Filomat 35:4 (2021), 1233–1240
https://doi.org/10.2298/FIL2104233G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Symmetric Bi-derivations and their Generalizations on Group Algebras

Mohammad Hossein Ahmadi Gandomani[a], **Mohammad Javad Mehdipour**[b]

[a]*Department of Mathematics, GhiroKarzin Branch, Islamic Azad University, GhiroKarzin 74765-118, Iran*
[b]*Department of Mathematics, Shiraz University of Technology, Shiraz 71555-313, Iran*

Abstract. Here, we investigate symmetric bi-derivations and their generalizations on $L_0^\infty(\mathfrak{G})^*$. For $\kappa \in \mathbb{N}$, we show that if $B : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ is a symmetric bi-derivation such that $[B(m, m), m^\kappa] \in Z(L_0^\infty(\mathfrak{G})^*)$ for all $m \in L_0^\infty(\mathfrak{G})^*$, then B is the zero map. Furthermore, we characterize symmetric generalized bi-derivations on group algebras. We also prove that any symmetric Jordan bi-derivation on $L_0^\infty(\mathfrak{G})^*$ is a symmetric bi-derivation.

1. Introduction

Let $\mathfrak{G}$ denote a locally compact abelian group with a fixed left Haar measure λ. The Banach algebras $L^1(\mathfrak{G})$ and $L^\infty(\mathfrak{G})$ are as defined in [7]. Let us remark that $L^\infty(\mathfrak{G})$ is the continuous dual of $L^1(\mathfrak{G})$. We denote by $L_0^\infty(\mathfrak{G})$ the subspace of $L^\infty(\mathfrak{G})$ consisting of all functions $g \in L^\infty(\mathfrak{G})$ that vanish at infinity; i.e. for each $\varepsilon > 0$, there is a compact subset K of $\mathfrak{G}$ for which

$$\|g\, \chi_{\mathfrak{G}\setminus K}\|_\infty < \varepsilon,$$

where $\chi_{\mathfrak{G}\setminus K}$ denotes the characteristic function of $\mathfrak{G} \setminus K$ on $\mathfrak{G}$. For every $n \in L_0^\infty(\mathfrak{G})^*$ and $g \in L_0^\infty(\mathfrak{G})$ we define the functional $ng \in L_0^\infty(\mathfrak{G})^*$ by $\langle ng, \phi\rangle = \langle n, g\phi\rangle$, in which $\langle g\phi, \psi\rangle = \langle g, \phi * \psi\rangle$ and

$$\phi * \psi(x) = \int_{\mathfrak{G}} \phi(y)\psi(y^{-1}x)\, d\lambda(y)$$

for all $\phi, \psi \in L^1(\mathfrak{G})$ and $x \in \mathfrak{G}$. This let us endow $L_0^\infty(\mathfrak{G})^*$ with the *first Arens product* " $\cdot$ " defined by the formula

$$\langle m \cdot n, g\rangle = \langle m, ng\rangle$$

for all $m, n \in L_0^\infty(\mathfrak{G})^*$ and $g \in L_0^\infty(\mathfrak{G})$. Then $L_0^\infty(\mathfrak{G})^*$ is a Banach algebra with this product. For an extensive study of $L_0^\infty(\mathfrak{G})^*$ see [8].

Let $\mathfrak{A}$ be an algebra and $B(.,.) : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a symmetric bi-linear mapping; that is, $B(x, y) = B(y, x)$, $B(\alpha x, y) = \alpha B(x, y)$ and

$$B(x + y, z) = B(x, z) + B(y, z)$$

2010 *Mathematics Subject Classification*. Primary 43A15; Secondary 47B47, 16W25
Keywords. Locally compact abelian group, bi-derivation, κ−centralizing mapping, κ−skew centralizing mapping
Received: 28 March 2020; Accepted: 28 June 2020
Communicated by Dragan S. Djordjević
Email addresses: ahmadig.mh@gmail.com (Mohammad Hossein Ahmadi Gandomani), mehdipour@sutech.ac.ir (Mohammad Javad Mehdipour)

for all $x, y, z \in \mathfrak{A}$ and $\alpha \in \mathbb{C}$. The mapping $f : \mathfrak{A} \to \mathfrak{A}$ defined by $f(x) = B(x, x)$ is called the *trace* of B. Let us recall that B is called a *symmetric bi-derivation* if

$$B(xy, z) = B(x, z)y + xB(y, z)$$

for all $x, y, z \in \mathfrak{A}$. Also, B is called a *symmetric generalized bi-derivation* if there exists a symmetric bi-derivation $\tilde{B}$ of $\mathfrak{A}$ such that

$$B(xy, z) = xB(y, z) + \tilde{B}(x, z)y$$

for all $x, y, z \in \mathfrak{A}$. A symmetric generalized bi-derivation B associated with a symmetric bi-derivation $\tilde{B}$ is denoted by $B_{\tilde{B}}$. Finally, B is called a *symmetric Jordan bi-derivation* if

$$B(x^2, y) = B(x, y)x + xB(x, y),$$

for all $x, y \in \mathfrak{A}$. For $\kappa \in \mathbb{N}$, a linear mapping $T : \mathfrak{A} \to \mathfrak{A}$ is called $\kappa-$(*skew*) *centralizing* if

$$[T(x), x^\kappa] \in Z(\mathfrak{A}) \quad (T(x) \circ x^\kappa \in Z(\mathfrak{A}))$$

for all $x \in \mathfrak{A}$, in a special case, if for every $x \in \mathfrak{A}$

$$[T(x), x^\kappa] = 0 \quad (T(x) \circ x^\kappa = 0)),$$

then T is called $\kappa-$(*skew*) *commuting*, where $Z(\mathfrak{A})$ is the center of $\mathfrak{A}$, $[x, y] = xy - yx$ and

$$x \circ y := x \cdot y + y \cdot x$$

for all $x, y \in \mathfrak{A}$. In the case that, $\kappa = 1$, T is called (*skew*) *centralizing* and (*skew*) *commuting*, respectively.

Symmetric bi-derivations on rings have been introduced and studied by Maksa [9, 10]. Several authors continued this investigations [2, 5, 15-18]. For example, Vukman [16] proved that if $B : R \times R \to R$ is a symmetric bi-derivation such that for every $x \in R$

$$[[f(x), x], x] \in Z(R),$$

then $B = 0$, where R is a noncommutative prime ring of characteristic not two and three. He conjectured that if there exists $\kappa \in \mathbb{N}$ such that for every $x \in R$ we have $f_\kappa(x) \in Z(R)$, then $B = 0$, where

$$f_{i+1}(x) = [f_i(x), x]$$

for $i > 1$ and $f_1(x) = f(x)$. Deng [5] gave an affirmative answer to the Vukman's conjecture. For related results on symmetric bi-derivations on Banach algebras see [3, 13]; see also [4, 6, 12] for study of generalized bi-derivations and Jordan bi-derivations.

An easy application of the Hahn-Banach's theorem shows that $L_0^\infty(\mathfrak{G})^*$ is not a semiprime ring, when $\mathfrak{G}$ is a non-discrete locally compact group. Also, note that if $\Lambda(\mathfrak{G})$ denotes the set of all weak*-cluster points of the canonical images of the bounded approximate identities, bounded by one, of $L^1(\mathfrak{G})$ in $L_0^\infty(\mathfrak{G})^*$, then for every nonzero element r in

$$\mathrm{Ann}_r(L_0^\infty(\mathfrak{G})^*) = \{n - u \cdot n : n \in L_0^\infty(\mathfrak{G})^*, u \in \Lambda(\mathfrak{G})\},$$

the mapping $B(.,.) : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ defined by

$$B(m, n) = r \cdot m \cdot n$$

is a nonzero bi-derivation. These facts lead us to investigate symmetric bi-derivations on $L_0^\infty(\mathfrak{G})^*$.

In this paper, we first study symmetric bi-derivations on $L_0^\infty(\mathfrak{G})^*$ and prove that they map $L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^*$ into the radical of $L_0^\infty(\mathfrak{G})^*$. We also show that if $B : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ is a symmetric bi-derivation and f is $\kappa-$centralizing for some $\kappa \in \mathbb{N}$, then B is zero map. In the case that, B is a symmetric generalized bi-derivation, we prove that there exists $\theta \in L_0^\infty(\mathfrak{G})^*$ such that $B(m, n) = m \cdot n \cdot \theta$ for all $m, n \in L_0^\infty(\mathfrak{G})^*$. Finally, we study symmetric Jordan bi-derivations on $L_0^\infty(\mathfrak{G})^*$ and establish that they are symmetric bi-derivations.

2. Symmetric bi-derivations and their generalizations

In the sequel, we use the symbols D, G_D and J for symmetric bi-derivations, symmetric generalized bi-derivations and symmetric Jordan bi-derivations, respectively. The following result is an analogue of Singer-Wermer conjecture [14] for bi-derivations.

Proposition 2.1. *Let $D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric bi-derivation. Then D maps $L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^*$ into the radical of $L_0^\infty(\mathfrak{G})^*$.*

Proof. For every $m \in L_0^\infty(\mathfrak{G})^*$ we define the mapping $\Delta_m : L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ by

$$\Delta_m(n) = D(m, n).$$

For every $m \in L_0^\infty(\mathfrak{G})^*$, Δ_m is a derivation on $L_0^\infty(\mathfrak{G})^*$ and hence Δ_m maps $L_0^\infty(\mathfrak{G})^*$ into its radical for all $m \in L_0^\infty(\mathfrak{G})^*$; see [11]. Since

$$D(L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^*) = \cup_m \Delta_m(L_0^\infty(\mathfrak{G})^*),$$

D maps $L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^*$ into the radical of $L_0^\infty(\mathfrak{G})^*$. □

Before, we prove the main result of this paper, let us remark that if $u \in \Lambda(\mathfrak{G})$, then $m \cdot u = m$ and $u \cdot \phi = \phi$ for all $m \in L_0^\infty(\mathfrak{G})^*$ and $\phi \in L^1(\mathfrak{G})$.

Theorem 2.2. *Let $D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric bi-derivation and f be the trace of D. Then the following assertions are equivalent.*

(a) *there exists $\kappa \in \mathbb{N}$ such that $f(m^\kappa) = 0$ for all $m \in L_0^\infty(\mathfrak{G})^*$;*
(b) *there exists $\kappa \in \mathbb{N}$ such that f is $\kappa-$commuting;*
(c) *there exists $\kappa \in \mathbb{N}$ such that f is $\kappa-$centralizing;*
(d) *there exists $\kappa \in \mathbb{N}$ such that f is $\kappa-$skew commuting;*
(e) *there exists $\kappa \in \mathbb{N}$ such that f is $\kappa-$skew centralizing;*
(f) *$D = 0$.*

Proof. Let $\kappa \in \mathbb{N}$ and $m \in L_0^\infty(\mathfrak{G})^*$. Choose $u \in \Lambda(\mathfrak{G})$ and set $m^0 = u$. Then

$$\begin{aligned} f(m^\kappa) &= D(m^\kappa, m^\kappa) \\ &= D(m, m^\kappa) \cdot m^{\kappa-1} + m \cdot D(m^{\kappa-1}, m^\kappa) \\ &= D(m, m \cdot m^{\kappa-1}) \cdot m^{\kappa-1} \\ &= D(m, m) \cdot m^{2\kappa-2} + m \cdot D(m, m^{\kappa-1}) \cdot m^{\kappa-1} \\ &= D(m, m) \cdot m^{2\kappa-2} \\ &= f(m) \cdot m^{2\kappa-2}. \end{aligned}$$

We also have

$$f(m) \cdot m^k = [f(m), m^k] = \langle f(m), m^k \rangle.$$

Theses facts imply that the assertions (a)-(e) are equivalent. To complete the proof, it suffices to show that (b)⇒(f). So let f be $\kappa-$commuting. Then

$$f(m) \cdot m^\kappa = 0 \tag{1}$$

for all $m \in L_0^\infty(\mathfrak{G})^*$. Hence $f(u) = 0$. Replacing m by $m + u$ in (1), we get

$$\begin{aligned} 0 &= f(m+u) \cdot (m+u)^\kappa \\ &= (f(m) + f(u) + 2D(m,u)) \cdot (m+u)^\kappa \qquad (2) \\ &= (f(m) + 2D(m,u)) \cdot (m+u)^\kappa. \end{aligned}$$

A simple calculation implies that

$$(m+u)^{\kappa}=\sum_{j=0}^{\kappa-1}\binom{\kappa-1}{j}m^{\kappa-j}+\sum_{j=0}^{\kappa-1}\binom{\kappa-1}{j}u\cdot m^{\kappa-j-1}.$$

This together with (1) and (2) shows that

$$\sum_{j=1}^{\kappa}\binom{\kappa}{j}f(m)\cdot m^{\kappa-j}+2\sum_{j=0}^{\kappa}\binom{\kappa}{j}D(m,u)\cdot m^{\kappa-j}=0. \tag{3}$$

Set

$$\mathfrak{A}(m):=\sum_{j\ even,\ j=2}^{\kappa}\binom{\kappa}{j}f(m)\cdot m^{\kappa-j},$$

$$\mathfrak{B}(m):=\sum_{j\ odd,\ j=1}^{\kappa}\binom{\kappa}{j}f(m)\cdot m^{\kappa-j},$$

$$\mathfrak{C}(m):=2\sum_{j\ even,\ j=0}^{\kappa}\binom{\kappa}{j}D(m,u)\cdot m^{\kappa-j}$$

and

$$\mathfrak{D}(m):=2\sum_{j\ odd,\ j=1}^{\kappa}\binom{\kappa}{j}D(m,u)\cdot m^{\kappa-j}.$$

Hence the relation (3) can be rewritten as

$$\mathfrak{A}(m)+\mathfrak{B}(m)+\mathfrak{C}(m)+\mathfrak{D}(m)=0. \tag{4}$$

Replacing m by $-m$ in (4), we arrive at

$$\mathfrak{A}(m)-\mathfrak{B}(m)-\mathfrak{C}(m)+\mathfrak{D}(m)=0. \tag{5}$$

Regarding (4) and (5) we deduce that

$$\mathfrak{A}(m)+\mathfrak{D}(m)=0 \tag{6}$$

and

$$\mathfrak{B}(m)+\mathfrak{C}(m)=0. \tag{7}$$

At this point, it is convenient to consider separately the cases κ even and odd. Suppose first that κ is even. According to (7), we infer that

$$\begin{aligned} 0 &= \mathfrak{B}(m)+\mathfrak{C}(m)\\ &= \sum_{j\ odd,\ j=1}^{\kappa-1}\binom{\kappa}{j}f(m)\cdot m^{\kappa-j}+2\sum_{j\ even,\ j=0}^{\kappa}\binom{\kappa}{j}D(m,u)\cdot m^{\kappa-j}\\ &= \sum_{j\ odd,\ j=1}^{\kappa-1}\binom{\kappa}{j}f(m)\cdot m^{\kappa-j}+2\sum_{j\ even,\ j=0}^{\kappa-2}\binom{\kappa}{j}D(m,u)\cdot m^{\kappa-j}\\ &+ 2D(m,u). \end{aligned} \tag{8}$$

Since for any $r\in \mathrm{Ann}_r(L_0^{\infty}(\mathfrak{G})^*)$

$$\sum_{j\ odd,\ j=1}^{\kappa-1}\binom{\kappa}{j}f(r)\cdot r^{\kappa-j}=\sum_{j\ even,\ j=0}^{\kappa-2}\binom{\kappa}{j}D(r,u)\cdot r^{\kappa-j}=0,$$

it follows from (8) that

$$D(r,u) = 0. \tag{9}$$

Taking $m - u \cdot m$ for r in (9), we arrive at

$$\begin{aligned} 0 &= D(m - u \cdot m, u) \\ &= D(m,u) - D(u \cdot m, u) \\ &= D(m,u) - D(u,u) \cdot m - u \cdot D(m,u) \\ &= D(m,u) - f(u) \cdot m. \end{aligned}$$

Since $f(u) = 0$, it follows that

$$D(m,u) = 0$$

for all $m \in L_0^\infty(\mathfrak{G})^*$. Hence $\mathfrak{D} = 0$. From this and (6) we infer that

$$\sum_{j\ even,\ j=2}^{\kappa} \binom{\kappa}{j} f(m) \cdot m^{\kappa-j} = 0. \tag{10}$$

Let i be even and $2 \le i \le \kappa - 2$. From (10) we conclude that

$$\begin{aligned} 0 &= \sum_{j\ even,\ j=2}^{\kappa} \binom{\kappa}{j} f(m) \cdot m^{\kappa+i-j} \\ &= \sum_{j\ even,\ j=2}^{j=i} \binom{\kappa}{j} f(m) \cdot m^{\kappa} \cdot m^{i-j} \\ &+ \sum_{j\ even,\ j=i+2}^{\kappa} \binom{\kappa}{j} f(m) \cdot m^{\kappa+i-j} \\ &= \sum_{j\ even,\ j=i+2}^{\kappa} \binom{\kappa}{j} f(m) \cdot m^{\kappa+i-j}. \end{aligned} \tag{11}$$

If $i = \kappa - 2$, then by (11)

$$f(m) \cdot m^{\kappa-2} = 0.$$

Hence (10) and (11) reduce to

$$\sum_{j\ even,\ j=4}^{\kappa} \binom{\kappa}{j} f(m) \cdot m^{\kappa-j} = 0$$

and

$$\sum_{j\ even,\ j=i+2}^{\kappa-2} \binom{\kappa}{j} f(m) \cdot m^{\kappa+i-j} = 0.$$

Continuing this procedure, we obtain $f(m) = 0$ for all $m \in L_0^\infty(\mathfrak{G})^*$ and therefore, $D = 0$.

Suppose now that κ is odd. By (6) we have

$$\mathfrak{A}(m) + \mathfrak{D}(m) = 0$$

for all $m \in L_0^\infty(\mathfrak{G})^*$. As before, we have $D(m,u) = 0$ for all $m \in L_0^\infty(\mathfrak{G})^*$. So $\mathfrak{C} = 0$. The same computation as for even κ yields $f = 0$ and therefore, $D = 0$. □

As an immediate consequence of Theorem 2.2 we give the following result.

Corollary 2.3. *Let $D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric bi-derivation and f be the trace of D. Then the following assertions are equivalent.*
(a) *f is (skew) centralizing;*
(b) *there exists $\kappa \in \mathbb{N}$ such that f is $\kappa-$(skew) centralizing;*
(c) *for every $\kappa \in \mathbb{N}$, f is $\kappa-$(skew) centralizing;*
(d) *$D = 0$.*

Corollary 2.4. *Let $D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric bi-derivation and f be the trace of D. Then the following assertions are equivalent.*
(a) *f is commuting;*
(b) *f is centralizing;*
(c) *f is skew commuting;*
(d) *f is skew centralizing;*
(e) *$D = 0$.*

In the following, we investigate the structure of symmetric generalized bi-derivations whose traces are $\kappa-$centralizing.

Theorem 2.5. *Let $G_D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric generalized bi-derivation and $\kappa \in \mathbb{N}$. If F is the trace of G, then the following assertions are equivalent.*
(a) *F is $\kappa-$commuting;*
(b) *F is $\kappa-$centralizing;*
(c) *there exists an element θ in $L_0^\infty(\mathfrak{G})^*$ such that $G(m, n) = m \cdot n \cdot \theta$ for all $m, n \in L_0^\infty(\mathfrak{G})^*$.*

Proof. Choose $u \in \Lambda(\mathfrak{G})$. First note that the Banach algebra $u \cdot L_0^\infty(\mathfrak{G})^*$ is isometrically isomorphic to the commutative Banach algebra $M(\mathfrak{G})$; see [8]. Hence for every $k, m, n \in L_0^\infty(\mathfrak{G})^*$, we have

$$\begin{aligned} k \cdot m \cdot n &= k \cdot u \cdot m \cdot u \cdot n \\ &= k \cdot u \cdot n \cdot u \cdot m \\ &= k \cdot n \cdot m. \end{aligned} \tag{12}$$

Also, for every $k, m, n \in L_0^\infty(\mathfrak{G})^*$, we have

$$\begin{aligned} G(k, n) &= k \cdot G(u, n) + D(k, n) \\ &= k \cdot G(n, u) + D(k, n) \\ &= k \cdot n \cdot G(u, u) + k \cdot D(n, u) + D(k, n) \\ &= k \cdot n \cdot G(u, u) + D(k, n). \end{aligned}$$

Now let f be the trace of D. Then

$$\begin{aligned} [F(m), m^\kappa] &= F(m) \cdot m^\kappa - m^\kappa \cdot F(m) \\ &= (m^2 \cdot G(u, u) + f(m)) \cdot m^\kappa - m^\kappa \cdot (m^2 \cdot G(u, u) + f(m)) \\ &= m^{\kappa+2} \cdot G(u, u) + f(m) \cdot m^\kappa - m^{\kappa+2} \cdot G(u, u) - m^\kappa \cdot f(m) \\ &= f(m) \cdot m^\kappa. \end{aligned}$$

So if F is $k-$centralizing, then f is $\kappa-$commuting. Hence $D = 0$ by Theorem 2.2. Thus

$$\begin{aligned} G(m, n) &= m \cdot n \cdot G(u, u) + D(m, n) \\ &= m \cdot n \cdot G(u, u) \end{aligned}$$

for all $m, n \in L_0^\infty(\mathfrak{G})^*$. That is, (b) implies (c). Also, the implication (a)⇒(b) is trivial. Finally, (c) implies (a) by (12). □

Theorem 2.6. *Let $G_D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric generalized bi-derivation and $\kappa \in \mathbb{N}$. If F is the trace of G, then the following statements hold.*

(i) *If F is $\kappa-$skew centralizing, then there exists an element θ in $L^1(\mathfrak{G})$ such that $G(m,n) = m \cdot n \cdot \theta$ for all $m, n \in L_0^\infty(\mathfrak{G})^*$.*

(ii) *If F is $\kappa-$skew commuting, then $G = 0$ on $L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^*$.*

Proof. (i) Suppose that F is $\kappa-$skew centralizing. So

$$\langle F(m), m^\kappa \rangle \in Z(L_0^\infty(\mathfrak{G})^*)$$

for all $m \in L_0^\infty(\mathfrak{G})^*$. Then

$$[F(m), m^{k+1}] = [\langle F(m), m^\kappa \rangle, m] = 0.$$

This implies that F is $(\kappa + 1)-$commuting. In view of Theorem 2.5, there exists $\theta \in L_0^\infty(\mathfrak{G})^*$ such that

$$G(m,n) = m \cdot n \cdot \theta$$

for all $m, n \in L_0^\infty(\mathfrak{G})^*$. Choose $u \in \Lambda(\mathfrak{G})$. Then

$$\begin{aligned} 2u \cdot \theta &= G(u,u) + u \cdot G(u,u) \\ &= F(u) \cdot u^\kappa + u^\kappa \cdot F(u) \\ &= \langle F(u), u^\kappa \rangle. \end{aligned}$$

Thus

$$u \cdot \theta \in Z(L_0^\infty(\mathfrak{G})^*).$$

The proof will be complete, if we note that $Z(L_0^\infty(\mathfrak{G})^*) = L^1(\mathfrak{G})$ and

$$G(m,n) = m \cdot n \cdot \theta = m \cdot n \cdot u \cdot \theta. \tag{13}$$

(ii) Let F be $\kappa-$skew commuting. By (i) there exists $\theta \in Z(L_0^\infty(\mathfrak{G})^*)$ such that

$$G(m,n) = m \cdot n \cdot \theta$$

for all $m, n \in L_0^\infty(\mathfrak{G})^*$. If $u \in \Lambda(\mathfrak{G})$, then

$$\begin{aligned} 0 &= \langle F(u), u^\kappa \rangle \\ &= F(u) \cdot u + u \cdot F(u) \\ &= u \cdot \theta + u \cdot \theta \\ &= 2\, u \cdot \theta. \end{aligned}$$

This together with (13) shows that $G = 0$. □

As an immediate corollary of Theorems 2.5 and 2.6 we present the next result.

Corollary 2.7. *Let $G_D : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric generalized bi-derivation and F be the trace of G. Then the following assertions are equivalent.*

(a) *F is (skew) centralizing;*

(b) *there exists $\kappa \in \mathbb{N}$ such that F is $\kappa-$(skew) centralizing;*

(c) *for every $\kappa \in \mathbb{N}$, F is $\kappa-$(skew) centralizing.*

We conclude the paper with the following result.

Theorem 2.8. *Let $J : L_0^\infty(\mathfrak{G})^* \times L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ be a symmetric Jordan bi-derivation. Then J is a symmetric bi-derivation.*

Proof. For every $m \in L_0^\infty(\mathfrak{G})^*$, we define the mapping $\Delta_m : L_0^\infty(\mathfrak{G})^* \to L_0^\infty(\mathfrak{G})^*$ by $\Delta_m(n) = J(m,n)$. Then

$$\begin{aligned}\Delta_m(n^2) &= J(m,n^2)\\ &= J(m,n)\cdot n + n\cdot J(m,n)\\ &= \Delta_m(n)\cdot n + n\cdot\Delta_m(n)\end{aligned}$$

for all $m,n \in L_0^\infty(\mathfrak{G})^*$. This shows that Δ_m is a Jordan derivation of $L_0^\infty(\mathfrak{G})^*$ for all $m \in L_0^\infty(\mathfrak{G})^*$. By [1] every Jordan derivation of $L_0^\infty(\mathfrak{G})^*$ is a derivation of $L_0^\infty(\mathfrak{G})^*$. Hence Δ_m is a derivation of $L_0^\infty(\mathfrak{G})^*$. Thus

$$\begin{aligned}J(m\cdot k,n) = \Delta_n(m\cdot k) &= \Delta_n(m)\cdot k + m\cdot\Delta_n(k)\\ &= J(m,n)\cdot k + m\cdot J(k,n).\end{aligned}$$

Consequently, J is a bi-derivation. □

Acknowledgement.
The authors would like to thank the referee of the paper for invaluable comments.

References

[1] M. H. Ahmadi Gandomani, M. J. Mehdipour, Jordan, Jordan right and Jordan left derivations on convolution algebras, Bull. Iranian Math. Soc. **45** (1) (2019) 189–204.
[2] M. Ashraf, On symmetric biderivations in rings, Rend. Istit. Mat. Univ. Trieste XXXI (1999) 25–36.
[3] J. Bae, W. Park, Approximate bi-homomorphisms and bi-derivations in C*−ternary algebras, Bull. Korean Math. Soc. 47 (2010) 195-–209.
[4] M. Bresar, On generalized bi-derivations and related maps, J. Algebra 122 (1995) 764–786.
[5] Q. Deng, On a conjecture of Vukman, Internat. J. Math. Math. Sci. 20 (2) (1997) 263–266.
[6] A. Fosner, On generalized α-bi derivations, Mediterr. J. Math. 12 (1) (2015) 1–7.
[7] E. Hewitt, K. Ross, Abstract Harmonic Analysis I, Springer-Verlag, New York, 1970.
[8] A. T. Lau, J. Pym, Concerning the second dual of the group algebra of a locally compact group, J. London Math. Soc. 41 (1990) 445–460.
[9] G. Maksa, A remark on symmetric biadditive function having nonnegative diagonalization, Glas. Mat. Ser. III 46 (1980) 279—282.
[10] G. Maksa, On the trace of symmetric bi-derivations, C. R. Math. Rep. Acad. Sci. Canada 9 (1987) 303—307.
[11] M. J. Mehdipour, Z. Saeedi, Derivations on group algebras of a locally compact abelian group, Monatsh. Math. 180 (3) (2016) 595-–605.
[12] N. M. Muthana, Left centralizer traces, generalized biderivations, left bi-multipliers and generalized Jordan biderivations, Aligarh Bull. Math. 26 (2007) 33–45.
[13] C. Park, Biderivations and bihomomorphisms in Banach algebras, Filomat 33 (8) (2019) 2317-–2328.
[14] I. M. Singer, J. Wermer, Derivations on commutative normed algebras, Math. Ann. 129 (1955) 260-–264.
[15] J. Vukman, Symmetric bi-derivations on prime and semi-prime rings, Aequationes Math. 38 (2-3) (1989) 245—254.
[16] J. Vukman, Two results concerning symmetric bi-derivations on prime rings, Aequations Math. 40 (1990) 181–189.
[17] Y. Wang, Biderivations of triangular rings, Lin. Multilin. Alg. 64 (10) (2016) 1952–1959.
[18] Y. Wang, Permuting triderivations of prime and semiprime rings, Miskolc Math. Notes 18 (2017) 489-–497.