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A Unified Common Fixed Point Theorem for a Family of Mappings

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Abstract. The main objective of the paper is to prove some unified common fixed point theorems for a family of mappings under a minimal set of sufficient conditions. Our results subsume and improve a host of common fixed point theorems for contractive type mappings available in the literature of the metric fixed point theory. Simultaneously, we provide some new answers in a general framework to the problem posed by Rhoades (Contemp Math 72, 233-245, 1988) regarding the existence of a contractive definition which is strong enough to generate a fixed point, but which does not force the mapping to be continuous at the fixed point. Concrete examples are also given to illustrate the applicability of our proved results.

1. Introduction and preliminaries

One of the fundamental questions in fixed point theory is to seek or identify sufficient conditions which on imposing on the set X and/or the mapping T, assure a nonempty set of fixed points, i.e., $Fix(T) \neq \phi$. Common fixed point theorems are natural extensions of fixed point theorems. It is more efficient to study fixed point theorems for a pair or a family of mappings satisfying some conditions rather than fixed point theorems satisfying an individual mapping. These conditions are generally sufficient conditions and include continuity or weaker form of continuity, containment of range of the mappings, a noncommuting condition besides a contractive condition and every substantial common fixed point theorem attempts to minimize the set of conditions by weakening one or more of these sufficient conditions. In addition to ensuring existence of a common fixed point, it may be necessary to prove its uniqueness. From a computational view, a constructive algorithm to calculate the value of a common fixed point is desirable. Such algorithms often require iterates of the given mappings.

The interdependence of common fixed points and commuting mappings was first observed by Jungck [24]. His result in the setting of complete metric spaces yields an abstraction of the Banach contraction principle and partially answers the historical open question (see [6, 21]): For a pair of commuting self mappings (S, T) on the $[0, 1]$, what additional conditions guarantee that S and T have a common fixed point?

Jungck's result motivated researcher to investigate common fixed point theorems for commuting and noncommuting pairs of mappings satisfying contractive conditions. The constructive technique used by

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Jungck has been further improved and extended by other researchers to establish common fixed point theorems for three, four and sequence of mappings.

Definition 1.1. *Let (X, d) be a metric space with $S, T : X \to X$. A pair of mappings (S, T) is said to be:*

(1) compatible [25] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

(2) α-compatible [56] if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0, \lim_{n\to\infty} d(SSx_n, TSx_n) = 0, \lim_{n\to\infty} d(STx_n, TTx_n) = 0, \lim_{n\to\infty} d(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

(3) quasi-α-compatible [56] provided every sequence $\{x_n\}$ in X satisfying $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = t$ for some t in X splits into most four subsequences such that any of these subsequences, say $\{x_{n_i}\}$, satisfies at least one of the four conditions $\lim\limits_{n_i\to\infty} d(STx_{n_i}, TSx_{n_i}) = 0$, $\lim\limits_{n_i\to\infty} d(SSx_{n_i}, TSx_{n_i}) = 0$, $\lim\limits_{n_i\to\infty} d(STx_{n_i}, TTx_{n_i}) = 0$, $\lim\limits_{n_i\to\infty} d(SSx_{n_i}, TTx_{n_i}) = 0$.

(4) semi α-compatible [57] if $\lim\limits_{n\to\infty} d(STx_n, TSx_n) = 0$ or $\lim\limits_{n\to\infty} d(SSx_n, TSx_n) = 0$ or $\lim\limits_{n\to\infty} d(STx_n, TTx_n) = 0$ or $\lim\limits_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X satisfying $Sx_n, Tx_n \in SX \cap TX$ and $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = t$ for some t in X.

(5) weakly compatible [26] if, for all $x \in X$, $Sx = Tx$ implies that $STx = TSx$.

(6) non-trivially weakly compatible if S and T commute on the set of coincidence points, whenever $C(S, T) = \{x \in X : Sx = Tx\} \neq \phi$, i.e., the set of coincidences is nonempty.

Remark 1.2. *It is well-known that the compatibility implies quasi-α-compatibility or semi α-compatibility but the converse need not be true [57]. However, quasi-α-compatibility and semi α-compatibility are independent to each other [57]. A systematic study of the relationship between various noncommuting conditions can be found in [20].*

Remark 1.3. *It may be observed that weakly compatible mappings commute at all the coincidence points, hence a minimal noncommuting condition for the existence of common fixed point for contractive type mapping pairs. However, the notion of semi α-compatibility is useful not only in establishing the existence of a coincidence point but also implies commutativity at coincidence points.*

Common fixed point theorems for a sequence of mappings have been studied by several authors. The best known results along these lines are the following theorems which encompass most of the results established in the literature of metric fixed point theory.

Theorem 1.4. *(Jungck et al. [27]) Let (X, d) be a complete metric space and S,T selfmaps of X with S or T continuous. Suppose their exist a sequence $\{A_i\}$ of selfmaps of X satisfying*

(i) either $A_i : X \to SX \cap TX$ for each i; or
(i') $S, T : X \to \cap_i A_i X$;
(ii) each A_i is compatible with S and T;
(iii) for any $\epsilon > 0$ their exist a $\delta > 0$, δ being lower semicontinuous, such that

$$\epsilon \leq \max\{d(Sx, Ty), d(A_i x, Sx), d(A_j y, Ty), [d(A_i x, Ty) + (A_j y, Sx)]/2\} < \epsilon + \delta \Rightarrow d(A_i x, A_j y) < \epsilon.$$

Then all the A_i, S and T have a unique common fixed point.

Theorem 1.4 is actually a correction of the result of Rhoades et al. [68].

Theorem 1.5. *(Jachymski [22]) Let S and T be selfmaps of a complete metric space (X, d) and either S or T continuous. Let $\{A_i\}_{i=0}^{\infty}$ be a sequence of selfmaps of X satisfying*

(i) $A_0X \subset TX$, $A_iX \subset SX$ *for* $i \in \mathbb{N}$;
(ii) *pairs of* (A_0, S) *and* (A_i, T), $i \in \mathbb{N}$, *are compatible;*
(iii) *for each* $i \in \mathbb{N}$ *there exists an upper semicontinuous function* $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\phi_i(t) < t$ *for all* $t > 0$ *and for any* $x, y \in X$,

$$d(A_0x, A_iy) \le \phi_i(\max\{d(Sx, Ty), d(A_0x, Sx), d(A_iy, Ty), [d(A_0x, Ty) + d(A_iy, Sx)]/2\}).$$

Then all the A_i, $i \in \mathbb{N} \cup \{0\}$, S and T have a unique common fixed point.

The following key lemma connects Theorems 1.4 and 1.5.

Lemma 1.6. *[22] Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a metric space (X, d). For any $x, y \in X$ and $i, j \in \mathbb{N}$ define*

$$M_{ij}(x, y) = \max\{d(Sx, Ty), d(A_ix, Sx), d(A_jy, Ty), [d(A_ix, Ty) + d(A_jy, Sx)]/2\}).$$

Then the following statements are same:

(I) There exists a lower semi continuous function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $\epsilon > 0, \delta(\epsilon) > \epsilon$ and for $x, y \in X$ and $i, j \in \mathbb{N}$ with $i \neq j$

$$\epsilon \le M_{ij}(x, y) < \delta(\epsilon) \text{ implies } d(A_ix, A_jy) < \epsilon.$$

(II) There exists an upper semi continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ϕ is nondecreasing, $\phi(t) < t$ for all $t > 0$, and

$$d(A_ix, A_jy) \le \phi(M_{ij}(x, y)), \text{ for } x, y \in X \text{ and } i, j \in \mathbb{N} \text{ with } i \neq j.$$

In 1996, Pant [53] proved the following theorem which is one of the most general fixed point theorem for a sequence of mappings.

Theorem 1.7. *(Pant [53]) Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) and any one of A_1, A_2, S and T be continuous such that*

(i) $A_1X \subset TX$, $A_2X \subset SX$;
(ii) *pairs of* (A_1, S) *and* (A_2, T) *are compatible;*
(iii) *there exists an upper semicontinuous function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\phi(t) < t$ *for all* $t > 0$ *and for any* $x, y \in X$
$d(A_1x, A_2y) \le \phi(\max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Ty) + d(A_iy, Sx)]/2\}))$;
(iv) $d(A_1x, A_iy) < \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Ty) + d(A_iy, Sx)]/2\})$.

Then all the A_i, S and T have a unique common fixed point.

If S and T are self-mappings of a metric space (X, d) and if $\{x_n\}$ is a sequence in X such that $Sx_n = Tx_{n+1}$, $n = 0, 1, 2, \ldots$, then the set $O(x_0, S, T) = \{Sx_n : n = 0, 1, 2, \ldots\}$ is called the (S, T)-orbit at x_0 and T (or S) is called (S, T)-orbitally continuous [12] if $\lim_n Sx_n = z$ implies $\lim_n TSx_n = Tz$ (or $\lim_n Sx_n = z$ implies $lim_n SSx_n = Sz$).

The main objective of this paper is to prove common fixed point theorems for a family of mappings satisfying a minimal set of sufficient conditions. Our results generalize the results of Ćirić [12, 13], Fisher [16–18], Boyd and Wong [7], Agarwal et al. [1], Husain and Sehgal [32], Browder [9], Chang [10], Jungck [24–26], Jungck et al. [27–29], Jachymski [22, 23], Sessa [70], Sessa et al. [72], Pant [53, 57], Pathak et al. [60, 61], Pathak and Khan [62], Singh [73], Singh and Singh [74], Singh and Tiwari [75], Hadjic [30], Iseki [34], Kaneko [35], Khan [38], Khan et al. [39, 40], Kubiak [41], Matkowski [43], Mukherjee [49], Kang and Kim [37], Meir and Keeler [44], Maiti and Pal [42], Park and Bae [58], Park and Moon [59], Rao and Rao [63, 64], Ray [65], Reilly [66], Rhoades et al. [68], Sehgal [70], Rus [69], Yeh [76, 77] and many others.

2. Main Results

In the following theorems we shall denote

$$M_{1i}(x,y) = \max\{d(Sx,Ty), d(A_1x,Sx), d(A_iy,Ty), [d(A_1x,Ty) + d(A_iy,Sx)]/2\}$$

for any $x, y \in X$. Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for all $t > 0$.

Theorem 2.1. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that*

(i) $A_1X \subset TX$, $A_iX \subset SX$ *when* $i > 1$;
(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x,y))$;
(iii) $d(A_1x, A_iy) < M_{1i}(x,y)$, *whenever* $M_{1i}(x,y) > 0$.

Let S be semi $\alpha-$compatible with A_1 and T be semi $\alpha-$compatible with A_k for some $k > 1$. If the mappings in one of the semi $\alpha-$compatible pairs (A_1, S) or (A_k, T) are orbitally continuous, then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i). if $A_1x_{2n} = A_2x_{2n+1}$ or $A_2x_{2n+1} = A_1x_{2n+2}$ for some value of $n \in \mathbb{N} \cup \{0\}$, it becomes easier to establish the existence of the fixed point. So let us assume that $A_1x_{2n} \neq A_2x_{2n+1}$ and $A_2x_{2n+1} \neq A_1x_{2n+2}$, for every value of $n \in \mathbb{N} \cup \{0\}$, then by virtue of ($ii$), we obtain

(1) $d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n})$ and
(2) $d(y_{2n-1}, y_{2n}) \leq \phi(d(y_{2n-2}, y_{2n-1})) < d(y_{2n-2}, y_{2n-1})$.

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \geq 0$. Suppose $r > 0$. Then relation (1) on making $n \to \infty$ and in view of upper semi continuity of ϕ yields $r \leq \phi(r) < r$, a contradiction. Hence $r = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. We show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist an $\epsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\epsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \epsilon$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < \epsilon + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\epsilon$$

a contradiction. If m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) \geq \epsilon$ then

$$\epsilon \leq d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}) < \epsilon + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}).$$

That is, there exists integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \epsilon$ and

(3) $\lim_{n_i\to\infty} d(y_{n_i}, y_{m_i}) = \epsilon$.

Without loss of generality we can assume that n_i is odd and m_i even . Now, by virtue of (1), we have

$$d(y_{n_i+1}, y_{m_i+1}) \leq \phi(d(y_{n_i}, y_{m_i}) + d(y_{n_i}, y_{n_i+1}).$$

Now, on letting $n_i \to \infty$ and in view of (3) and upper semi continuity of ϕ, the above relation yields $\epsilon \leq \phi(\epsilon) < \epsilon$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = A_1x_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2} \to z.$$

We show that $A_ix_{2n+1} \to z$ for each $i > 2$. If $\lim_n A_ix_{2n+1} \neq z$ for some $i > 2$, then there exists a subsequence $\{A_ix_{2m+1}\}$ of $\{A_ix_{2n+1}\}$, a number $r > 0$ and a positive integer M such that for each $m \geq M$ we have $d(A_1x_{2m}, A_ix_{2m+1}) \geq r$, $d(A_ix_{2m+1}, z) \geq r$ and

$$d(A_1x_{2m}, A_ix_{2m+1}) < M_{1i}(x_{2m}, x_{2m+1}) = d(A_1x_{2m}, A_ix_{2m+1}),$$

a contradiction. Hence $A_i x_{2n+1} \to z$ for each $i > 1$.

Suppose that T is semi $\alpha-$compatible with A_k for some $k > 1$ and T and A_k are orbitally continuous. Then orbital continuity of A_k and T implies that $A_k T x_{2n+1} \to A_k z, A_k A_k x_{2n+1} \to A_k z, TA_k x_{2n+1} \to Tz$ and $TTx_{2n+1} \to Tz$. Semi $\alpha-$compatibility of A_k and T yields $\lim_{n\to\infty} d(A_k T x_{2n+1}, TA_k x_{2n+1}) = 0$ or $\lim_{n\to\infty} d(A_k T x_{2n+1}, TTx_{2n+1}) = 0$ or $\lim_{n\to\infty} d(TA_k x_{2n+1}, A_k A_k x_{2n+1}) = 0$ or $\lim_{n\to\infty} d(A_k A_k x_{2n+1}, TTx_{2n+1}) = 0$, whenever $\{x_n\}$ is a sequence in X satisfying $A_k x_n, Tx_n \in A_k X \cap TX$ and $\lim_{n\to\infty} A_k x_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = z$ for some z in X. Hence $A_k z = Tz$.

Since $A_k X \subset SX$, there exists a point u in X such that $A_k z = Su$. We show that $Su = A_1 u$. If not, then the inequality

$$d(A_1 u, A_k z) < M_{1k}(u, z) = d(A_1 u, A_k z)$$

yields a contradiction. Hence $Tz = A_k z = Su = Au$.

Semi $\alpha-$compatibility of A_1 and S implies that $A_1 A_1 u = A_1 Su = SA_1 u = SSu$. Similarly, Semi $\alpha-$compatibility of A_k and T implies that $A_k A_k z = A_k Tz = TA_k z = TTz$. If $A_1 u \neq A_1 A_1 u$ using (iii) we get

$$d(A_1 u, A_1 A_1 u) = d(A_1 A_1 u, A_k z) < M_{1k}(A_1 u, z) = d(A_1 A_1 u, A_k z),$$

a contradiction. Hence $A_1 u = A_1 A_1 u$ and $A_1 u = A_1 A_1 u = SA_1 u$, i.e., $A_1 u$ is a common fixed point of A_1 and S. Similarly, using (iii) we find that $A_k z (= A_1 u)$ is a common fixed point of A_k and T. Moreover, if $A_k z \neq A_i A_k z$ for some $i > 1$, using (iii) we get

$$d(A_k z, A_i A_k z) = d(A_1 u, A_i A_k z) < M_{1i}(u, A_k z) = d(A_1 u, A_i A_k z),$$

a contradiction. Hence $A_k z (= A_1 u)$ is a common fixed point of T and A_i for $i > 1$. The proof is similar when A_1 and S are assumed semi $\alpha-$compatible and orbitally continuous. Uniqueness of the common fixed point follows easily. □

Remark 2.2. *Theorem 2.1 is also true if we replace the condition (i), i.e.,* $A_1 X \subset TX$ *and* $A_i X \subset SX$ *when* $i > 1$ *by the following condition: Given* x_0 *in* X *there exist* x_1 *and* x_2 *in* X *such that* $A_1 x_0 = Tx_1$ *and* $A_i x_1 = Sx_2$ *when* $i > 1$.

Theorem 2.3. *Let* $\{A_i\}$, $i = 1, 2, 3, \ldots$, *S and T be selfmappings of a complete metric space* (X, d) *such that*

(i) $A_1 X \subset TX$, $A_i X \subset SX$ *when* $i > 1$;
(ii) $d(A_1 x, A_2 y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1 x, A_i y) < M_{1i}(x, y)$, *whenever* $M_{1i}(x, y) > 0$.

Let S be $\alpha-$compatible with A_1 and T be $\alpha-$compatible with A_k for some $k > 1$. If the mappings in one of the $\alpha-$compatible pairs (A_1, S) or (A_k, T) are orbitally continuous, then all the A_i, S and T have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 2.1. □

We now give a common fixed point theorem in which the notion of semi $\alpha-$compatibility is replaced by an independent notion of quasi $\alpha-$compatibility.

Theorem 2.4. *Let* $\{A_i\}$, $i = 1, 2, 3, \ldots$, *S and T be selfmappings of a complete metric space* (X, d) *such that*

(i) $A_1 X \subset TX$, $A_i X \subset SX$ *when* $i > 1$;
(ii) $d(A_1 x, A_2 y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1 x, A_i y) < M_{1i}(x, y), A_1 x \neq A_i y$.

Let S be quasi $\alpha-$compatible with A_1 and T be quasi $\alpha-$compatible with A_k for some $k > 1$. If the mappings in one of the quasi $\alpha-$compatible pairs (A_1, S) or (A_k, T) are orbitally continuous, then all the A_i, S and T have a unique common fixed point.

Proof. The proof can be written in the same lines of the proof of Theorem 2.1. □

The following corollaries are easy consequences of Theorem 2.1.

Corollary 2.5. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that*

(i) $A_1X \subset TX$, $A_iX \subset SX$ *when* $i > 1$;
(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1x, A_iy) < M_{1i}(x, y), A_1x \neq A_iy$.

Let S be quasi $\alpha-$compatible with A_1 and T be quasi $\alpha-$compatible with A_k for some $k > 1$. If the mappings in one of the quasi $\alpha-$compatible pairs (A_1, S) or (A_k, T) are continuous, then all the A_i, S and T have a unique common fixed point.

Corollary 2.6. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that*

(i) $A_1X \subset TX$, $A_iX \subset SX$ *when* $i > 1$;
(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1x, A_iy) < M_{1i}(x, y), A_1x \neq A_iy$.

Let S be quasi $R-$commuting with A_1 and T be quasi $R-$commuting with A_k for some $k > 1$. If the mappings in one of the quasi $R-$commuting pairs (A_1, S) or (A_k, T) are orbitally continuous, then all the A_i, S and T have a unique common fixed point.

Putting $k = 1, 2$ in Theorem 2.1, we get the following result for four mappings.

Theorem 2.7. *Let A_1, A_2, S and T be selfmappings of a complete metric space (X, d) such that for all $x, y \in X$,*

(i) $A_1X \subset TX$, $A_2X \subset SX$;
(ii) $d(A_1x, A_2y) \leq \phi(\max\{d(Sx, Ty), d(A_1x, Sx), d(A_2y, Ty), [d(A_1x, Ty) + d(A_2y, Sx)]/2\})$.

Let S be semi $\alpha-$compatible with A_1 and T be semi $\alpha-$compatible with A_2. If the mappings in one of the semi $\alpha-$compatible pairs (A_1, S) or (A_2, T) are orbitally continuous, then A_1, A_2, S and T have a unique common fixed point.

Taking $A_2 = A_1$ and $T = S$ in Theorem (2.7), we get the following fixed point theorem for a pair of mappings:

Theorem 2.8. *Let A_1 and S be selfmappings of a complete metric space (X, d) such that for all $x, y \in X$,*

(i) $A_1X \subset SX$;
(ii) $d(A_1x, A_1y) \leq \phi(\max\{d(Sx, Sy), d(A_1x, Sx), d(A_1y, Sy), [d(A_1x, Sy) + d(A_1y, Sx)]/2\})$.

Let S be semi $\alpha-$compatible with A_1 and (A_1, S) be orbitally continuous. Then A_1 and S have a unique common fixed point.

By putting in Theorem 2.7, $S = T = I$, the identity mapping on X, we get a distinct category of common fixed point theorems where we do not require the mappings to satisfy any commuting or noncommuting conditions.

Theorem 2.9. *Let A_1 and A_2 be selfmappings of a complete metric space (X, d) such that for all $x, y \in X$,*

$$d(A_1x, A_2y) \leq \phi(max\{d(x, y), d(A_1x, x), d(A_2y, y), [d(A_1x, y) + d(A_2y, x)]/2\}).$$

Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous function such that $\phi(t) < t$ for all $t > 0$. Then A_1 and A_2 have a unique common fixed point.

Putting $A_2 = A_1$ and $S = T = I$, the identity mapping on X in Theorem (2.9), we get the following result:

Corollary 2.10. *[Agarwal et al [1]] Let A_1 be a selfmapping of a complete metric space (X, d) such that for all $x, y \in X$,*

$$d(A_1x, A_1y) \leq \phi(max\{d(x, y), d(A_1x, x), d(A_1y, y), [d(A_1x, y) + d(A_1y, x)]/2\}).$$

Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous function such that $\phi(t) < t$ for all $t > 0$. Then A_1 has a unique fixed point.

Corollary 2.11 (Boyd and Wong [7]). *Let A_1 be a selfmapping of a complete metric space (X, d) such that for all $x, y \in X$,*

$$d(A_1x, A_1y) \leq \phi\{d(x, y)\}.$$

Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semicontinuous function such that $\phi(t) < t$ for all $t > 0$. Then A_1 has a unique fixed point.

In the next theorem we obtain a generalization of Theorem 2.1 by dropping the assumption on orbital continuity and semi $\alpha-$compatibility of the mappings and completeness of the space an replacing the later two by nontrivial weak compatibility and completeness of the range of one of the mappings.

Theorem 2.12. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a metric space (X, d) such that*

(i) $A_1X \subset TX$, $A_iX \subset SX$ when $i > 1$;
(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1x, A_iy) < M_{1i}(x, y)$, whenever $M_{1i}(x, y) > 0$.

Let S be nontrivially weakly compatible with A_1 and T be nontrivially weakly compatible with A_i for some $i > 1$. If the range of one of the mappings be a complete subspace of X, then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

Then proceeding exactly as in Theorem (2.1) it follows that $\{y_n\}$ is a Cauchy sequence and $\{y_{2n} = A_1x_{2n} = Tx_{2n+1}\}$ and $\{y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}\}$ are also Cauchy sequences. Suppose that the range of T is a complete subspace of X. Then, since $\{y_{2n} = Tx_{2n+1}\}$ is a Cauchy sequence in TX, there exists some $u \in X$ such that $Tx_{2n+1} \to Tu$. Thus $A_1x_{2n} = Tx_{2n+1} \to Tu$ and $A_2x_{2n+1} = Sx_{2n+2} \to Tu$. We now show that $A_2u = Tu$. If not, using (ii), for large values of n we get

$$d(A_1x_{2n}, A_2u) \leq \phi(M_{12}(x_{2n}, u)) = \phi(d(A_2u, Tu)).$$

On making $n \to \infty$ this yields $d(Tu, A_2u) \leq \phi(d(A_2u, Tu)) < d(Tu, A_2u)$, a contradiction. Hence $A_2u = Tu$. Since $A_2X \subset SX$, there exists some $w \in X$ such that $Tu = A_2u = sw$. We claim that $A_1w = Sw$. If $A_1w \neq Sw$, we have $d(A_1w, A_2u) < \phi(M_{12}(w, u)) = d(A_1w, A_2u)$, a contradiction. Hence $A_1w = A_2u = Tu = Sw$. If $A_1w \neq A_iu$, for some $i > 2$, by (iii) we get $d(A_1w, A_iu) < M_{12}(w, u) = d(A_1w, A_iu)$, a contradiction. Hence for each $i > 1$ we get $A_1w = A_iu = Tu = Sw$. Now nontrivial weak compatibility of A_1, S and A_i, T and the contractive conditions imply that A_1w is a common fixed point of A_1, A_k, S and T. Moreover, if $A_1w \neq A_iA_1w$ for some $i > 1$, we get $d(A_1w, A_iA_1w) < M_{12}(w, A_1w) = d(A_1w, A_iA_1w)$, a contradiction. Hence A_1w is a common fixed point of all A_i, S and T. The proof is similar when the range of S is assumed a complete subspace of X. This completes the proof of the theorem. □

Putting $k = 1, 2$ in Theorem 2.12, we get the following result for four mappings.

Theorem 2.13. *Let A_1, A_2, S and T be selfmappings of a complete metric space (X, d) such that for all $x, y \in X$,*

(i) $A_1X \subset TX$, $A_2X \subset SX$;
(ii) $d(A_1x, A_2y) \leq \phi(\max\{d(Sx, Ty), d(A_1x, Sx), d(A_2y, Ty), [d(A_1x, Ty) + d(A_2y, Sx)]/2\})$.

Let S be nontrivially weakly compatible with A_1 and T be nontrivially weakly compatible with A_2. If the range of one of the mappings be a complete subspace of X, then A_1, A_2, S and T have a unique common fixed point.

Taking $A_2 = A_1$ and $T = S$ in Theorem 2.13, we get the following fixed point theorem for a pair of mappings:

Theorem 2.14. *Let A_1 and S be selfmappings of a complete metric space (X, d) such that for all $x, y \in X$,*

(i) $A_1X \subset SX$;
(ii) $d(A_1x, A_1y) \leq \phi(\max\{d(Sx, Sy), d(A_1x, Sx), d(A_1y, Sy), [d(A_1x, Sy) + d(A_1y, Sx)]/2\})$.

Let S be nontrivially weakly compatible with A_1. If SX is a complete subspace of X, then A_1 and S have a unique common fixed point.

The following result is a consequence of Theorem 2.12, since nontrivially weakly compatible mappings are semi $\alpha-$compatible.

Theorem 2.15. *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that*

(i) $A_1X \subset TX$, $A_iX \subset SX$ *when* $i > 1$;
(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$;
(iii) $d(A_1x, A_iy) < M_{1i}(x, y)$, *whenever* $M_{1i}(x, y) > 0$.

Let S be semi $\alpha-$compatible with A_1 and T be semi $\alpha-$compatible with A_k for some $k > 1$. If the range of one of the mappings be a complete subspace of X, then all the A_i, S and T have a unique common fixed point.

3. Examples

The following examples illustrate Theorems 2.3 and 2.12.

Example 3.1. *Let $X = [2, 20]$ with usual metric d. Define mappings $A_i, S, T{:}X \to X$, $i = 1, 2, 3, \ldots$, by*
$A_1 = 2$, $\quad A_1x = 3$ *if* $x > 2$,
$A_2x = 2$ *if* $x = 2$ *or* $x > 5$, $A_2x = 6$ *if* $2 < x \leq 5$,
$S2 = 2 \quad Sx = 6$ *if* $x > 2$
$T2 = 2$, $\quad Tx = 12$ *if* $2 < x \leq 5$, $\quad Tx = x - 3$ *if* $x > 5$,
and for $i > 2$,
$A_ix = 2$ *if* $x \leq 2 + \frac{1}{i}$ *or* > 5, $\quad A_ix = 6$ *if* $2 + \frac{1}{i} < x \leq 5$.

Then $\{A_i\}$, S and T satisfy all the conditions of Theorems 2.3 and 2.12 and have a unique common fixed point $x = 2$. It is also easy to observe that A_1 and S are orbitally continuous and $\alpha-$compatible mappings. But neither A_1 nor S is continuous, not even at their common fixed point $x = 2$. It may also be verified that T and A_i are $\alpha-$compatible when $i > 2$. However, one can easily verified that T and A_i are quasi $\alpha-$compatible.

Example 3.2. *Let $X = [2, \infty)$ with usual metric d. Define mappings $A_i, S, T{:}X \to X$, $i = 1, 2, 3, \ldots$, by*
$A_1x = A_2x = 2$ *for all* x,
$Sx = 2$ *if* $x \geq 2$,
$T2 = 2x$ *if* $x \geq 3 \quad Tx = 2$ *if* $x < 3$,
and for $i > 2$,
$A_ix = 2(3 + \frac{1}{i})$ *if* $x > 3 + \frac{1}{i}$, $\quad A_ix = 2$ *if* $x \leq 3 + \frac{1}{i}$.

Then $\{A_i\}$, S and T satisfy all the conditions of Theorems 2.1 and 2.3 and have a unique common fixed point $x = 2$. It is also easy to observe that A_1 and S are orbitally continuous and $\alpha-$compatible mappings. It may also be verified that T and A_i are $\alpha-$compatible when $i > 2$. However, T and A_i are quasi $\alpha-$compatible when $i > 2$.

The following example [57] illustrates Theorem 2.8.

Example 3.3. *Let $X = [0,10]$ be equipped with the usual metric on X. Define mappings $A_1, S : X \to X$ by*

$$A_1x = (6-x)/2 \text{ if } x \le 2, \quad A_1x = 3 \text{ if } 2 < x \le 5, \quad A_1x = 2 \text{ if } x > 5,$$
$$Sx = x \text{ if } x \le 2, \quad Sx = 10 \text{ if } 2 < x \le 5, \quad Sx = (x+1)/3 \text{ if } x > 5.$$

Then A_1 and S satisfy all the conditions of Theorem 2.8 and have a unique common fixed point $x = 2$. It can be seen in this example that $\lim_{n\to\infty} d(A_1A_1x_n, SA_1x_n) = 0$, whenever $\{x_n\}$ is a sequence in X satisfying $A_1x_n, Sx_n \in A_1X \cap SX$ and $\lim_{n\to\infty} A_1x_n = \lim_{n\to\infty} Sx_n = t$ for some t in X. It can also be verified that A_1 and S satisfy the contractive condition $d(A_1x, A_1y) \le \phi(d(Sx, Sy))$ for all $x, y \in X$ whenever $\phi(t) = t/2$. Moreover, it is also easy to observe that A_1 and S are orbitally continuous mappings. It may be seen in this example that A_1 and S are neither compatible, nor A_1-compatible, nor $S-$compatible nor compatible of type (P).

Remark 3.4. *In Theorem 2.8, the notion of nontrivial weak compatibility can not be replaced by weak compatibility. The following example illustrates this fact:*

Example 3.5. *Let $X = [2,20]$ be equipped with the usual metric on X. Define mappings $A_1, S : X \to X$ by*

$$A_1x = 6 \text{ if } 2 \le x \le 5, \quad A_1x = (x+5)/5 \text{ if } x > 5,$$
$$Sx = 12 \text{ if } 2 \le x \le 5, \quad Sx = (x+1)/3 \text{ if } x > 5.$$

Then A_1 and S satisfy all the conditions of Theorem 2.8 but do not have a common fixed point. It can be seen in this example that A_1 and S are trivially weakly compatible.

Remark 3.6. *In Theorem 2.8 we cannot replace the notion of semi α-compatibility by weak compatibility. This can be seen from Example 3.5 above.*

4. Discussions

Our proved theorems apply to a wider class of mappings than the results of compatible and continuous maps since our theorems apply to semi $\alpha-$compatible or nontrivially weakly compatible and orbitally continuous maps also. Moreover, as compared to the analogous results, the present theorems have been proved under considerably weaker assumptions. A few observations regarding the above proved theorems are in order.

(i) Theorem 2.1 assumes S to be semi $\alpha-$compatible with A_1 and T to be compatible with A_i where $i > 1$. In comparison to this, Theorem 5.1 of Jachymski [22] assumes T to be compatible with A_i for each $i > 1$ while Rhoades et al. [68] (see also Jungck et al. [27]) assume each A_i to be compatible with both S and T. In Example 3.1, T and A_2 are not compatible.

(ii) In Theorem 2.1 for each $i > 2$ the mappings A_1, A_i satisfy the plain contractive condition $d(A_1x, A_iy) < M_{1i}(x, y)$. On the other hand, Theorem 5.1 of Jachymski [23] requires A_1, A_i, $i > 2$, to satisfy the contractive condition $d(A_1x, A_iy) < \phi_i(M_{1i}(x, y))$ where $\phi_i : R_+ \to R_+$ is an upper semicontinuous function such that $\phi_i(t) < t$ for each $t > 0$. This condition is not satisfied in Example 3.2 for any $i > 2$ since the required function ϕ_i would not be upper semicontinuous at $t = 4 + (2/i)$. Similarly the theorem of Rhoades et al. [68] (see also Jungck et al. [27]) requires each A_i, A_j to satisfy a Meir-Keeler type (ϵ, δ) contractive condition and δ is required to be lower semicontinuous. However, in Example 3.2 above, A_1 and A_i fail to satisfy the (ϵ, δ) condition at $\epsilon = 4 + (2/i)$ when $i > 2$.

(iii) Theorem 2.1 assumes the mappings in one of the semi $\alpha-$compatible pairs (A_1, S) or (A_k, T), where $k > 1$ be orbitally continuous. In comparison to this, Theorem 5.1 of Jachymski [22, 23] assumes S or T to be continuous while Rhoades et al. [27] (see also Jungck et al. [27]) also assumes S or T be continuous . Likewise, the theorems of Fisher [18] and Pant [53] assume one of the mappings to be continuous. In Example 3.1 none of the mappings are continuous.

Remark 4.1. *Our work provides a possibility to extend our proved results in other generalized metric spaces, namely, $b-$metric space, $b-$rectangular metric space, $b_v(s)-$ metric spaces[2, 3, 9, 14, 15, 19, 33, 45–48, 78].*

Remark 4.2. *In all the theorems established in this paper, we have not assumed any mapping to be continuous. In fact, the mappings assumed by us are not only discontinuous in their domain of definition but also discontinuous at their common fixed point. Thus, we provide more answers to the problem posed by Rhoades [67] regarding the existence of a contractive definition which generate a fixed point, but does not force the mapping to be continuous at the fixed point. The first answer to this problem was given by Pant [52]. The new answers of the Rhoades's problem are distinct from [4, 5, 57, 60, 62].*

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