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On Approximation of Bernstein-Durrmeyer-Type Operators in Movable Interval

Fengfeng Wang[a], Dansheng Yu[a]

[a]*Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China*

Abstract. In the present paper, we introduce a new type of Bernstein-Durrmeyer operators preserving linear functions in movable interval. The approximation rate of the new operators for continuous functions and Voronovskaja's asymptotic estimate are obtained.

1. Introduction

For any given $f \in C_{[0,1]}$, define

$$U_n(f,x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt, \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, k = 0,1,\cdots,n$, are the Bernstein fundamental polynomials. The operators $U_n(f,x)$ first appeared in papers of Goodman and Sharma [12] and Chen [4]. The operators $U_n(f)$ are limits of the Bernstein-Durrmeyer operators with Jacobi weights. One of the advantages of $U_n(f,x)$ compared to the usual Bernstein-Durrmeyer operators is that $U_n(f,x)$ can reproduce the linear functions. Lots of authors have done many excellent works on degree of approximation, Voronovskaja's asymptotic estimate, the eigenstructure of the $U_n(f,x)$ and its generalizations (see [2], [8]-[13], [20]-[22], [31]). Analogue constructions of various families of approximation operators such as Szász-Mirakjan operators, Szász-Mirakjan Beta type operators, Srivastava-Gupta operators, Bleimann-Butzer-Hann operators, and so on, were given by many mathematicians ([15], [23]-[26], [28]). The approximation properties of these operators are well investigated.

Recently, Srivastava, Özarslan and Mohiuddine ([27]) introduced a Stancu-type Bernstein operators with three parameters λ, α, β as follows:

$$B^{\lambda}_{n,\alpha,\beta}(f,x) := \sum_{k=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right)\tilde{p}_{n,k}(\lambda,x),$$

where

$$\tilde{p}_{n,0}(\lambda,x) = p_{n,0}(x) - \frac{\lambda}{n+1}p_{n+1,1},$$

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Corresponding author: Dansheng Yu
Email addresses: flfwang@163.com (Fengfeng Wang), dsyu@hznu.edu.cn (Dansheng Yu)

$$\tilde{p}_{n,k}(\lambda,x)=p_{n,k}(x)+\frac{n^2-2k+1}{n^2-1}\lambda p_{n+1,k}(x)-\frac{n-2k-1}{n^2-1}\lambda p_{n+1,k+1}(x),\ k=1,2,\cdots,n-1,$$

$$\tilde{p}_{n,n}(\lambda,x)=p_{n,n}(x)-\frac{\lambda}{n+1}p_{n+1,n}.$$

When $\lambda=0$, $B^{\lambda}_{n,\alpha,\beta}(f,x)$ reduce to the classical Bernstein-Stancu operators; When $\alpha=\beta=0$, $B^{\lambda}_{n,\alpha,\beta}(f,x)$ become the $\lambda-$Bernstein operator given by Cai et al ([3]). Srivastava, Özarslan and Mohiuddine ([27]) calculated the moments of $B^{\lambda}_{n,\alpha,\beta}(f,x)$ and prove global approximation formula in terms of Ditzian-Totik modulus of smoothness.

In 2010, Gadjiev and Ghorbanalizadeh ([7]) generalized Bernstein-Stancu operators by using shifted knots as follows:

$$S_{n,\alpha,\beta}(f;x)=\left(\frac{n+\beta_2}{n}\right)^n\sum_{k=0}^{n}f\left(\frac{k+\alpha_1}{n+\beta_1}\right)q_{n,k}(x), \tag{2}$$

where $x\in A_n:=\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$, and $q_{n,k}(x)=\binom{n}{k}\left(x-\frac{\alpha_2}{n+\beta_2}\right)^k\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)^{n-k}$, $k=0,1,\cdots,n$ with $\alpha_k,\beta_k,k=1,2$ are positive real numbers satisfying $0\le\alpha_1\le\beta_1$, $0\le\alpha_2\le\beta_2$. When $\alpha_2=\beta_2=0$, $S_{n,\alpha,\beta}(f;x)$ reduces to the well known Bernstein-Stancu operators, when $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, it reduces to the classical Bernstein operators.

It should be noted that , $S^*_{n,\alpha,\beta}(f,x)$ is positive and linear in the set A_n. Although, $S^*_{n,\alpha,\beta}(f,x)$ is also well defined on $[0,1]\backslash A_n$, but it is not positive in this case. Wang, Yu and Zhou ([30]) proved that $S_{n,\alpha,\beta}(f;x)$ can also be used to approximate the continuous functions on $[0,1]$, and obtained both the direct and the converse results. Many generalizations of $S_{n,\alpha,\beta}(f;x)$ such as its Kantorovich variant and Durrmeyer variant were introduced. Some studies on such operators can be founded in [1], [6], [14]-[19], [29] and [30].

In the present paper, our main goal is to introduce a new type of genuine Bernstein-Durrmeyer operators in movable interval A_n, and establish the approximation rate for continuous functions and Voronovskaja's asymptotic estimate of the new operators.

Our new operators are defined as follows:

$$U_n^{(\alpha,\beta)}(f,x):=\left(\frac{n+\beta}{n}\right)^n f\left(\frac{\alpha}{n+\beta}\right)q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n f\left(\frac{n+\alpha}{n+\beta}\right)q_{n,n}(x)$$

$$+\left(\frac{n+\beta}{n}\right)^n\lambda_{n-2}^{-1}\sum_{k=1}^{n-1}q_{n,k}(x)\int_{A_{n-2}}q_{n-2,k-1}(t)f\left(\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right)\right)dt, \tag{3}$$

where $0\le\alpha\le\beta$ are fixed numbers,

$$\lambda_{n-2}:=\left(\frac{n-2}{n-2+\beta}\right)^{n-1}\frac{1}{n-1},$$

$$q_{n,k}(x):=\binom{n}{k}\left(x-\frac{\alpha}{n+\beta}\right)^k\left(\frac{n+\alpha}{n+\beta}-x\right)^{n-k},\ k=0,1,\cdots,n.$$

For convenience, write $\bar{x}:=x-\frac{\alpha}{n+\beta}$. Then, $q_{n,k}(x)$ can be rewritten as follows:

$$q_{n,k}(x)=\binom{n}{k}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k},\ k=0,1,\cdots,n.$$

When $t\in A_{n-2}$, we have

$$\frac{n(n-2+\beta)}{(n-2)(n+\beta)}\left(t-\frac{2\alpha}{n(n-2+\beta)}\right)\in A_n.$$

So, $U_n^{(\alpha,\beta)}(f,x)$ is well defined in A_n. We will show that $U_n^{(\alpha,\beta)}(f,x)$ keeps the linear functions (see Lemma 4 in section 2). For the approximation rate of $U_n^{(\alpha,\beta)}(f,x)$ for $f\in C(A_n)$, we have

Theorem 1.1. *Let $0 \le \lambda \le 1$ be a fixed number. For any $f \in C(A_n)$, there is a positive constant only depending on λ, α and β such that*

$$\left| U_n^{(\alpha,\beta)}(f,x) - f(x) \right| \le C\omega_{\varphi^\lambda}^2 \left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{4}$$

where $\varphi(x) := \sqrt{\bar{x}\left(\frac{n}{n+\beta} - \bar{x}\right)}$.

We also have the following Voronovskaja's asymptotic estimate of $U_n^{(\alpha,\beta)}(f,x)$:

Theorem 1.2. *Let $f \in C(A_n)$. If f'' exists at a point $x \in A_n$, then*

$$\lim_{n\to\infty} n \left(U_n^{(\alpha,\beta)}(f,x) - f(x) \right) = 2\varphi^2(x) f''(x).$$

Throughout the paper, C denotes either a positive absolute constant or a positive constant may depend on some parameters but not on f, x and n . Their values may be deferent in different situations.

2. Auxiliary Lemmas

We need the following some auxiliary lemmas.

Lemma 2.1. *It holds that*

$$\sum_{k=1}^{n} \frac{k}{n} q_{n,k}(x) = \left(\frac{n}{n+\beta} \right)^{n-1} \bar{x}, \tag{5}$$

$$\sum_{k=1}^{n} \frac{k^2}{n^2} q_{n,k}(x) = \frac{n-1}{n} \left(\frac{n}{n+\beta} \right)^{n-2} \bar{x}^2 + \frac{1}{n} \left(\frac{n}{n+\beta} \right)^{n-1} \bar{x}. \tag{6}$$

$$\sum_{k=1}^{n} \frac{k^3}{n^3} q_{n,k}(x) = \frac{(n-1)(n-2)}{n^2} \left(\frac{n}{n+\beta} \right)^{n-3} \bar{x}^3 + \frac{3(n-1)}{n^2} \left(\frac{n}{n+\beta} \right)^{n-2} \bar{x}^2 + \frac{1}{n^2} \left(\frac{n}{n+\beta} \right)^{n-1} \bar{x}. \tag{7}$$

$$\begin{aligned} \sum_{k=1}^{n} \frac{k^4}{n^4} q_{n,k}(x) &= \frac{(n-1)(n-2)(n-3)}{n^3} \left(\frac{n}{n+\beta} \right)^{n-4} \bar{x}^4 + \frac{6(n-1)(n-2)}{n^3} \left(\frac{n}{n+\beta} \right)^{n-3} \bar{x}^3 \\ &\quad + \frac{7(n-1)}{n^3} \left(\frac{n}{n+\beta} \right)^{n-2} \bar{x}^2 + \frac{1}{n^3} \left(\frac{n}{n+\beta} \right)^{n-1} \bar{x}. \end{aligned} \tag{8}$$

Proof. Direct calculations yield that

$$\begin{aligned} \sum_{k=1}^{n} \frac{k}{n} q_{n,k}(x) &= \sum_{k=1}^{n} \binom{n-1}{k-1} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x} \right)^{n-k} \\ &= \bar{x} \sum_{k=0}^{n-1} \binom{n-1}{k} \bar{x}^k \left(\frac{n}{n+\beta} - \bar{x} \right)^{n-1-k} \\ &= \left(\frac{n}{n+\beta} \right)^{n-1} \bar{x}, \end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\frac{k^2}{n^2}q_{n,k}(x) &= \sum_{k=1}^{n}\frac{k}{n}\binom{n-1}{k-1}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &= \frac{n-1}{n}\sum_{k=1}^{n}\frac{k-1}{n-1}\binom{n-1}{k-1}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{1}{n}\sum_{k=1}^{n}\binom{n-1}{k-1}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &= \frac{n-1}{n}\bar{x}^2\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^{k-2}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{\bar{x}}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &= \frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sum_{k=1}^{n}\frac{k^3}{n^3}q_{n,k}(x)=\frac{(n-1)(n-2)}{n^2}\bar{x}^3\sum_{k=0}^{n-3}\binom{n-3}{k}\bar{x}^{k-3}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &+\frac{3(n-1)}{n^2}\bar{x}^2\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^{k-2}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{\bar{x}}{n^2}\sum_{k=0}^{n-1}\binom{n-1}{k}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ =\ &\frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3+\frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x},\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{n}\frac{k^4}{n^4}q_{n,k}(x)=\frac{(n-1)(n-2)(n-3)}{n^3}\bar{x}^4\sum_{k=0}^{n-4}\binom{n-4}{k}\bar{x}^{k-4}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &+\frac{6(n-1)(n-2)}{n^3}\bar{x}^3\sum_{k=0}^{n-3}\binom{n-3}{k}\bar{x}^{k-3}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ &+\frac{7(n-1)}{n^3}\bar{x}^2\sum_{k=0}^{n-2}\binom{n-2}{k}\bar{x}^{k-2}\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}+\frac{\bar{x}}{n^3}\sum_{k=0}^{n-1}\binom{n-1}{k}\bar{x}^k\left(\frac{n}{n+\beta}-\bar{x}\right)^{n-k}\\ =\ &\frac{(n-1)(n-2)(n-3)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-4}\bar{x}^4+\frac{6(n-1)(n-2)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-3}\bar{x}^3\\ &+\frac{7(n-1)}{n^3}\left(\frac{n}{n+\beta}\right)^{n-2}\bar{x}^2+\frac{1}{n^3}\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}.\end{aligned}$$

□

Lemma 2.2. *For $k=1,2,\cdots,n$, $j=0,1,2,\cdots$, it holds that*

$$\begin{aligned}&\int_{A_{n-2}}q_{n-1,k-1}(t)\left(t-\frac{2\alpha}{n(n-2+\beta)}\right)^j dt\\ =\ &\lambda_{n-2}\left(\frac{n-2}{n-2+\beta}\right)^j\sum_{s=0}^{j}\binom{j}{s}\left(\frac{\alpha}{n}\right)^{j-s}\frac{(n-1)!(k+s-1)!}{(n+s-1)!(k-1)!}\end{aligned}\tag{9}$$

Proof. Let $t = \frac{n-2}{n-2+\beta}u + \frac{\alpha}{n-2+\beta}$. Then

$$\begin{aligned}
& \int_{A_{n-2}} q_{n-1,k-1}(t)\left(t - \frac{2\alpha}{n(n-2+\beta)}\right)^j dt \\
= & \int_{A_{n-2}} q_{n-1,k-1}(t)\left(t - \frac{\alpha}{n-2+\beta} + \frac{(n-2)\alpha}{n(n-2+\beta)}\right)^j dt \\
= & \int_{A_{n-2}} \binom{n-2}{k-1}\sum_{s=0}^{j}\binom{j}{s}\left(\frac{(n-2)\alpha}{n(n-2+\beta)}\right)^{j-s}\left(t-\frac{\alpha}{n-2+\beta}\right)^{k+s-1}\left(\frac{n-2+\alpha}{n-2+\beta}-t\right)^{n-k-1} dt \\
= & \left(\frac{n-2}{n-2+\beta}\right)^{n+j-1}\binom{n-2}{k-1}\sum_{s=0}^{j}\binom{j}{s}\left(\frac{\alpha}{n}\right)^{j-s}\int_0^1 u^{k+s-1}(1-u)^{n-k-1}du \\
= & \lambda_{n-2}\left(\frac{n-2}{n-2+\beta}\right)^j\sum_{s=0}^{j}\binom{j}{s}\left(\frac{\alpha}{n}\right)^{j-s}\frac{(n-1)!(k+s-1)!}{(n+s-1)!(k-1)!},
\end{aligned}$$

which proves (9). □

Lemma 2.3. *It holds that*
(i). $U_n^{(\alpha,\beta)}(1,x) = 1$;
(ii). $U_n^{(\alpha,\beta)}(t,x) = x$;
(iii). $U_n^{(\alpha,\beta)}(t^2,x) = \frac{n-1}{n+1}\bar{x}^2 + \frac{n}{n+\beta}\left(\frac{2}{n+1}+\frac{2\alpha}{n}\right)\bar{x} + \left(\frac{\alpha}{n+\beta}\right)^2$;
(iv).

$$\begin{aligned}
U_n^{(\alpha,\beta)}(t^3,x) = & \frac{(n-1)(n-2)}{(n+1)(n+2)}\bar{x}^3 + \left(\frac{6(n-1)n}{(n+1)(n+2)(n+\beta)} + 3\frac{(n-1)\alpha}{(n+1)(n+\beta)}\right)\bar{x}^2 \\
& + \left(\frac{6}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2 + \frac{6}{n+1}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} + 3\left(\frac{\alpha}{n+\beta}\right)^2\right)\bar{x} + \left(\frac{\alpha}{n+\beta}\right)^3;
\end{aligned}$$

(v).

$$\begin{aligned}
U_n^{(\alpha,\beta)}(t^4,x) = & \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}\bar{x}^4 + \left(\frac{12n(n-1)(n-2)}{(n+1)(n+2)(n+3)(n+\beta)}\right. \\
& \left. +4\frac{(n-1)(n-2)}{(n+1)(n+2)}\left(\frac{\alpha}{n+\beta}\right)\right)\bar{x}^3 + \left(\frac{36(n-1)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^2\right. \\
& \left. +\frac{24(n-1)}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} + 6\frac{n-1}{n+1}\left(\frac{\alpha}{n+\beta}\right)^2\right)\bar{x}^2 \\
& +\left(\frac{24}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^3 + \frac{24}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^3\frac{\alpha}{n}\right. \\
& \left. +\frac{12}{n+1}\left(\frac{n}{n+\beta}\right)^3\left(\frac{\alpha}{n}\right)^2 + 4\left(\frac{\alpha}{n+\beta}\right)^3\right)\bar{x} + \left(\frac{\alpha}{n+\beta}\right)^4.
\end{aligned}$$

Proof. Set

$$A_j(x) = \sum_{k=1}^{n-1}\frac{(n-1)!(k+j-1)!}{(n+j-1)!(k-1)!}q_{n,k}(x),\ j = 0,1,\cdots.$$

Thus, by Lemma 2.1, we have

$$\begin{aligned} A_0(x) &= \sum_{k=1}^{n-1} q_{n,k}(x) = \sum_{k=0}^{n} q_{n,k}(x) - q_{n,n}(x) - q_{n,0}(x) \\ &= \left(\frac{n}{n+\beta}\right)^n - q_{n,n}(x) - q_{n,0}(x); \end{aligned} \tag{10}$$

$$A_1(x) = \sum_{k=1}^{n} \frac{k}{n} q_{n,k}(x) - q_{n,n}(x) = \left(\frac{n}{n+\beta}\right)^{n-1} \bar{x} - q_{n,n}(x); \tag{11}$$

$$\begin{aligned} A_2(x) &= \frac{n}{n+1} \sum_{k=1}^{n-1} \frac{k^2}{n^2} q_{n,k}(x) + \frac{1}{n+1} \sum_{k=1}^{n-1} \frac{k}{n} q_{n,k}(x) \\ &= \frac{n}{n+1}\left(\frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}\right) + \frac{1}{n+1}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x} - q_{n,n}(x) \\ &= \frac{n-1}{n+1}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{2}{n+1}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x} - q_{n,n}(x); \end{aligned} \tag{12}$$

$$\begin{aligned} A_3(x) &= \frac{n^2}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{k^3}{n^3} q_{n,k}(x) + \frac{3n}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{k^2}{n^2} q_{n,k}(x) \\ &\quad + \frac{2}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{k}{n} q_{n,k}(x) - q_{n,n}(x) \\ &= \frac{n^2}{(n+1)(n+2)}\left(\frac{(n-1)(n-2)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-3} \bar{x}^3 + \frac{3(n-1)}{n^2}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2\right. \\ &\quad \left.+ \frac{1}{n^2}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}\right) + \frac{3n}{(n+1)(n+2)}\left(\frac{n-1}{n}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 + \frac{1}{n}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x}\right) \\ &\quad + \frac{2}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x} - q_{n,n}(x) \\ &= \frac{(n-1)(n-2)}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^{n-3} \bar{x}^3 + \frac{6(n-1)}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^{n-2} \bar{x}^2 \\ &\quad + \frac{6}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^{n-1} \bar{x} - q_{n,n}(x) \end{aligned} \tag{13}$$

$$\begin{aligned} A_4(x) &= \frac{n^3}{(n+1)(n+2)(n+3)}\left(\sum_{k=1}^{n} \frac{k^4}{n^4} q_{n,k}(x) - q_{n,n}(x)\right) \\ &\quad + \frac{6n^2}{(n+1)(n+2)(n+3)}\left(\sum_{k=1}^{n} \frac{k^3}{n^3} q_{n,k}(x) - q_{nn}(x)\right) \\ &\quad + \frac{11n}{(n+1)(n+2)(n+3)}\left(\sum_{k=1}^{n} \frac{k^2}{n^2} q_{n,k}(x) - q_{n,n}(x)\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{6}{(n+1)(n+2)(n+3)}\left(\sum_{k=1}^{n}\frac{k}{n}q_{n,k}(x)-q_{n,n}(x)\right)\\
=\ &\frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^{n-4}\overline{x}^4\\
&+\frac{12(n-1)(n-2)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^{n-3}\overline{x}^3\\
&+\frac{36(n-1)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^{n-2}\overline{x}^2\\
&+\frac{24}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^{n-1}\overline{x}-q_{n,n}(x). \qquad (14)
\end{aligned}
$$

Now, we begin to prove lemma. By (9), we have

$$
U_n^{(\alpha,\beta)}(1,x)=\left(\frac{n+\beta}{n}\right)^n\sum_{k=0}^{n}q_{n,k}(x)=1.
$$

By (9), (10) and (11), we have

$$
\begin{aligned}
U_n^{(\alpha,\beta)}(t,x)\ =\ &\left(\frac{n+\beta}{n}\right)^n\frac{\alpha}{n+\beta}q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n\frac{n+\alpha}{n+\beta}q_{n,n}(x)\\
&+\left(\frac{n+\beta}{n}\right)^{n-1}\left(A_1(x)+\frac{\alpha}{n}A_0(x)\right)\\
=\ &\left(\frac{n+\beta}{n}\right)^n\frac{\alpha}{n+\beta}q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n\frac{n+\alpha}{n+\beta}q_{n,n}(x)\\
&+\left(\frac{n+\beta}{n}\right)^{n-1}\left(\left(\frac{n}{n+\beta}\right)^{n-1}\bar{x}-q_{n,n}(x)\right)\\
&+\left(\frac{n+\beta}{n}\right)^{n-1}\frac{\alpha}{n}\left(\left(\frac{n}{n+\beta}\right)^{n}-q_{n,0}(x)-q_{n,n}(x)\right)\\
=\ &x-\frac{\alpha}{n+\beta}+\frac{\alpha}{n+\beta}=x, \qquad (15)
\end{aligned}
$$

which implies the second equation.

Similarly, by (9), (10)-(14), we deduce that

$$
\begin{aligned}
U_n^{(\alpha,\beta)}(t^2,x)\ =\ &\left(\frac{n+\beta}{n}\right)^n\left(\frac{\alpha}{n+\beta}\right)^2q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n\left(\frac{n+\alpha}{n+\beta}\right)^2q_{n,n}(x)\\
&+\left(\frac{n+\beta}{n}\right)^{n-2}\left(A_2(x)+\frac{2\alpha}{n}A_1(x)+\frac{\alpha^2}{n^2}A_0(x)\right)\\
=\ &\frac{n-1}{n+1}\bar{x}^2+\frac{n}{n+\beta}\left(\frac{2}{n+1}+\frac{2\alpha}{n}\right)\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
U_n^{(\alpha,\beta)}(t^3,x) &= \left(\frac{n+\beta}{n}\right)^n\left(\frac{\alpha}{n+\beta}\right)^3 q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n\left(\frac{n+\alpha}{n+\beta}\right)^3 q_{n,n}(x)\\
&\quad+\left(\frac{n+\beta}{n}\right)^{n-3}\left(A_3(x)+3\frac{\alpha}{n}A_2(x)+3\frac{\alpha^2}{n^2}A_1(x)+\frac{\alpha^3}{n^3}A_0(x)\right)\\
&= \frac{(n-1)(n-2)}{(n+1)(n+2)}\overline{x}^3+\left(\frac{6(n-1)}{(n+1)(n+2)}\frac{n}{n+\beta}+3\frac{n-1}{n+1}\frac{\alpha}{n+\beta}\right)\\
&\quad\overline{x}^2+\left(\frac{6}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2+\frac{6}{n+1}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n}\right.\\
&\quad\left.+3\left(\frac{\alpha}{n+\beta}\right)^2\right)\overline{x}+\left(\frac{\alpha}{n+\beta}\right)^3,
\end{aligned}
$$

and

$$
\begin{aligned}
U_n^{(\alpha,\beta)}(t^4,x) &= \left(\frac{n+\beta}{n}\right)^n\left(\frac{\alpha}{n+\beta}\right)^4 q_{n,0}(x)+\left(\frac{n+\beta}{n}\right)^n\left(\frac{n+\alpha}{n+\beta}\right)^4 q_{n,n}(x)\\
&\quad+\left(\frac{n+\beta}{n}\right)^{n-4}\left(A_4(x)+\frac{4\alpha}{n}A_3(x)+\frac{6\alpha^2}{n^2}A_2(x)+\frac{4\alpha^3}{n^3}A_1(x)+\frac{\alpha^4}{n^4}A_0(x)\right)\\
&= \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}\overline{x}^4+\left(\frac{12(n-1)(n-2)}{(n+1)(n+2)(n+3)}\frac{n}{n+\beta}\right.\\
&\quad\left.+4\frac{(n-1)(n-2)}{(n+1)(n+2)}\left(\frac{\alpha}{n+\beta}\right)\right)\overline{x}^3+\left(\frac{36(n-1)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^2\right.\\
&\quad\left.+\frac{24(n-1)}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n}+6\frac{n-1}{n+1}\left(\frac{\alpha}{n+\beta}\right)^2\right)\overline{x}^2\\
&\quad+\left(\frac{24}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^3+\frac{24}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^3\frac{\alpha}{n}\right.\\
&\quad\left.+\frac{12}{n+1}\left(\frac{n}{n+\beta}\right)^3\left(\frac{\alpha}{n}\right)^2+4\left(\frac{\alpha}{n+\beta}\right)^3\right)\overline{x}+\left(\frac{\alpha}{n+\beta}\right)^4.
\end{aligned}
$$

□

Lemma 2.4. *It holds that*

$$U_n^{(\alpha,\beta)}(t-x,x)=0, \tag{16}$$

$$U_n^{(\alpha,\beta)}((t-x)^2,x)=\frac{2}{n+1}\varphi^2(x), \tag{17}$$

$$U_n^{(\alpha,\beta)}((t-x)^4,x)=\frac{12(n-7)\varphi^4(x)+24\varphi^2(x)\left(\frac{n}{n+\beta}\right)^2}{(n+1)(n+2)(n+3)}. \tag{18}$$

Proof. It is obvious that (16) is valid by (i) and (ii) of Lemma 2.3.

By using (i)-(iii) of Lemma 2.3, we have

$$
\begin{aligned}
U_n^{(\alpha,\beta)}((t-x)^2,x) &= U_n^{(\alpha,\beta)}(t^2,x)-2xU_n^{(\alpha,\beta)}(t,x)+x^2\\
&= \frac{n-1}{n+1}\bar{x}^2+\frac{n}{n+\beta}\left(\frac{2}{n+1}+\frac{2\alpha}{n}\right)\bar{x}+\left(\frac{\alpha}{n+\beta}\right)^2-\left(\bar{x}+\frac{\alpha}{n+\beta}\right)^2\\
&= -\frac{2}{n+1}\bar{x}^2+\frac{n}{n+\beta}\frac{2}{n+1}\bar{x}=\frac{2}{n+1}\varphi^2(x),
\end{aligned}
$$

which proves (17).

The proof of (18) is similar, but is rather complicated. By Lemma 2.3, we deduce that

$$
\begin{aligned}
& U_n^{(\alpha,\beta)}((t-x)^4,x) = U_n^{(\alpha,\beta)}(t^4,x) - 4xU_n^{(\alpha,\beta)}(t^3,x) + 6x^2U_n^{(\alpha,\beta)}(t^2,x) - 4x^4 + x^4 \\
= \; & \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}\overline{x}^4 + \Big(\frac{12(n-1)(n-2)}{(n+1)(n+2)(n+3)}\frac{n}{n+\beta} + \frac{4(n-1)(n-2)}{(n+1)(n+2)}\frac{\alpha}{n+\beta}\Big)\overline{x}^3 \\
& + \left(\frac{n}{n+\beta}\right)^2\Big(\frac{36(n-1)}{(n+1)(n+2)(n+3)} + \frac{24(n-1)}{(n+1)(n+2)}\frac{\alpha}{n} + \frac{6(n-1)}{n+1}\frac{\alpha^2}{n^2}\Big)\overline{x}^2 \\
& + \left(\frac{n}{n+\beta}\right)^3\Big(\frac{24}{(n+1)(n+2)(n+3)} + \frac{24}{(n+1)(n+2)}\frac{\alpha}{n} + \frac{12}{n+1}\frac{\alpha^2}{n^2} + 4\frac{\alpha^3}{n^3}\Big)\overline{x} \\
& + \left(\frac{\alpha}{n+\beta}\right)^4 - 4x\Big[\frac{(n-1)(n-2)}{(n+1)(n+2)}\overline{x}^3 + \Big(\frac{6(n-1)}{(n+1)(n+2)}\frac{n}{n+\beta} + \frac{3(n-1)}{n+1}\frac{\alpha}{n+\beta}\Big)\overline{x}^2 \\
& + \Big(\frac{6}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2 + \frac{6}{n+1}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} + 3\left(\frac{\alpha}{n+\beta}\right)^2\Big]\overline{x} \\
& -4x\left(\frac{\alpha}{n+\beta}\right)^3 + 6x^2\Big(\frac{n-1}{n+1}\overline{x}^2 + \frac{n}{n+\beta}\left(\frac{2}{n+1} + \frac{2\alpha}{n}\right)\Big)\overline{x} + 6x^2\left(\frac{\alpha}{n+\beta}\right)^2 - 4x^4 + x^4 \\
= \; & \Big(\frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)} + 1\Big)\overline{x}^4 + \Big(\frac{12(n-1)(n-2)}{(n+1)(n+2)(n+3)}\frac{n}{n+\beta} \\
& + \frac{4(n-1)(n-2)}{(n+1)(n+2)}\frac{\alpha}{n+\beta} - 4x\frac{(n-1)(n-2)}{(n+1)(n+2)}\Big)\overline{x}^3 \\
& \quad + \Big(\frac{36(n-1)}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^2 + \frac{24(n-1)}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} \\
& \quad + \frac{6(n-1)}{n+1}\left(\frac{\alpha}{n+\beta}\right)^2 - \frac{24x(n-1)}{(n+1)(n+2)}\frac{n}{n+\beta} - \frac{12x(n-1)}{n+1}\frac{\alpha}{n+\beta} \\
& \quad + \frac{6x^2(n-1)}{n+1}\Big)\overline{x}^2 + \Big(\frac{24}{(n+1)(n+2)(n+3)}\left(\frac{n}{n+\beta}\right)^3 \\
& \quad + \frac{24}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^3\frac{\alpha}{n} + \frac{12}{n+1}\left(\frac{n}{n+\beta}\right)^3\left(\frac{\alpha}{n}\right)^2 + 4\left(\frac{\alpha}{n+\beta}\right)^3 \\
& \quad - \frac{24x}{(n+1)(n+2)}\left(\frac{n}{n+\beta}\right)^2 - \frac{24x}{n+1}\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} - 12x\left(\frac{\alpha}{n+\beta}\right)^2 \\
& \quad + \frac{12x^2}{n+1}\frac{n}{n+\beta} + 12x^2\frac{\alpha}{n+\beta} - 4x^3\Big)\overline{x} \\
= \; & \left(\frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)} + 1\right)\overline{x}^4 + \frac{(n-1)(n-2)}{(n+1)(n+2)(n+3)}\Big(12\frac{n}{n+\beta} \\
& \quad +4(n+3)\frac{\alpha}{n+\beta} - 4(n+3)x\Big)\overline{x}^3 + \frac{n-1}{(n+1)(n+2)(n+3)} \\
& \quad \Big(36\left(\frac{n}{n+\beta}\right)^2 + 24(n+3)\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n} + 6(n+2)(n+3)\left(\frac{\alpha}{n+\beta}\right)^2 \\
& \quad -24(n+3)x\frac{n}{n+\beta} - 12(n+2)(n+3)x\frac{\alpha}{n+\beta} + 6(n+2)(n+3)x^2\Big)\overline{x}^2
\end{aligned}
$$

$$+\frac{1}{(n+1)(n+2)(n+3)}\Bigg(24\left(\frac{n}{n+\beta}\right)^3+24(n+3)\left(\frac{n}{n+\beta}\right)^3\frac{\alpha}{n}+12(n+2)(n+3)$$
$$\left(\frac{n}{n+\beta}\right)^3\left(\frac{\alpha}{n}\right)^2+4(n+1)(n+2)(n+3)\left(\frac{\alpha}{n+\beta}\right)^3-24(n+3)x\left(\frac{n}{n+\beta}\right)^2$$
$$-24(n+2)(n+3)\left(\frac{n}{n+\beta}\right)^2\frac{\alpha}{n}-12(n+1)(n+2)(n+3)x\left(\frac{\alpha}{n+\beta}\right)^2$$
$$+12(n+2)(n+3)x^2\frac{n}{n+\beta}+12(n+2)(n+3)x^2\frac{\alpha}{n+\beta}-4(n+1)(n+2)(n+3)x^3\Bigg)\overline{x}$$
$$=\left(\frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}+1\right)\overline{x}^4+\frac{(n-1)(n-2)}{(n+1)(n+2)(n+3)}M_1$$
$$+\frac{n-1}{(n+1)(n+2)(n+3)}M_2+\frac{1}{(n+1)(n+2)(n+3)}M_3,$$

where

$$M_1:=12\left(\frac{n}{n+\beta}-\bar{x}\right)\bar{x}^3-4n\bar{x}^4$$

$$M_2:=36\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2+6n^2\bar{x}^4+30n\bar{x}^4-24n\frac{n}{n+\beta}\bar{x}^3$$

$$M_3:=24\left(\frac{n}{n+\beta}-\bar{x}\right)^3\bar{x}-24n^2\bar{x}^4-4n^3\bar{x}^4-44n\bar{x}^4+12n^2\frac{n}{n+\beta}\bar{x}^3$$
$$+60n\frac{n}{n+\beta}\bar{x}^3-24n\left(\frac{n}{n+\beta}\right)^2\bar{x}^2.$$

Therefore,

$$U_n^{(\alpha,\beta)}((t-x)^4,x)=\frac{1}{(n+1)(n+2)(n+3)}\Bigg(\Big((n-1)(n-2)(n-3)+(n+1)(n+2)(n+3)$$
$$-4n(n-1)(n-2)+6n^2(n-1)+30n(n-1)-24n^2-4n^3-44n\Big)\bar{x}^4$$
$$+\left(12n^2\frac{n}{n+\beta}+60n\frac{n}{n+\beta}+12(n-1)(n-2)\left(\frac{n}{n+\beta}-\bar{x}\right)-24n(n-1)\frac{n}{n+\beta}\right)\bar{x}^3$$
$$+36(n-1)\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2-24n\left(\frac{n}{n+\beta}\right)^2\bar{x}^2+24\left(\frac{n}{n+\beta}-x\right)^3\bar{x}\Bigg)$$
$$=\frac{1}{(n+1)(n+2)(n+3)}\Bigg(48n\frac{n}{n+\beta}\bar{x}^3-24n\bar{x}^4+24\left(\frac{n}{n+\beta}-\bar{x}\right)\bar{x}^3$$
$$+36(n-1)\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2-24n\left(\frac{n}{n+\beta}\right)^2\bar{x}^2+24\left(\frac{n}{n+\beta}-\bar{x}\right)^3\bar{x}\Bigg)$$
$$=\frac{1}{(n+1)(n+2)(n+3)}\Bigg(-24n\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2+36(n-1)\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2$$
$$-48\left(\frac{n}{n+\beta}-\bar{x}\right)^2\bar{x}^2+24\left(\frac{n}{n+\beta}-\bar{x}\right)\bar{x}\left(\frac{n}{n+\beta}\right)^2\Bigg)$$

$$= \quad \frac{12(n-7)\varphi^4(x) + 24\varphi^2(x)\left(\frac{n}{n+\beta}\right)^2}{(n+1)(n+2)(n+3)}.$$

□

3. Proof of results

3.1. Proof of Theorem 1.1

It is obvious that

$$\|U_n^{(\alpha,\beta)}(f)\| \le \|f\|, \tag{19}$$

where $\|f\|$ is the uniform norm of f in A_n.

Set $D_\lambda^2 := \left\{f \in C(A_n), f' \in A.C._{loc}, \|\varphi^{2\lambda} f''\| < +\infty\right\}$. Define

$$K_{\varphi^\lambda}(f, t^2) = \inf_{g \in D_\lambda^2} \{\|f-g\| + t^2\|\varphi^{2\lambda} g''\|\}.$$

It is well known that (see [5]) $K_{\varphi^\lambda}(f,t^2) \sim \omega^2_{\varphi^\lambda}(f,t)$. Therefore, for any fixed x, λ and x, we may choose a $g_{n,x,\lambda}(t) \in D_\lambda^2$ such that

$$\|f-g\| \le C\omega^2_{\varphi^\lambda}\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{20}$$

$$\frac{\varphi^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda} g''\| \le C\omega^2_{\varphi^\lambda}\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{21}$$

By (19) and (20), we have

$$\begin{aligned}
|U_n^{(\alpha,\beta)}(f,x) - f(x)| &\le |U_n^{(\alpha,\beta)}(f-g,x)| + |f(x)-g(x)| + |U_n^{(\alpha,\beta)}(g,x) - g(x)| \\
&\le 2\|f-g\| + |U_n^{(\alpha,\beta)}(g,x) - g(x)| \\
&\le C\omega^2_{\varphi^\lambda}\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right) + |U_n^{(\alpha,\beta)}(g,x) - g(x)|. \qquad (22)
\end{aligned}$$

By using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u)du,$$

Lemma 1.3, and the following inequality (see [6]):

$$\frac{|t-u|}{\varphi^{2\lambda}(u)} \le \frac{|t-x|}{\varphi^{2\lambda}(x)}, \text{ for any } u \text{ between } x \text{ and } t,$$

we have

$$\begin{aligned}
|U_n^{(\alpha,\beta)}(g,x) - g(x)| &= \left|U_n^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\le C\|\varphi^{2\lambda} g''\| U_n^{(\alpha,\beta)}\left(\frac{(t-x)^2}{\varphi^{2\lambda}(x)}, x\right) \\
&\le C\frac{\varphi^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda} g''\| \\
&\le C\omega^2_{\varphi^\lambda}\left(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right), \qquad (23)
\end{aligned}$$

where in the last inequality, (21) is used.

By combining (22) and (23), we obtain Theorem 1.1.

3.2. Proof of Theorem 1.2

Using Taylor's expansion of f:

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \epsilon(t,x)(t-x)^2,$$

where $\epsilon(t,x) \to 0$ as $t \to x$. By (16) and (17), we have

$$U_n^{(\alpha,\beta)}(f,x) - f(x) = \frac{\varphi^2(x)}{2(n+1)}f''(x) + U_n^{(\alpha,\beta)}\left(\epsilon(t,x)(t-x)^2, x\right). \tag{24}$$

It follows from (18) that $U_n^{(\alpha,\beta)}\left((t-x)^4, x\right) = O\left(\frac{1}{n^2}\right)$. Then by using Cauchy-Schwarz inequality, we have

$$|U_n^{(\alpha,\beta)}\left(\epsilon(t,x)(t-x)^2, x\right)| \le U_n^{(\alpha,\beta)}\left(\epsilon(t,x)^2, x\right)^{1/2} U_n^{(\alpha,\beta)}\left((t-x)^4, x\right)^{1/2} = o\left(\frac{1}{n}\right) \tag{25}$$

as $n \to \infty$.

By (24) and (25), we complete the proof of Theorem 1.2.

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