Multipliers and Closures of Besov-Type Spaces in the Bloch Space

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Abstract. Let $p > 1$ and let $\rho$ be a non-negative function defined in $\mathbb{R}^+$. A function $f \in H(D)$ belongs to the space $B_p(\rho)$ (see [4]) if

$$
\|f\|_{B_p(\rho)}^p = |f(0)|^p + \int_D \left| (1 - |z|^2) f(z) \right|^p \frac{\rho(1 - |z|^2)}{(1 - |z|^2)^2} dA(z) < \infty.
$$

In this paper, motivated by the works of Békoellé and Bao and Gögüs, under some conditions on the weight function $\rho$, we investigate the closures $C_B(B \cap B_p(\rho))$ of the spaces $B \cap B_p(\rho)$ in the Bloch space. Moreover, we prove that interpolating Blaschke products in $C_B(B \cap B_p(\rho))$ are multipliers of $B_p(\rho) \cap BMOA$.

1. Introduction

We denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ by $D$ and its boundary $\{z \in \mathbb{C} : |z| = 1\}$ by $\partial D$. Let $H(D)$ be the space of all analytic functions in $D$.

Let $H^p$ (see [11]) denote the space of those analytic functions $f \in H(D)$ such that

$$
\|f\|_{H^p}^p = \sup_{0 < r < 1} M^p_r(f) = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.
$$

Let $BMOA$ denote the space of those analytic functions $f$ in the Hardy space $H^p$ whose boundary functions have bounded mean oscillation on $\partial D$. $BMOA$ ([17, 19]) is a Banach space under the following norm:

$$
\|f\|_{BMOA} = |f(0)|^p + \sup_{a \in D} \|f \circ \varphi_a - f(a)\|_{H^p}^p,
$$

where $\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$, $a, z \in D$ and $p \geq 1$.

Recall that the Bloch space ([2, 34]) is the class of functions $f \in H(D)$ satisfying

$$
\|f\|_B = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.
$$

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Let $p > 1$ and let $\rho$ be a non-negative function defined in $\mathbb{R}^+$. A function $f \in H(\mathbb{D})$ belongs to the space $B_p(\rho)$ if
\[\|f\|_{B_p(\rho)}^p = |f(0)|^p + \int_{\mathbb{D}} \left(1 - |z|^2\right)^{\rho} dA(z) < \infty,\]
where $dA(z)$ is the usual normalized Lebesgue measure on $\mathbb{D}$. This space is introduced by Arcozzi, Rochberg and Sawyer in [4]. They considered Carleson measures for $B_p(\rho)$ spaces under the condition that the weight function $\rho$ is $\rho$-admissible, or admissible, that is, $\rho$ satisfies the following conditions:

(i) $\rho(z)$ is regular, i.e., there exist $\epsilon > 0$, $C > 0$ such that $\rho(z) \leq C\rho(w)$ when $z$ and $w$ are within hyperbolic distance $\epsilon$. Equivalently, there are $\delta < 1$, $C' > 0$ so that $\rho(z) \leq C'\rho(w)$ whenever
\[\left|\frac{z - w}{1 - \overline{z}w}\right| \leq \delta < 1.\]

(ii) The weight $\rho_p(z) = (1 - |z|^2)^{\gamma-2}\rho(z)$ satisfies the Békollé-Bonami $B_p$ condition([7, 8]): There is a $C(\rho, p)$ so that for all $a \in \mathbb{D}$ we have
\[
\left(\int_{S(a)} \rho_p(z)dA(z)\right)^{1/(\gamma-1)} \leq C(\rho, p)\left(\int_{S(a)} dA(z)\right)^{\gamma}.
\]
Where $1/p + 1/q = 1$, and
\[S(a) = \{z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \arg(az) \leq \frac{1 - |a|}{2}\}, \quad a \in \mathbb{D}.
\]

In the case $\rho(t) = t^s$, $0 \leq s < \infty$, the space $B_p(\rho)$ gives the usual Besov type space $B_p(s)$. In particular, if $s = 0$, this gives the classical Besov space $B_p$. We refer to [5], [9], [10] and [12] for $B_p(s)$ spaces and [30], [31] and [32] for $B_2(s) = D_s$ spaces. The space $B_p(\rho)$ has been extensively studied. For example, under some conditions on $\rho$, N. Arcozzi, R. Kerman and E. Sawyer [4] give many results on $B_p(\rho)$ spaces using Carleson measures. When $p = 2$, $B_2(\rho) = D_\rho$, weighted Dirichlet spaces. Using maximal operators, R. Kerman and E. Sawyer [21] characterized the Carleson measures and multipliers of $D_\rho$ spaces. For more informations on $D_\rho$ spaces, we refer to [1] and the paper referinthere.

Let us recall that a weight $\rho$ is of upper (resp. lower) type $\gamma$ ($0 \leq \gamma < \infty$) ([20]), if
\[\rho(s) \leq Cs^\gamma \rho(t), \quad s \geq 1 \quad (\text{resp. } s \leq 1) \quad \text{and} \quad 0 < t < \infty.
\]
We say that $\rho$ is of upper type less than $\gamma$ if it is of upper type $\delta$ for some $\delta < \gamma$ and similarly for lower type greater than $\delta$. From [20], we see that an increasing function $\rho$ is of finite upper type if and only if $\rho(2t) \leq C\rho(t)$ for some positive constant $C$ and all $t$. It is not hard to verify that $\rho$ satisfies (i) and (ii), if $\rho$ is of upper type less than 1 and lower type greater than 0.

In [2], Anderson, Clunie and Pommerenke raised the question of determining the closure of $H^\rho$ in the Bloch norm. Until now, the problem is still unsolved. Jones [3, Theorem 9] gave an unpublished characterization of the closure of BMOA in Bloch space. Zhao [33] studied the closures of some Möbius invariant spaces in the Bloch space. Lou and Chen [22] generalize [33] later. For $1 < p < \infty$, Monreal Galán and Nicolau in [24] characterized the closure in the Bloch norm of the space $H^p \cap B$. Galanopoulos, Monreal Galán and Pau [16] generalize [24] to $0 < p < \infty$ later. Recently, Bao and Göğüs [6] and Galanopoulos and Girela [15] have investigated the closures in $B$ of $B \cap D_\rho$ for certain spaces of Dirichlet Type $D_\rho^p$. For more results on closures of analytic function spaces in the Bloch space, we refer to [28] and [29]. In this paper, we study the closures of the $B_\rho(\rho)$ spaces, generalizing the main results in [6] and [15]. Meanwhile, interpolating Blaschke products in $C_\rho(B \cap B_\rho(\rho))$ as multipliers of $B_\rho(\rho) \cap BMOA$ are also investigated.

Throughout this paper, let $\rho : [0, \infty) \rightarrow [0, \infty)$ be a right continuous and nondecreasing function with $\rho(0) = 0$ and $\rho(t) > 0$ if $t > 0$. The symbol $A \approx B$ means that $A \leq B \leq A$. We say that $A \leq B$ if there exist a constant $C$ such that $A \leq CB$. 
Remark 1. Using [20, Lemma 4], the fact that \( \rho \) is increasing, and the above mentioned fact that \( \rho \) is of finite upper if and only if \( \rho(2t) \leq C \rho(t) \) (\( t \geq 0 \)), we deduce the following:

If \( \rho \) is of finite upper type, then \( \rho \) is of upper type less than \( p \) if and only if

\[
\frac{\rho(t)}{t^p} \approx \int_t^\infty \rho(s) \frac{ds}{s^{1+p}}.
\]

Remark 2. Let \( 0 \leq a < 1 \) and \( p > 1 \). If \( \rho \) is of finite upper type \( a \), we can deduce that \( B_p(\rho) \subseteq H^p \). Indeed, take \( b \) with \( a < b < 1 \), using Remark 1, we deduce that

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2} \rho(1 - |z|^2) \, dA(z)
\]

\[
= \int_D |f'(z)|^p (1 - |z|^2)^{p-2} \frac{\rho(1 - |z|^2)}{(1 - |z|^2)^p} (1 - |z|^2) \, dA(z)
\]

\[
\approx \int_D |f'(z)|^p (1 - |z|^2)^{p-2} \left( \int_{1-|z|^2}^\infty \rho(s) \frac{ds}{s^{1+p}} \right) (1 - |z|^2) \, dA(z)
\]

\[
\leq \int_D |f'(z)|^p (1 - |z|^2)^{p-2} \left( \int_{1-|z|^2}^\infty \rho(s) \frac{ds}{s^{1+p}} \right) (1 - |z|^2) \, dA(z)
\]

\[
\approx \rho(1) \int_D |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |z|^2) \, dA(z).
\]

Thus, \( B_p(\rho) \subseteq B_p(b) \). Then the inclusion \( B_p(\rho) \subseteq H^p \) follows from the well known fact that \( B_p(b) \subseteq H^p \) because \( 0 < b < 1 \).

2. Equivalent Characterizations of closures of \( B_p(\rho) \) spaces in Bloch space

Theorem 1. Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( 1 < p < \infty \). Then the following conditions are equivalent.

(1) \( f \in C_0(B_p(\rho) \cap D) \).

(2) For any \( \epsilon > 0 \),

\[
\int_{\Omega_\epsilon(f)} \rho(1 - |z|^2) \frac{dA(z)}{(1 - |z|^2)^2} < \infty,
\]

where

\[
\Omega_\epsilon(f) = \{ z \in D : (1 - |z|^2)|f'(z)| \geq \epsilon \}.
\]

(3) For \( \epsilon > 0 \) and \( s > 1 \),

\[
\int_{\Gamma_{\epsilon s}(f)} \rho(1 - |z|^2) \frac{dA(z)}{(1 - |z|^2)^2} < \infty,
\]

where

\[
\Gamma_{\epsilon s}(f) = \left\{ z \in D : \int_D |f'(w)|^p (1 - |w|^2)^{p-2} (1 - |\rho_z(w)|^2)^2 dA(w) \geq \epsilon \right\}.
\]

Proof. (2) \( \Rightarrow \) (1). Following [33], without loss of generality, we may assume that \( f(0) = 0 \). By Proposition 4.27 in [34], we have that

\[
f(z) = \frac{1}{(a + 1)} \int_D \frac{f'(w)(1 - |w|^2)^{1+a}}{w(1 - zw)^{2+a}} dA(w), \ z \in D,
\]
where \( \alpha > 0 \). Set \( f(z) = f_1(z) + f_2(z) \), where
\[
f_1(z) = \frac{1}{(\alpha + 1)} \int_{\Omega,1'} f'(w)(1 - |w|^2)^{1+\alpha} \frac{dA(w)}{|w(1 - zw)^{2+\alpha}}
\]
and
\[
f_2(z) = \frac{1}{(\alpha + 1)} \int_{D,1'} f'(w)(1 - |w|^2)^{1+\alpha} \frac{dA(w)}{|w(1 - zw)^{2+\alpha}}.
\]
Clearly,
\[
|f_1'(z)| \leq \int_{\Omega,1'} \frac{|f'(w)(1 - |w|^2)^{1+\alpha}}{|w(1 - zw)^{3+\alpha}} dA(w)
\]
and
\[
|f_2'(z)| \leq \int_{D,1'} \frac{|f'(w)(1 - |w|^2)^{1+\alpha}}{|w(1 - zw)^{3+\alpha}} dA(w).
\]
Let \( F = f_1 - f_1(0) \). Then \( F(0) = 0 \) and
\[
\|f - F\|_B = \sup_{z \in D} (1 - |z|^2)|f_1'(z)|
\]
\[
\leq \sup_{z \in D} (1 - |z|^2) \int_{D,1'} \frac{|f'(w)(1 - |w|^2)^{1+\alpha}}{|w(1 - zw)^{3+\alpha}} dA(w)
\]
\[
\leq \epsilon \sup_{z \in D} (1 - |z|^2) \int_{D} (1 - |w|^2)^{\alpha} \frac{dA(w)}{1 - z\alpha^{3+\alpha}}.
\]
Using \cite[Lemma 3.10]{34} with \( t = \alpha \) and \( c = 1 \), we see that \( \|f - F\|_B \leq \epsilon \). It remains to prove that \( F \in B_p(\rho) \).

Using Fubini’s theorem, we have
\[
\int_{D} |f'(z)|^p (1 - |z|^2)^{\alpha - 2} \rho(1 - |z|^2) dA(z)
\]
\[
= \int_{D} |f'(z)|^p (1 - |z|^2)^{\alpha - 2} \rho(1 - |z|^2) dA(z)
\]
\[
\leq \|f_1\|_B^{p-1} \int_{D} |f_1'(z)| (1 - |z|^2)^{-1} \rho(1 - |z|^2) dA(z)
\]
\[
\leq \int_{D} (1 - |z|^2)^{-1} \rho(1 - |z|^2) \left[ \int_{\Omega,1'} \frac{|f'(w)(1 - |w|^2)^{1+\alpha}}{|w(1 - zw)^{3+\alpha}} dA(w) \right] dA(z)
\]
\[
\leq \int_{\Omega,1'} |f'(w)| (1 - |w|^2)^{1+\alpha} \left[ \int_{D} \frac{\rho(1 - |z|^2)}{|w(1 - zw)^{\alpha+3}(1 - |z|^2)} dA(z) \right] dA(w)
\]
\[
\leq \|f\|_B \int_{\Omega,1'} (1 - |w|^2)^{\alpha} \left[ \int_{D} \frac{\rho(1 - |z|^2)}{1 - z\alpha^{\alpha+3}(1 - |z|^2)} dA(z) \right] dA(w).
\]
Since \( \rho \) is of finite lower type greater than 0 and upper type less than 1, there exist \( \gamma \) and \( \delta \) with \( 0 < \gamma < \delta < 1 \), such that
\[
\rho(st) \leq \delta^s \rho(t), \quad s \leq 1
\]
and
\[
\rho(st) \leq \delta^s \rho(t), \quad s \geq 1,
\]
where \(0 < t < \infty\). Using this and [34, Lemma 3.10], we obtain
\[
\int_{D} \frac{\rho(1-|z|^{2})}{|1-\bar{z}w|^{s+3}(1-|z|^{2})} dA(z) \\
= \rho(1-|w|^{2}) \int_{D} \frac{\rho(1-|z|^{2})}{|1-\bar{z}w|^{s+3}(1-|z|^{2})} dA(z) \\
\leq \rho(1-|w|^{2}) \int_{D} \left( \frac{1}{|1-\bar{z}w|^{s+3}} \right)^{\gamma} (1-|z|^{2})^{\delta} dA(z) \\
\leq \frac{\rho(1-|w|^{2})}{(1-|w|^{2})^{s+2}}.
\]
Combining this with (2), we have
\[
\int_{D} |f'(z)|^{p}(1-|z|^{2})^{p-2} \rho(1-|z|^{2}) dA(z) \leq \int_{D} \rho(1-|z|^{2}) (1-|z|^{2})^{p} dA(z) < \infty.
\]
Hence, \(F \in B_{p}(\rho)\). This finishes the proof.

(1) \(\Rightarrow\) (3). It is well known that \(\|f\|_{B_{p}}\) is equivalent to
\[
\|f\|_{B_{p}} = \|f(0)\| + \left( \sup_{z \in D} \int_{D} |f'(w)|^{p}(1-|w|^{2})^{p-2} (1-|\varphi_{\omega}(w)|^{2})^{\gamma} dA(w) \right)^{1/p},
\]
where \(p > 1\) and \(s > 1\). Let \(f \in C_{B}(B_{p}(\rho) \cap \mathcal{B})\). Then for any \(\epsilon > 0\), there exists \(g \in B_{p}(\rho) \cap \mathcal{B}\) such that \(\|f-g\|_{B_{p}} \leq \frac{\epsilon}{C_{3}}\). For any \(z \in D\), we have
\[
\int_{D} |f'(w)|^{p}(1-|w|^{2})^{p-2} (1-|\varphi_{\omega}(w)|^{2})^{\gamma} dA(w) \\
\leq C \int_{D} |f'(w) - g'(w)|^{p}(1-|w|^{2})^{p-2} (1-|\varphi_{\omega}(w)|^{2})^{\gamma} dA(w) + C \int_{D} |g'(w)|^{p}(1-|w|^{2})^{p-2} (1-|\varphi_{\omega}(w)|^{2})^{\gamma} dA(w).
\]
Thus, \(\Gamma_{P_{r},f}(g) \leq \Gamma_{P_{r},\varphi_{\omega}}(g)\). Note that
\[
1-|\varphi_{\omega}(w)|^{2} = \frac{(1-|z|^{2})(1-|w|^{2})}{|1-\bar{z}w|^{2}}.
\]
Using Fubini’s theorem, we have
\[
\int_{\Gamma_{P_{r},f}} \frac{\rho(1-|z|^{2})}{(1-|z|^{2})^{2}} dA(z) \\
\leq 2^{p} C \int_{P_{r},f} \left( 1-|z|^{2} \right)^{-2} \rho(1-|z|^{2}) \left[ \int_{D} |g'(w)|^{p}(1-|w|^{2})^{p-2} \frac{(1-|w|^{2})^{\gamma}}{|1-\bar{z}w|^{2}} dA(w) \right] dA(z) \\
\leq \int_{D} |g'(w)|^{p}(1-|w|^{2})^{p-2} \left[ \int_{D} \left( 1-|z|^{2} \right)^{-2} \rho(1-|z|^{2}) \frac{(1-|w|^{2})}{|1-\bar{z}w|^{2}} dA(z) \right] dA(w).
\]
Combining (A) with (B), we deduce that
\[
\int_{D} (1-|z|^{2})^{s-2} \rho(1-|z|^{2}) \frac{(1-|w|^{2})}{|1-\bar{z}w|^{2s}} dA(z) \leq \frac{\rho(1-|w|^{2})}{(1-|w|^{2})^{s}}, \quad s > 1.
\]
Thus,
\[
\int_{\Gamma_{P_{r},f}} \frac{\rho(1-|z|^{2})}{(1-|z|^{2})^{2}} dA(z) \leq \int_{D} |g'(w)|^{p}(1-|w|^{2})^{p-2} \rho(1-|w|^{2}) dA(w) < \infty.
\]
(3) ⇒ (2). Let \( E(z, 1/2) = \{ w \in \mathbb{D} : |\varphi_z(w)| < 1/2 \} \) be a pseudo-hyperbolic disk of center \( z \in \mathbb{D} \) and radius 1/2. Recall that
\[
1 - |w| \approx |1 - \overline{z}w| \approx 1 - |z|, \quad w \in E(z, 1/2)
\]
and \( |E(z, 1/2)| \approx (1 - |z|^2) \) (see [34, Page 69]). Using the subharmonicity of \(|f'|^p\), we obtain
\[
\int_D |f'(w)|^p (1 - |w|^2)^{p-2} (1 - |\varphi_z(w)|^2) \, dA(w)
\geq \int_{E(z, 1/2)} |f'(w)|^p (1 - |w|^2)^{p-2} (1 - |\varphi_z(w)|^2) \, dA(w)
\geq (1 - |z|^2)|f'(z)|^p.
\]
Therefore, \( \Omega_\epsilon(f) \subseteq \Gamma_{p, \eta}(f) \). The proof is complete.

\(\square\)

3. Interpolating Blaschke product in \( C_\beta(B_p(\rho) \cap \mathcal{B}) \) as multipliers

An analytic function in the unit disc \( \mathbb{D} \) is called an inner function if it is bounded and has radial limits of absolute value 1 at almost every point of the boundary \( \partial \mathbb{D} \). It is well known that every inner function has a factorization \( e^{i\gamma} B(z) S(z) \), where \( \gamma \in \mathbb{R} \), \( B(z) \) is a Blaschke product and \( S(z) \) is a singular inner function. A Blaschke product \( B \) with sequence of zeros \( \{ a_k \}_{k=1}^\infty \) is called interpolating if there exists a positive constant \( \delta \) such that
\[
\prod_{j=0}^{\infty} |\varphi_{a_j}(a_0)| \geq \delta, \quad k = 1, 2, \cdots.
\]
We also say that \( \{ a_k \}_{k=1}^\infty \) is an interpolating sequence or an uniformly separated sequence. The following notions was introduced by Dyakonov [14]:

Suppose \( X \) and \( Y \) are two classes of analytic functions on \( \mathbb{D} \), and \( X \subseteq Y \). Let \( \theta \) be an inner function, \( \theta \) is said to be \((X, Y)\)-improving, if every function \( f \in X \) satisfying \( f \theta \in Y \) must actually satisfy \( f \theta \in X \). For more information related to improving multipliers, we refer to [27]. Motivated by the works of Dyakonov and Peláez, we have the following result.

**Theorem 2.** Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( 1 < p < \infty \) and \( B(z) \) is an interpolating Blaschke product with zeros \( \{ a_k \}_{k=1}^\infty \). Then following are equivalent:

1. \( B \in C_\beta(B_p(\rho) \cap \mathcal{B}) \).
2. \( \sum_{k=1}^{\infty} \rho(1 - |a_k|^2) < \infty \).
3. \( B \) is \((B_p(\rho) \cap \text{BMOA}), \text{BMOA}\)-improving.
4. \( B \) is \((B_p(\rho) \cap \text{BMOA}), \mathcal{B}\)-improving.

Before we get into the proof, we need some lemmas.

**Lemma 1.** ([25, Lemma 2.5]) Let \( s > -1, r, t > 0, \) and \( t < s + 2 < r \). Then
\[
\int_D \frac{(1 - |w|^2)^r}{|1 - \overline{z}w|^t |1 - \overline{w}z|^t} \, dA(w) \leq \frac{(1 - |z|^2)^{2r-s-t}}{|1 - \overline{w}z|^t}, \quad z, \xi \in \mathbb{D}.
\]

**Lemma 2.** Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( f \in H(\mathbb{D}) \) and \( a \in \mathbb{D} \), then
\[
\int_D |f(z) - f(0)|^p \rho \left[ \frac{1 - |\varphi_a(z)|^2}{1 - |z|^2} \right] \, dA(z)
\leq \int_D |f'(z)|^p (1 - |z|^2)^{p-1} \rho \left[ 1 - |\varphi_a(z)|^2 \right] \, dA(z).
\]
Proof. Let \( \epsilon > 0 \) be sufficiently small. From the proof of Lemma 2.1 of [9], we see that
\[
|f(z) - f(0)|^p \leq \left( \int_{\mathbb{D}} |f'(w)|^p \frac{(1 - |w|^2)^{(2 + \rho - 1)}}{|1 - \overline{w}z|^{2 + \rho}} dA(w) \right) (1 - |z|^2)^{-(p - 1)},
\]
where \( \sigma - \epsilon < -1 \). Using Fubini's theorem, we have
\[
\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{1 - |\varphi_z(z)|^2}{1 - |z|^2} dA(z)
\]
\[
\leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f'(w)|^p \frac{(1 - |w|^2)^{(2 + \rho - 1)}}{|1 - \overline{w}z|^{2 + \rho}} dA(w) \right) \frac{1 - |\varphi_z(z)|^2}{1 - |z|^2} dA(z)
\]
\[
= \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p - 1} \left( \int_{\mathbb{D}} \frac{1 - |\varphi_z(z)|^2}{(1 - |z|^2)^{1 + \rho - 1}|1 - \overline{w}z|^{2 + \rho}} dA(z) \right) dA(w).
\]
Using conditions (A) and (B), combining (C) with Lemma 1, we deduce
\[
\int_{\mathbb{D}} \frac{1 - |\varphi_z(z)|^2}{(1 - |z|^2)^{1 + \rho - 1}|1 - \overline{w}z|^{2 + \rho}} dA(z)
\]
\[
= \rho \left( 1 - |\varphi_z(w)|^2 \right) \int_{\mathbb{D}} \frac{1 - |\varphi_z(z)|^2}{(1 - |w|^2)^{1 + \rho - 1}|1 - \overline{w}z|^{2 + \rho}} dA(z)
\]
\[
\leq \rho \left( 1 - |\varphi_z(w)|^2 \right) \int_{\mathbb{D}} \left( \frac{1 - |\varphi_z(z)|^2}{(1 - |z|^2)^{1 + \rho - 1}|1 - \overline{w}z|^{2 + \rho}} \right)^{\gamma} + \left( \frac{1 - |\varphi_z(z)|^2}{(1 - |z|^2)^{1 + \rho - 1}|1 - \overline{w}z|^{2 + \rho}} \right)^{\delta} dA(z)
\]
\[
\leq \rho \left( 1 - |\varphi_z(w)|^2 \right) (1 - |w|^2)^{-1 + \rho - 1},
\]
where \( \gamma + \epsilon(p - 1) < \delta + \epsilon(p - 1) < 1 \). Thus,
\[
\int_{\mathbb{D}} |f(z) - f(0)|^p \frac{1 - |\varphi_z(z)|^2}{1 - |z|^2} dA(z)
\]
\[
\leq \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p - 1} \rho (1 - |\varphi_z(w)|^2) dA(w).
\]
The proof is complete. \( \square \)

**Lemma 3.** ([23]) Let \( \{a_k\}_{k=1}^\infty \) be a sequence in \( \mathbb{D} \). Then the measure \( d\mu_n = \sum_{k=1}^\infty (1 - |a_k|^2) \delta_{a_k} \) is a Carleson measure, i.e.
\[
\sup_{w \in \mathbb{D}} \sum_{k=1}^\infty (1 - |\varphi(w)(a_k)|^2) < \infty,
\]
if and only if \( \{a_k\}_{k=1}^\infty \) is a finite union of interpolating sequences.

**Lemma 4.** Let \( \rho \) be of finite lower type greater than 0 and upper type less than 1. Suppose that \( 1 < p < \infty \), \( B(z) \) is an interpolating Blaschke product with zeros \( \{a_k\}_{k=1}^\infty \) and \( f \in B_p(\rho) \). If \( \sum_{k=1}^\infty |f(a_k)|^p \rho (1 - |a_k|^2) < \infty \), then \( fB \in B_p(\rho) \).
Proof. Suppose that \( f \in B_p(\rho) \) and \( B(z) \) is an interpolating Blaschke product with zeros \( \{a_k\}_{k=1}^{\infty} \). Since

\[
\int_{\mathcal{D}} |(fB)'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z)
\leq \int_{\mathcal{D}} |f'(z)|^p |B(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) + \int_{\mathcal{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z)
\leq \int_{\mathcal{D}} |f'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) + \int_{\mathcal{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z)
\]

It is enough to prove

\[
\int_{\mathcal{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z) < \infty.
\]

Notice the fact that

\[(1-|z|^2) |B'(z)| \leq 1\]

and

\[|B'(z)| \leq \sum_{k=1}^{\infty} \frac{1-|a_k|^2}{1-\overline{a_k}z^2},\]

we have

\[
\int_{\mathcal{D}} |f(z)|^p |B'(z)|^p (1-|z|^2)^{p-2} \rho(1-|z|^2) dA(z)
\leq \int_{\mathcal{D}} |f(z)|^p |B'(z)|(1-|z|^2)^{-1} \rho(1-|z|^2) dA(z)
\leq \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathcal{D}} \frac{|f(a_k)|^p}{|1-\overline{a_k}z|^2} (1-|z|^2)^{-1} \rho(1-|z|^2) dA(z)
\leq \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathcal{D}} \frac{|f(x) - f(a_k)|^p}{|1-\overline{a_k}z|^2} (1-|z|^2)^{-1} \rho(1-|z|^2) dA(z)
= M + N.
\]

Since

\[
\int_{\mathcal{D}} \frac{\rho(1-|z|^2)}{|1-\overline{a_k}z|^2(1-|z|^2)} dA(z) \leq \frac{\rho(1-|a_k|^2)}{(1-|a_k|^2)},
\]

we deduce that

\[
M := \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathcal{D}} \frac{|f(a_k)|^p}{|1-\overline{a_k}z|^2(1-|z|^2)} \rho(1-|z|^2) dA(z)
\leq \sum_{k=1}^{\infty} |f(a_k)|^p \rho(1-|a_k|^2) < \infty.
\]

Making the change of variables \( z = \varphi_n(w) \), we obtain

\[
N := \sum_{k=1}^{\infty} (1-|a_k|^2) \int_{\mathcal{D}} \frac{|f(z) - f(a_k)|^p}{|1-\overline{a_k}z|^2(1-|z|^2)} \rho(1-|z|^2) dA(z)
= \sum_{k=1}^{\infty} \int_{\mathcal{D}} |f \circ \varphi_n(w) - f \circ \varphi_n(0)|^p \frac{\rho(1-|\varphi_n(w)|^2)}{(1-|w|^2)} dA(w).
\]
Since we obtain \[ \delta > \}\quad (18, \text{Page 681}), \] we know that there exist a

Proof.

Proof of Theorem 2.

Using Fubini’s theorem and Lemma 2, we have

\[
N = \sum_{k=1}^{\infty} \int_{E} |f \circ \varphi_{a_k}(w) - f \circ \varphi_{a_k}(0)| \rho(1 - |\varphi_{a_k}(w)|^2) \frac{dA(w)}{(1 - |w|^2)}
\]

\[
\leq \sum_{k=1}^{\infty} \int_{E} |(f \circ \varphi_{a_k})'(w)| \rho(1 - |\varphi_{a_k}(w)|^2)(1 - |w|^2)^{p-1} dA(w)
\]

\[
= \sum_{k=1}^{\infty} \int_{E} |f'(w)|((1 - |w|^2)^{p-2} \rho(1 - |w|^2)(1 - |\varphi_{a_k}(w)|^2) dA(w).
\]

Since \( \{a_k\}_{k=1}^{\infty} \) is an interpolating sequences, using Lemma 3, we have \( N \leq \|f\|_{A_p}^p \), that is,

\[
\int_{E} \int_{E} |f(z)||B(z)|((1 - |z|^2)^{2-p} \rho(1 - |z|^2)dA(z)
\]

\[
\leq \sum_{k=1}^{\infty} \|f(a_k)|^p \rho(1 - |a_k|^2) + \|f\|_{A_p}^p.
\]

The proof is complete. \( \square \)

We also need the following lemma.

Lemma 5. ([13, Theorem 1]) If \( f \in BMOA \) and \( \theta \) is an inner function, then the following conditions are equivalent:

1. \( \theta \in \text{BMOA} \);
2. \( \sup_{z \in E} |f(z)|((1 - |\theta(z)|^2) < \infty; \)
3. \( \sup_{z \in \mathbb{D}(\theta, \epsilon)} |f(z)| < \infty, \text{for every } \epsilon, 0 < \epsilon < 1; \)
4. \( \sup_{z \in \mathbb{D}(\theta, \epsilon)} |f(z)| < \infty, \text{for some } \epsilon, 0 < \epsilon < 1. \)

Proof of Theorem 2.

Proof. \( (1) \Rightarrow (2). \) Let \( B \) be an interpolating Blaschke product with zeros \( \{a_k\}_{k=1}^{\infty} \) and \( \theta \in C_{\mathbb{D}}(\mathbb{B} \cap \mathbb{B}_{\rho}(\rho)) \). From [18, Page 681], we know that there exist a \( \delta > 0 \), such that

\[
(1 - |z|^2)|B(z)| \geq \frac{\delta(1 - \delta)}{8}, \quad z \in E(a_k, \frac{\delta}{4}).
\]

Thus,

\[
\bigcup_{k=1}^{\infty} E(a_k, \frac{\delta}{4}) \subseteq \left\{ z \in \mathbb{D} : (1 - |z|^2)|B(z)| \geq \frac{\delta(1 - \delta)}{8} \right\}.
\]

Since \( \bigcup_{k=1}^{\infty} E(a_k, \frac{\delta}{4}) \) are pairwise disjoint, using the fact that

\[
|E(a_k, \frac{\delta}{4})| \approx (1 - |z|^2)^2, \quad z \in E(a_k, \frac{\delta}{4}),
\]

we obtain

\[
\sum_{k=1}^{\infty} \rho(1 - |a_k|^2) \leq \sum_{k=1}^{\infty} \int_{E(a_k, \frac{\delta}{4})} \rho(1 - |z|^2) \frac{dA(z)}{(1 - |z|^2)^2}
\]

\[
\leq \int_{|z| \in \mathbb{D}(1 - |\theta(\epsilon)||B(\epsilon)||E(\frac{\delta}{4})|} \rho(1 - |z|^2) \frac{dA(z)}{(1 - |z|^2)^2} < \infty.
\]
(2) ⇒ (3). Suppose that \( f \in B_p(\rho) \cap BMOA \) and \( fB \in BMOA \). We only need to prove that \( fB \in B_p(\rho) \). Using Lemma 5, we obtain
\[
\sum_{k=1}^{\infty} |f(a_k)|^p \rho(1-|a_k|^2) \leq \sup_{a} |f(a_k)|^p \sum_{k=1}^{\infty} \rho(1-|a_k|^2) < \infty.
\]
By Lemma 4, we have \( fB \in B_p(\rho) \).

(3) ⇒ (4). Let \( f \in B_p(\rho) \cap BMOA \subseteq BMOA \) and \( fB \in B \). From [27, Corollary 1], we see that every interpolating Blaschke product \( B \) is \((BMOA, B)\)-improving. Hence, we have \( fB \in BMOA \). Notice that \( B \) is \((B_p(\rho) \cap BMOA, BMOA)\)-improving, we have \( fB \in B_p(\rho) \cap BMOA \). Thus, \( B \) is \((B_p(\rho) \cap BMOA, B)\)-improving.

(4) ⇒ (1). Suppose that \( B \) is \((B_p(\rho) \cap BMOA, B)\)-improving. Note that \( 1 \in B_p(\rho) \cap BMOA \) and \( B \in H^\infty \subseteq B \). Thus, \( B \in B_p(\rho) \cap BMOA \subseteq B_p(\rho) \cap B \subseteq C_B(\rho) \cap B \). The proof is complete.

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