the existence of positive periodic solution is considered. Finally, numerical simulations are carried to verify and extend the validity and feasibility of the all theoretical results. Numerical simulations show that the stochastic non-autonomous population model has more complex dynamic behavior than the corresponding deterministic periodic model and the autonomous stochastic model. Specifically, the seasonal variation leads to an oscillation periodic behavior of the model solution, and the white noises have an obvious adverse effects on the persistence and extinction of population. How can effectively reduce the negative impacts of white noises and maximally adapt to seasonal variation for populations make population achieve sustainable development will be an important project.

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