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On the Value Distribution of the Differential Polynomial $Af^nf^{(k)} + Bf^{n+1} - 1$

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Abstract. In the paper, we study the value distribution of the differential polynomial $Af^nf^{(k)} + Bf^{n+1} - 1$, where f is a transcendental meromorphic function and $n(\geq 2), k(\neq 2)$ are positive integers. We prove an inequality for the Nevanlinna characteristic function $T(r, f)$ in terms of reduced counting function only. The result of the paper not only improves the result due to Q.D. Zhang [J. Chengdu Ins. Meteor., 20(1992), 12-20], also partially improves a recent result of H. Karmakar and P. Sahoo [Results Math., (2018),73:98].

1. Introduction, Definitions and Results

In this paper by meromorphic function we shall always mean meromorphic function in the complex plane $\mathbb{C}$. We shall use standard notations of the Nevanlinna theory of meromorphic functions as explained in [2, 6, 12, 13]. We denote by $T(r, f)$ the Nevanlinna characteristic function of a nonconstant meromorphic function f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r possibly outside a set of finite logarithmic measure. A meromorphic function ξ is said to be a small function of f, if $T(r, \xi) = S(r, f)$.

In this research work the following definitions will be needed.

Definition 1.1. *[13] Let f be a nonconstant meromorphic function and p be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{p)}(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are not greater than p and by $\overline{N}_{p)}(r, \frac{1}{f-a})$ the corresponding reduced counting function. We denote by $N_{(p+1}(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are greater than p and by $\overline{N}_{(p+1}(r, \frac{1}{f-a})$ the corresponding reduced counting function. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of those zeros of $f(z) - a$ whose multiplicities are exactly p.*

Definition 1.2. *[13] Suppose that f is a nonconstant meromorphic function in the complex plane $\mathbb{C}$, and α is a small function of f. Let $n_0, n_1, \cdots, n_k$ be nonnegative integers. We denote by $M(f) = \alpha f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k}$ the differential monomial in f and by $n = \sum_{j=0}^{k} n_j$ the degree of $M(f)$. Also let $M_1(f), M_2(f), \cdots, M_k(f)$ be differential monomials in f of degree $m_1, m_2, \cdots, m_k$ respectively. The summation $P(f) = \sum_{j=1}^{k} M_j(f)$ is said to be the differential polynomial in f and $m = max\{m_1, m_2, \cdots, m_k\}$, the degree of $P(f)$.*

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A great number of research works have been done on value distribution of differential polynomials of meromorphic functions by many mathematicians across the world (See [4, 8–11, 14, 15]). In 1979, E. Mues [7] proved a qualitative result in this direction which is as follows.

Theorem 1.1. *Let f be a transcendental meromorphic function in the complex plane. Then $f^2f' - 1$ has infinitely many zeros.*

In 1992, Q.D. Zhang [14] proved the following quantitative result related to Theorem 1.1.

Theorem 1.2. *Let f be a transcendental meromorphic function in the complex plane and $f(z)$ is not of the form $C.e^{-\frac{B}{A}z}$, where $A(\neq 0)$, B, C are complex constants. Then*

$$T(r, f) \leq 6N\left(r, \frac{1}{Af^2f' + Bf^3 - 1}\right) + S(r, f).$$

In 2005, X. Huang and Y. Gu [3] proved the following result related to Theorem 1.2.

Theorem 1.3. *Let f be a transcendental meromorphic function in the complex plane and k be a positive integer. Then*

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2f^{(k)} - 1}\right) + S(r, f).$$

To find whether the above inequality holds if the counting function is replaced by corresponding reduced counting function, in 2009, J.F. Xu, H.X. Yi and Z.L. Zhang [10] proved the following theorem.

Theorem 1.4. *Let f be a transcendental meromorphic function in the complex plane and $L[f] = a_kf^{(k)} + a_{k-2}f^{(k-2)} + \cdots + a_0f$, where $a_0, a_1, \cdots, a_{k-2}, a_k(\not\equiv 0)$ are small functions of f. For $c(\neq 0)$, let $F = f^2L[f] - c$. Then*

$$T(r, f) \leq M\overline{N}\left(r, \frac{1}{F}\right) + S(r, f),$$

where $M > 0$ is a constant which does not depend on f.

Remark 1.1. *In the same paper, assuming $N_1(r, \frac{1}{f}) = S(r, f)$, the authors also proved the inequality*

$$T(r, f) \leq 2\overline{N}\left(r, \frac{1}{f^2f^{(k)} - 1}\right) + S(r, f).$$

In 2011, the same authors [11] improved the above result by eliminating the restriction on simple zeros of f and proved the following result.

Theorem 1.5. *Let f be a transcendental meromorphic function in the complex plane. Then*

$$T(r, f) \leq M\overline{N}\left(r, \frac{1}{f^2f^{(k)} - 1}\right) + S(r, f),$$

where M is 6 if $k = 1$ or $k \geq 3$ and M is 10 if $k = 2$.

Recently, H. Karmakar and P. Sahoo [5] proved the following result which certainly improves Theorems 1.3 and 1.5.

Theorem 1.6. *Let f be a transcendental meromorphic function and $n(\geq 2)$, $k(\geq 1)$ be integers. Then*

$$T(r, f) \leq \frac{6}{2n - 3}\overline{N}\left(r, \frac{1}{f^nf^{(k)} - 1}\right) + S(r, f).$$

Now it is natural to ask the following question.

Question 1.1. *What happens if we replace $f^n f^{(k)} - 1$ by $Af^n f^{(k)} + Bf^{n+1} - 1$ in Theorem 1.6, where $A(\neq 0)$ and B are complex constants?*

In this paper we investigate to find out a partial answer of the above question and obtain the following result.

Theorem 1.7. *Let f be a transcendental meromorphic function, $n(\geq 2)$, $k(\neq 2)$ be positive integers and $f(z)$ is not of the form $\sum_{i=1}^{k} C_i.e^{m_i z}$, where $m_i^k + \frac{B}{A} = 0$, C_i $(i = 1, 2, \cdots, k)$ are arbitrary constants and $A(\neq 0), B$ are complex constants. Then*

$$T(r, f) \leq \frac{6}{2n-3}\overline{N}\left(r, \frac{1}{Af^n f^{(k)} + Bf^{n+1} - 1}\right) + S(r, f).$$

Remark 1.2. *Theorem 1.7 improves and generalizes Theorem 1.6, except for $k = 2$.*

Remark 1.3. *Obviously, Theorem 1.7 improves Theorem 1.2.*

Remark 1.4. *The authors do not know about the validity of the conclusion of Theorem 1.7 when $k = 2$. So it remains open for further research.*

2. Lemmas

Suppose that f is a transcendental meromorphic function in the complex plane. Let us define $g = Af^n f^{(k)} + Bf^{n+1} - 1$ and $h = \frac{g'}{f^{n-1}}$ where $n(\geq 2)$, $k(\neq 2)$ are positive integers and $A(\neq 0), B$ are complex constants. Also, let

$$F = a_1\left(\frac{g'}{g}\right)^2 + a_2\left(\frac{g'}{g}\right)' + a_3\frac{g'}{g}\cdot\frac{h'}{h} + a_4\left(\frac{h'}{h}\right)^2 + a_5\left(\frac{h'}{h}\right)' + a_6\frac{B}{A}\cdot\frac{g'}{g} + a_7\frac{B}{A}\cdot\frac{h'}{h} + a_8\left(\frac{B}{A}\right)^2, \tag{2.1}$$

where for $k = 1$,
$a_1 = 2(4n^2 + 8n + 7)$, $a_2 = 2(n+2)(4n^2 - 1)$,
$a_3 = -2(n+2)(2n^2 + 3n + 4)$, $a_4 = (n+2)^2(n+1)$,
$a_5 = -(n+2)^2(2n-1)$, $a_6 = 2(n+2)(2n^2 + 9n + 1)$,
$a_7 = -(n+2)^2(n+1)(2n+1)$, $a_8 = 2n(n+1)(n+2)^2$,

and for $k > 2$,
$a_1 = \{(n-1)k^3 - 3(n^3 - 2n + 1)k^2 - 3(6n^3 - 3n + 1)k - (27n^3 - 4n + 1)\}$,
$a_2 = (n+k+1)\{(n-1)k + (3n-1)\}\{(n-1)k^2 - (3n^2 - 5n + 2)k - (9n^2 - 4n + 1)\}$,
$a_3 = -2n(n+k+1)\{(n-1)k^2 - (3n^2 - 5n + 2)k - (9n^2 - 4n + 1)\}$,
$a_4 = n^2(n-1)(k+1)(n+k+1)^2$,
$a_5 = -n(n-1)(k+1)(n+k+1)^2\{(n-1)k + (3n-1)\}$, $a_6 = a_7 = a_8 = 0$.

Lemma 2.1. *[1] Suppose that f is a transcendental meromorphic function and $f^n P(f) = Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in $f(z)$ with functions of small proximity related to f as the coefficient and the degree of $Q(f)$ is at most n. Then $m(r, P(f)) = S(r, f)$.*

Lemma 2.2. *[5] For two integers $n(\geq 2)$, $k(\geq 2)$, if*

$$\begin{aligned} f(x) &= (n-1)\Big[\Big\{(k+1)n^4 + 2(k^2+5k+10)n^3 + (k+1)^2(k+2)n^2 - (k+1)^2(2k+5)n + (k+1)^3\Big\}x^2 \\ &+ (n+k+1)(k+1)\Big\{(k+1)n^3 + (k^2+4k+9)n^2 - (2k^2+7k+5)n + (k+1)^2\Big\}x \\ &- n(n+k+1)^2(k+1)\Big\{(n-1)k + (2n-1)\Big\}\Big], \end{aligned}$$

then $f(x) = 0$ has no solution in $\mathbb{Z}_+$.

Lemma 2.3. *Let f and g be defined as in the beginning of the section and $f(z)$ be not of the form $\sum_{i=1}^{k} C_i e^{m_i z}$, where $m_i^k + \frac{B}{A} = 0$ and C_i $(i = 1, 2, \cdots, k)$ are arbitrary constants. Then g is not equivalently constant.*

Proof. Suppose $Af^n f^{(k)} + Bf^{n+1} \equiv D$(a constant). Obviously, $D \neq 0$, otherwise, we get $f(z) = \sum_{i=1}^{k} C_i e^{m_i z}$, a contradiction. Hence we have

$$\frac{1}{f^{n+1}} = \frac{A}{D} \cdot \frac{f^{(k)}}{f} + \frac{B}{D}.$$

Therefore

$$m\left(r, \frac{1}{f^{n+1}}\right) = m\left(r, \frac{A}{D} \cdot \frac{f^{(k)}}{f} + \frac{B}{D}\right).$$

i.e.,

$$(n+1)m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{A}{D}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{B}{D}\right) + O(1) = S(r, f).$$

Also, since

$$\frac{1}{Af^n f^{(k)} + Bf^{n+1}} = \frac{1}{D},$$

we have,

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{Af^n f^{(k)} + Bf^{n+1}}\right) = N\left(r, \frac{1}{D}\right) = S(r, f).$$

Therefore,

$$T(r, f) = S(r, f),$$

a contradiction. Thus $Af^n f^{(k)} + Bf^{n+1}$ is not equivalently constant and hence g is not equivalently constant. □

Lemma 2.4. *Let f and g be defined as in the beginning of the section. Then*

$$(n+1)T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + N_{k)}\left(r, \frac{1}{f}\right) + k\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r, f) \tag{2.2}$$

and

$$\left\{N(r, f) - \overline{N}(r, f)\right\} + \left\{N\left(r, \frac{1}{f}\right) - \overline{N}\left(r, \frac{1}{f}\right)\right\} + \left\{N_{(k+1}\left(r, \frac{1}{f}\right) - k\overline{N}_{(k+1}\left(r, \frac{1}{f}\right)\right\} + (n-2)N\left(r, \frac{1}{f}\right)$$
$$+m(r, f) + n\, m\left(r, \frac{1}{f}\right) \leq \overline{N}\left(r, \frac{1}{g}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r, f), \tag{2.3}$$

where $N_0\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which are not zero of f or g.

Proof. Given $g = Af^n f^{(k)} + Bf^{n+1} - 1$. By Lemma 2.3, we have g is not equivalently constant. Therefore we can write

$$\frac{1}{f^{n+1}} = A \cdot \frac{f^{(k)}}{f} - \frac{g'}{f^{n+1}} \cdot \frac{g}{g'} + B. \tag{2.4}$$

Now

$$g' = Af^n f^{(k+1)} + nAf^{n-1} f' f^{(k)} + B(n+1) f^n f'.$$

Therefore

$$\frac{g'}{f^{n+1}} = A \cdot \frac{f^{(k)}}{f} + nA \cdot \frac{f'}{f} \cdot \frac{f^{(k)}}{f} + B(n+1) \cdot \frac{f'}{f}$$

and hence

$$m\left(r, \frac{g'}{f^{n+1}}\right) = S(r, f). \tag{2.5}$$

From (2.4) and (2.5) we get

$$\begin{aligned}
(n+1)m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{g'}{f^{n+1}}\right) + m\left(r, \frac{g}{g'}\right) + O(1) \\
&\leq m\left(r, \frac{g}{g'}\right) + S(r, f) \\
&\leq T\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g}{g'}\right) + S(r, f) \\
&\leq N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + S(r, f) \\
&\leq \overline{N}(r, f) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f). \qquad (2.6)
\end{aligned}$$

Let

$$N\left(r, \frac{1}{g'}\right) = N_{000}\left(r, \frac{1}{g'}\right) + N_{00}\left(r, \frac{1}{g'}\right) + N_0\left(r, \frac{1}{g'}\right) + S(r, f), \tag{2.7}$$

where $N_{000}\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which come from the zeros of g and $N_{00}\left(r, \frac{1}{g'}\right)$ denotes the counting function of those zeros of g' which come from the zeros of f. Therefore

$$N\left(r, \frac{1}{g}\right) - N_{000}\left(r, \frac{1}{g'}\right) = \overline{N}\left(r, \frac{1}{g}\right). \tag{2.8}$$

Let z_0 be a zero of f with multiplicity p. If $p \leq k$, then z_0 is a zero of g' with multiplicity at least $(np-1)$. If $p \geq k+1$, then z_0 is zero of g' with multiplicity at least $(n+1)p - k - 1$. Therefore

$$\begin{aligned}
N_{00}\left(r, \frac{1}{g'}\right) &\geq nN_{k)}\left(r, \frac{1}{f}\right) - \overline{N}_{k)}\left(r, \frac{1}{f}\right) + (n+1)N_{(k+1}\left(r, \frac{1}{f}\right) - (k+1)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \\
&= nN\left(r, \frac{1}{f}\right) - \overline{N}\left(r, \frac{1}{f}\right) + N_{(k+1}\left(r, \frac{1}{f}\right) - k\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) + S(r, f). \qquad (2.9)
\end{aligned}$$

From (2.6) - (2.9), we get

$$\begin{aligned}(n+1)T(r,f) &= (n+1)m\left(r,\frac{1}{f}\right)+(n+1)N\left(r,\frac{1}{f}\right)+O(1)\\ &\leq (n+1)N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g}\right)-nN\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)\\ &- N_{(k+1}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f)\\ &= \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+N_{k)}\left(r,\frac{1}{f}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f), \qquad (2.10)\end{aligned}$$

which is (2.2). Also

$$\begin{aligned}(n+1)T(r,f) &= T(r,f)+n\,T\left(r,\frac{1}{f}\right)+O(1)\\ &= N(r,f)+m(r,f)+(n-2)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+N_{k)}\left(r,\frac{1}{f}\right)\\ &+ N_{(k+1}\left(r,\frac{1}{f}\right)+n\,m\left(r,\frac{1}{f}\right)+S(r,f). \qquad (2.11)\end{aligned}$$

Combining (2.10) and (2.11) we get

$$\left\{N(r,f)-\overline{N}(r,f)\right\}+\left\{N\left(r,\frac{1}{f}\right)-\overline{N}\left(r,\frac{1}{f}\right)\right\}+\left\{N_{(k+1}\left(r,\frac{1}{f}\right)-k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\}+(n-2)N\left(r,\frac{1}{f}\right)$$
$$+m(r,f)+n\,m\left(r,\frac{1}{f}\right)\leq \overline{N}\left(r,\frac{1}{g}\right)-N_0\left(r,\frac{1}{g'}\right)+S(r,f),$$

which is (2.3). This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let f, g, h, F, a_i's$(i=1,2,\cdots,8)$ be defined as in the beginning of the section. Then the simple poles of f are zero of F.*

Proof. Let z_0 be a simple pole of f. Then in some neighbourhood of of z_0, we write

$$f(z)=\frac{a}{z-z_0}\Big[1+b_0(z-z_0)+b_1(z-z_0)^2+b_2(z-z_0)^3+O\Big((z-z_0)^4\Big)\Big],$$

where $a(\neq 0)$, b_0, b_1, b_2 are constants. Therefore we get

$$f'(z)=\frac{a}{(z-z_0)^2}\Big[-1+b_1(z-z_0)^2+2b_2(z-z_0)^3+O\Big((z-z_0)^4\Big)\Big];$$

$$f^{(k)}(z)=\frac{(-1)^k ak!}{(z-z_0)^{k+1}}\Big[1+(-1)^k b_k(z-z_0)^{k+1}+O\Big((z-z_0)^{k+2}\Big)\Big];$$

$$f^n(z)=\frac{a^n}{(z-z_0)^n}\Big[1+nb_0(z-z_0)+\frac{1}{2}\Big\{n(n-1)b_0^2+2nb_1\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]$$

and

$$f^{n+1}(z)=\frac{a^{n+1}}{(z-z_0)^{n+1}}\Big[1+(n+1)b_0(z-z_0)+\frac{1}{2}\Big\{n(n+1)b_0^2+2(n+1)b_1\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big].$$

Now we discuss the following two cases separately.

Case 1. Let k=1. Then

$$\begin{aligned} g(z) &= Af^n(z)f'(z)+Bf^{n+1}(z)-1=\frac{(-1)Aa^{n+1}}{(z-z_0)^{n+2}}\Big[1+\Big(nb_0-\frac{B}{A}\Big)(z-z_0)\\ &+ \frac{1}{2}\Big\{n(n-1)b_0^2+2(n-1)b_1-2(n+1)b_0\frac{B}{A}\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big] \end{aligned}$$

and

$$\begin{aligned} h(z) &= \frac{g'(z)}{f^{n-1}(z)}=\frac{Aa^2}{(z-z_0)^4}\Big[(n+2)+\Big\{2b_0-(n+1)\frac{B}{A}\Big\}(z-z_0)\\ &- \Big\{2(n-1)b_1+(n+1)b_0\frac{B}{A}\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{g'(z)}{g(z)} &= \frac{-1}{z-z_0}\Big[(n+2)-\Big(nb_0-\frac{B}{A}\Big)(z-z_0)+\Big\{nb_0^2-2(n-1)b_1+2b_0\frac{B}{A}+\Big(\frac{B}{A}\Big)^2\Big\}(z-z_0)^2\\ &+ O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \tag{2.12}$$

$$\begin{aligned} \Big(\frac{g'(z)}{g(z)}\Big)^2 &= \frac{1}{(z-z_0)^2}\Big[(n+2)^2-2(n+2)\Big(nb_0-\frac{B}{A}\Big)(z-z_0)+\Big\{n(3n+4)b_0^2-4(n-1)(n+2)b_1\\ &+ 2(n+4)b_0\frac{B}{A}+(2n+5)\Big(\frac{B}{A}\Big)^2\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \tag{2.13}$$

$$\Big(\frac{g'(z)}{g(z)}\Big)' = \frac{1}{(z-z_0)^2}\Big[(n+2)-\Big\{nb_0^2-2(n-1)b_1+2b_0\frac{B}{A}+\Big(\frac{B}{A}\Big)^2\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]; \tag{2.14}$$

$$\begin{aligned} \frac{h'(z)}{h(z)} &= \frac{-1}{z-z_0}\Big[4-\frac{2b_0-(n+1)\frac{B}{A}}{n+2}(z-z_0)+\Big\{\frac{4b_0^2+4(n+2)(n-1)b_1+2n(n+1)b_0\frac{B}{A}}{(n+2)^2}\\ &+ \frac{(n+1)^2\left(\frac{B}{A}\right)^2}{(n+2)^2}\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Big(\frac{h'(z)}{h(z)}\Big)^2 &= \frac{1}{(z-z_0)^2}\Big[16-8\,\frac{2b_0-(n+1)\frac{B}{A}}{n+2}(z-z_0)+\Big\{4\,\frac{9b_0^2+8(n+2)(n-1)b_1}{(n+2)^2}\\ &+ \frac{4(n+1)(4n-1)b_0\frac{B}{A}+9(n+1)^2\left(\frac{B}{A}\right)^2}{(n+2)^2}\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \tag{2.16}$$

$$\Big(\frac{h'(z)}{h(z)}\Big)' = \frac{1}{(z-z_0)^2}\Big[4-\Big\{\frac{4b_0^2+2n(n+1)b_0\frac{B}{A}+(n+1)^2\left(\frac{B}{A}\right)^2}{(n+2)^2}+\frac{4(n-1)b_1}{n+2}\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big] \tag{2.17}$$

and

$$\begin{aligned} \frac{g'(z)}{g(z)}\cdot\frac{h'(z)}{h(z)} &= \frac{1}{(z-z_0)^2}\Big[4(n+2)-\Big\{2(2n+1)b_0-(n+5)\frac{B}{A}\Big\}(z-z_0)\\ &+ \Big\{2(2n+1)b_0^2-4(n-1)b_1+(n+7)b_0\frac{B}{A}+(n+5)\Big(\frac{B}{A}\Big)^2\Big\}(z-z_0)^2+O\Big((z-z_0)^3\Big)\Big]. \end{aligned} \tag{2.18}$$

Now substituting these values from (2.12) - (2.18) in the expression (2.1) we get $F(z) = O\Big((z-z_0)\Big)$, which shows that z_0 is a zero of F.

Case 2. Let $k > 2$. Then

$$\begin{aligned} g(z) &= Af^n(z)f^{(k)}(z) + Bf^{n+1}(z) - 1 = \frac{(-1)^k k! A a^{n+1}}{(z-z_0)^{n+k+1}}\Big[1 + nb_0(z-z_0) + \frac{1}{2}\Big\{n(n-1)b_0^2 + 2nb_1\Big\}(z-z_0)^2 \\ &+ O\Big((z-z_0)^3\Big)\Big] \end{aligned}$$

and

$$h(z) = \frac{g'(z)}{f^{n-1}(z)} = \frac{(-1)^{k+1}k! A a^2}{(z-z_0)^{k+3}}\Big[(n+k+1) + (k+1)b_0(z-z_0) - (n-k-1)b_1(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big].$$

Therefore

$$\frac{g'(z)}{g(z)} = \frac{-1}{z-z_0}\Big[(n+k+1) - nb_0(z-z_0) + \Big\{nb_0^2 - 2nb_1\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big]; \quad (2.19)$$

$$\begin{aligned} \left(\frac{g'(z)}{g(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\Big[(n+k+1)^2 - 2n(n+k+1)b_0(z-z_0) + \Big\{n(3n+2k+2)b_0^2 \\ &- 4n(n+k+1)b_1\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \quad (2.20)$$

$$\left(\frac{g'(z)}{g(z)}\right)' = \frac{1}{(z-z_0)^2}\Big[(n+k+1) - \Big\{nb_0^2 - 2nb_1\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big]; \quad (2.21)$$

$$\begin{aligned} \frac{h'(z)}{h(z)} &= \frac{-1}{z-z_0}\Big[(k+3) - \frac{(k+1)b_0}{n+k+1}(z-z_0) + \Big\{\frac{(k+1)^2b_0^2}{(n+k+1)^2} + 2\,\frac{(n-k-1)b_1}{n+k+1}\Big\}(z-z_0)^2 \\ &+ O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \quad (2.22)$$

$$\begin{aligned} \left(\frac{h'(z)}{h(z)}\right)^2 &= \frac{1}{(z-z_0)^2}\Big[(k+3)^2 - 2\,\frac{(k+1)(k+3)b_0}{n+k+1}(z-z_0) + \Big\{\frac{(k+1)^2(2k+7)b_0^2}{(n+k+1)^2} \\ &+ 4\,\frac{(k+3)(n-k-1)b_1}{n+k+1}\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big]; \end{aligned} \quad (2.23)$$

$$\left(\frac{h'(z)}{h(z)}\right)' = \frac{1}{(z-z_0)^2}\Big[(k+3) - \Big\{\frac{(k+1)^2b_0^2}{(n+k+1)^2} + 2\,\frac{(n-k-1)b_1}{n+k+1}\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big] \quad (2.24)$$

and

$$\begin{aligned} \frac{g'(z)}{g(z)}\cdot\frac{h'(z)}{h(z)} &= \frac{1}{(z-z_0)^2}\Big[(n+k+1)(k+3) - (nk+k+3n+1)b_0(z-z_0) \\ &+ \Big\{(nk+k+3n+1)b_0^2 - 2(nk+k+2n+1)b_1\Big\}(z-z_0)^2 + O\Big((z-z_0)^3\Big)\Big]. \end{aligned} \quad (2.25)$$

Now substituting these values from (2.19) - (2.25) in the expression (2.1) we get $F(z) = O\Big((z-z_0)\Big)$. This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let f, g, h, F, a_i's $(i = 1, 2, \cdots, 8)$ be defined as in the beginning of this section. Then $F(z) \not\equiv 0$.*

Proof. If possible, we assume that $F(z) \equiv 0$. Under this hypothesis we first show that
i) g has no zero,
ii) h has no zero.

Suppose that z_1 is a zero of g of multiplicity $l_1(\geq 1)$. Then it is clear that $f(z_1) \neq 0, \infty$ and z_1 is a zero of h with multiplicity $(l_1 - 1)$. Then from the Laurent series expansion of $F(z)$ we get the coefficient of $(z - z_1)^{-2}$ as

$$A(l_1) = (a_1 + a_3 + a_4)l_1^2 - (a_2 + a_3 + 2a_4 + a_5)l_1 + (a_4 + a_5).$$

For $k = 1$, putting the values of a_i 's $(i = 1, 2, \cdots, 5)$ we get

$$A(l_1) = -\Big\{(n+1)(3n^2 - 2n - 2)l_1^2 + (n+2)(4n^2 - 3n - 4)l_1 + (n+2)^2(n-2)\Big\}.$$

Clearly $A(l_1) \neq 0$ for any positive integral value of l_1.

For $k > 2$, we get

$$\begin{aligned} A(l_1) &= (n-1)\Big[\Big\{(k+1)n^4 + 2(k^2+5k+10)n^3 + (k+1)^2(k+2)n^2 - (k+1)^2(2k+5)n + (k+1)^3\Big\}l_1^2 \\ &+ (n+k+1)(k+1)\Big\{(k+1)n^3 + (k^2+4k+9)n^2 - (2k^2+7k+5)n + (k+1)^2\Big\}l_1 \\ &- n(n+k+1)^2(k+1)\Big\{(n-1)k + (2n-1)\Big\}\Big]. \end{aligned}$$

By Lemma 2.2, we get $A(l_1) \neq 0$ for any positive integral value of l_1. Therefore z_1 is a pole of F, a contradiction to our hypothesis. Thus, z_1 is not a zero of g and hence g has no zero.

Let z_2 be a zero of h of multiplicity l_2. Then z_2 is neither a zero nor a pole of g. Then from the Laurent series expansion of $F(z)$ we obtain the coefficient of $(z - z_2)^{-2}$ as

$$B(l_2) = a_4 l_2^2 - a_5 l_2.$$

Now for $k = 1$, $\frac{a_5}{a_4} = -\frac{2n-1}{n+1}$ and for $k > 2$, $\frac{a_5}{a_4} = -(k+3) + \frac{k+1}{n}$. Clearly, $B(l_2) \neq 0$ for any positive integer value of l_2. Then z_2 is a pole of F, a contradiction. Hence h has no zero.

Set

$$\psi = \frac{g'}{g} \cdot \frac{1}{f} = \frac{h}{g} \cdot f^{n-2} = \frac{A\Big\{ff^{(k+1)} + nf'f^{(k)}\Big\} + (n+1)Bff'}{Af^n f^{(k)} + Bf^{n+1} - 1} \cdot f^{n-2}.$$

Also,

$$\frac{g'}{g} = \psi f \quad \text{and} \quad \frac{h'}{h} = \psi f - (n-2)\frac{f'}{f} + \frac{\psi'}{\psi}.$$

Now substituting these values in the expression (2.1) we get

$$\begin{aligned} \Big\{(n-2)(a_3+2a_4) - (a_2+a_5)\Big\}\psi f' &= \Big(a_1+a_3+a_4\Big)\psi^2 f^2 + \Big\{\Big(a_2+a_3+2a_4+a_5\Big)\frac{\psi'}{\psi} + (a_6+a_7)\frac{B}{A}\Big\}\psi f \\ &+\Big[a_4\left(\frac{\psi'}{\psi} - (n-2)\frac{f'}{f}\right)^2 + a_5\Big\{\left(\frac{\psi'}{\psi}\right)' - (n-2)\left(\frac{f'}{f}\right)'\Big\} + a_7\frac{B}{A}\left(\frac{\psi'}{\psi} - (n-2)\frac{f'}{f}\right) + a_8\left(\frac{B}{A}\right)^2\Big]. \end{aligned}$$

From this we obtain

$$f' = \frac{l_1}{\psi} + l_2 f + l_3\psi f^2, \tag{2.26}$$

where l_1, l_2, l_3 are differential polynomials of $\frac{\psi'}{\psi}$ and $\frac{f'}{f}$.

Now let z_3 be a zero of f. Then z_3 is a pole of l_1 and atmost a zero of ψ. Hence from (2.26), z_3 is a pole of f', which is a contradiction.
Therefore $N\left(r, \frac{1}{f}\right) = 0$. Then from (2.3) we get $m\left(r, \frac{1}{f}\right) = S(r, f)$. Therefore

$$T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = S(r, f),$$

a contradiction. Hence $F(z) \not\equiv 0$. □

3. Proof of the Theorem

Proof. By Lemmas 2.5 and 2.6 we have seen that the simple poles of f are zeros of F and $F(z) \not\equiv 0$. Now

$$g = Af^n f^{(k)} + Bf^{n+1} - 1 \quad \text{and} \quad h = \frac{g'}{f^{n-1}} = A\left\{ff^{(k+1)} + nf'f^{(k)}\right\} + (n+1)Bff'. \tag{3.1}$$

Let

$$\beta = -\frac{h}{g} = -\frac{A\left\{ff^{(k+1)} + nf'f^{(k)}\right\} + (n+1)Bff'}{Af^n f^{(k)} + Bf^{n+1} - 1}. \tag{3.2}$$

Therefore

$$\beta f^{n-1} = -\frac{g'}{g}. \tag{3.3}$$

Now we consider the poles of $\beta^2 F$. From Lemma 2.5 we observe that the poles of F are of multiplicities at most 2 and come from the zeros and poles of g or h. From (3.2) we can see that the poles of β are zeros of g or poles of h. Now poles of g and h come from the poles of f. But we see that a pole of f of order $s(\geq 2)$ is a zero of β of order $(n-1)s - 1 \geq 1$. Therefore poles of f can not be a pole of $\beta^2 F$. Also from (3.1) we can see that zeros of h comes from multiple zeros of f. But multiple zeros of f are pole of F of order at most 2 and zero of β^2 of order at least 2. Therefore multiple zeros of f can not be a pole of $\beta^2 F$. Hence poles of $\beta^2 F$ comes only from zeros of g.

Let us suppose that z_4 be a zero of g of multiplicity t. Then $f(z_4) \neq 0$ or ∞. Therefore z_4 is a zero of g' and h with multiplicity $(t-1)$ and hence a simple pole of β. Also we remember that the zeros of g and h can be a pole of F of order at most 2. Therefore z_4 is a pole of $\beta^2 F$ of order at most 4. Therefore

$$N(r, \beta^2 F) \leq 4\overline{N}\left(r, \frac{1}{g}\right) + S(r, f). \tag{3.4}$$

Now from the expression (2.1) we get $m(r, F) = S(r, f)$. Also using Lemma 2.1 we get from (3.3) that $m(r, \beta^2) = S(r, f)$. Thus $m(r, \beta^2 F) = S(r, f)$. Therefore

$$T(r, \beta^2 F) \leq 4\overline{N}\left(r, \frac{1}{g}\right) + S(r, f). \tag{3.5}$$

Now the zeros of f of order $\mu(\geq k+1)$ are zero of β of order atleast $(2\mu - k - 1)$. Also zeros of f are not zero of g but a zero of h of order $(2\mu - k - 1)$ and then a pole of F of order 2. Therefore zeros of $\beta^2 F$ are of multiplicity at least $(4\mu - 2k - 4)$. Also simple poles of f are zero of $\beta^2 F$. Therefore

$$N_1(r, f) + 4N_{(k+1}\left(r, \frac{1}{f}\right) - 2(k+2)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\beta^2 F}\right) \leq T\left(r, \frac{1}{\beta^2 F}\right) \leq 4\overline{N}\left(r, \frac{1}{g}\right) + S(r, f). \tag{3.6}$$

Combining (3.6) with twice of (2.2) we obtain

$$2(n+1)T(r,f) - 2\overline{N}(r,f) - 2\overline{N}\left(r,\frac{1}{f}\right) - 2N_{k)}\left(r,\frac{1}{f}\right) - 2k\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) + N_1(r,f) + 4N_{(k+1}\left(r,\frac{1}{f}\right)$$
$$-2(k+2)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) \le 6\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \quad (3.7)$$

Now

$$(2n+2)T(r,f) = (2n-3)T(r,f) + T(r,f) + 4T\left(r,\frac{1}{f}\right) \ge (2n-3)T(r,f) + N(r,f) + 4N\left(r,\frac{1}{f}\right). \quad (3.8)$$

From (3.7) and (3.8) we get

$$(2n-3)T(r,f) + \left\{N(r,f) + N_1(r,f) - 2\overline{N}(r,f)\right\} + \left\{4N\left(r,\frac{1}{f}\right) - 2\overline{N}\left(r,\frac{1}{f}\right) - 2N_{k)}\left(r,\frac{1}{f}\right)\right.$$
$$\left. + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 4(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\} \le 6\overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \quad (3.9)$$

Now

$$N(r,f) + N_1(r,f) - 2\overline{N}(r,f) \ge N_1(r,f) + N_{(2}(r,f) + N_1(r,f) - 2\overline{N}_1(r,f) - 2\overline{N}_{(2}(r,f)$$
$$= N_{(2}(r,f) - 2\overline{N}_{(2}(r,f) \ge 0$$

and

$$4N\left(r,\frac{1}{f}\right) - 2\overline{N}\left(r,\frac{1}{f}\right) - 2N_{k)}\left(r,\frac{1}{f}\right) + 4N_{(k+1}\left(r,\frac{1}{f}\right) - 4(k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)$$
$$= 2\left\{N\left(r,\frac{1}{f}\right) - \overline{N}\left(r,\frac{1}{f}\right)\right\} + 2\left\{N\left(r,\frac{1}{f}\right) - N_{k)}\left(r,\frac{1}{f}\right)\right\} + 4\left\{N_{(k+1}\left(r,\frac{1}{f}\right) - (k+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\right\} \ge 0.$$

Therefore from (3.9) we have

$$T(r,f) \le \frac{6}{2n-3}\overline{N}\left(r,\frac{1}{g}\right) + S(r,f).$$

This completes the proof of the theorem. □

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