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A New Iteration Process for Approximation of Fixed Points of $\alpha-\psi-$Contractive Type Mappings in CAT(0) Spaces

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Abstract. In this paper we introduce a new iterative algorithm for approximating fixed points of $\alpha-\psi-$contractive type mappings in CAT(0) spaces. We prove a $\triangle$-convergence theorem under suitable conditions. The result we obtain improves and extends several recent results stated by many others; they also complement many known recent results in the literature. We then provide some numerical examples to illustrate our main result and to display the efficiency of the proposed algorithm.

1. Introduction

The Banach contraction principle states that every contraction on a complete metric space has a unique fixed point, moreover, the fixed point can be approximated by the Picard's iterates. F. E. Browder [5] and D. Gohde [13] independently proved in 1965 that every nonexpansive self-mapping of a closed, convex, and bounded subset of a uniformly convex Banach space has a fixed point. W. A. Kirk (see [16, 17]) studied for the first time the fixed point theory in Cartan-Alexandrov-Toponogov spaces, or briefly, in CAT(0) spaces. He proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Many others studied concerning fixed point theorems for various mappings in CAT(0) space [11, 18, 20, 22, 26, 27].

Let (X,d) be a metric space and x,y be two fixed elements in X such that $d(x,y)=l$. A geodesic path from x to y is an isometry $c:[0,l]\to c([0,1])\subset X$ such that $c(0)=x$, $c(l)=y$. The image of a geodesic path between two points is called a geodesic segment. A metric space (X,d) is called a geodesic space if every two points of X are joined by a geodesic segment. A geodesic triangle represented by $\triangle(x,y,z)$ in a geodesic space consists of three points x,y,z and the three segments joining each pair of the points. A comparison triangle of a geodesic triangle $\triangle(x,y,z)$, denoted by $\overline{\triangle}(x,y,z)$ or $\triangle(\overline{x},\overline{y},\overline{z})$, is a triangle in the Euclidean space $\mathbb{R}^2$ such that $d(x,y)=d_{\mathbb{R}^2}(\overline{x},\overline{y})$, $d(x,z)=d_{\mathbb{R}^2}(\overline{x},\overline{z})$, and $d(y,z)=d_{\mathbb{R}^2}(\overline{y},\overline{z})$. This is obtainable by using the triangle inequality, and it is unique up to isometry on $\mathbb{R}^2$. Bridson and Haefliger [6] have shown that such a triangle always exists. A geodesic segment joining two points x,y in a geodesic space X is represented by $[x,y]$. Every point z in the segment is represented by $\alpha x\oplus(1-\alpha)y$, where $\alpha\in[0,1]$, that is, $[x,y]:=\{\alpha x\oplus(1-\alpha)y:\ \alpha\in[0,1]\}$. A subset C of a metric space X is called convex if for all $x,y\in C$,

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$[x, y] \subset C$. A geodesic space is called a CAT(0) space if for every geodesic triangle $\triangle$ and its comparison triangle $\overline{\triangle}$, the following inequality, called CAT(0) inequality, is satisfied: $d(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y})$ for all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. Complete CAT(0) spaces are often called Hadamard spaces (see [15] and [23]). Examples of CAT(0) spaces include the $\mathbb{R}$-tree, Hadamard manifold, and Hilbert ball equipped with hyperbolic metric. For more details on these spaces, see for example [2, 4, 7]. A geodesic space (X, d) is called hyperbolic (see [12, 24]) if, for any $x, y, z \in X$,

$$d(\frac{1}{2}z \oplus \frac{1}{2}x, \frac{1}{2}z \oplus \frac{1}{2}y) \le \frac{1}{2}d(x, y).$$

Also, Bruhat and Tits [8] proved (CN) inequality in CAT(0) spaces as below:

$$d\Big(\frac{x \oplus y}{2}, z\Big)^2 \le \frac{1}{2}d(x, z)^2 + \frac{1}{2}d(y, z)^2 - \frac{1}{4}d(x, y)^2.$$

The class of hyperbolic spaces include the normed spaces, CAT(0) spaces, and some others. Bashir Ali in [3] presented an example of a hyperbolic space that is not a normed space. Therefore the class of hyperbolic spaces is more general than the class of normed spaces.

We now turn to recall some well-known iteration processes. The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$.

Further, the Ishikawa iteration process is defined

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are some sequences in $(0, 1)$.

Let C be a nonempty subset of (X, d). A mapping $T : C \to C$ is said to be mean nonexpansive if

$$d(Tx, Ty) \le a\, d(x, y) + b\, d(x, Ty), \quad \forall x, y \in C,$$

where a and b are two nonnegative real numbers such that $a + b \le 1$.

In 2017 Abkar and Rastgoo iteration process [1] is defined

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \alpha_n)x_n \oplus \alpha_n T(x_n), \\ y_n = T((1 - \beta_n)z_n \oplus \beta_n T(z_n)), \\ x_{n+1} = T(y_n), \end{cases}$$

where T is a mean nonexpansive mapping and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are some sequences in $(0, 1)$.

In this paper, we give the definition of a new type of mappings, namely, $\alpha - \psi-$contractive mappings and prove some fixed point theorem for this type of mappings in CAT(0) spaces. Then, we introduce a new iterative algorithm for approximating fixed points of $\alpha - \psi-$contractive type mappings in CAT(0) spaces. Under suitable conditions, we prove a $\triangle$-convergence theorem for our algorithm. To be more precise, Let (X, d) be a complete CAT(0) space and C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a given mapping. We say that T is an $\alpha - \psi-$contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(T(x), T(y)) \le \psi(d(x, y)), \text{ for all } x, y \in X.$$

The following example shows that there are $\alpha-\psi-$contractive mappings which are not mean nonexpansive, therefore the results of this paper cannot be deduced from results already appeared in [1].
Example 1. Let $X = [0,1]$ be endowed with the usual metric $d(x,y) = |x-y|$. Define $T : X \to X$ by $T(x) = 1$ if x is rational, and $T(x) = 0$ if x is irrational.

$$Tx = \begin{cases} 1 & x \in [0,1] \textit{ is rational;} \\ \\ 0 & x \in [0,1] \textit{ is irrational.} \end{cases}$$

Now, we consider the mapping $\alpha : X \times X \to [0,1]$ by

$$\alpha(x,y) = \frac{|x-y|}{3}.$$

Clearly T is an $\alpha-\psi-$contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$, but not mean nonexpansive; suppose that T is mean nonexpansive, then

$$d(Tx,Ty) \leq a\,d(x,y) + b\,d(x,Ty), \quad \forall x,y \in C, \tag{1}$$

where a and b are two nonnegative real numbers such that $a+b \leq 1$. Now, let $x = 0$ and $y \in [0,1]$ is irrational, so due to the above inequality we can write: $1 \leq ay$, but since $a \leq 1$ and $0 < y < 1$, this is a contradiction.

The above example shows that this paper improves and extends several recent results in the literature, in particular, the result in [1]. Finally, we provide some numerical examples to illustrate our main result, displaying on this way the efficiency of our proposed algorithm.

2. Preliminaries

Throughout this article, (X,d) will stand for a metric space. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers.
Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contains $\{x_n\}$. Suppose $\Phi(x) = \limsup_{n\to\infty} d(x_n,x)$ $\big[\Phi : C \to \mathbb{R}\big]$. We denote the notation

$$\{x_n\} \rightharpoonup \omega \Longleftrightarrow \Phi(\omega) = \inf_{x\in C} \Phi(x).$$

We start by recalling some basic Lemmas and definitions.

Lemma 2.1. *([21], Lemma 2.1) Let (X,d) be a CAT(0) space. Then*

$$d((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d(x,z) + \alpha d(y,z)$$

for all $\alpha \in [0,1]$ and $x,y,z \in X$.

Lemma 2.2. *([21], Lemma 4.5) Let x be a given point in a CAT(0) space (X,d) and $\{t_n\}$ be a sequence in a closed interval $[a,b]$ with $0 < a \leq b < 1$ and $0 < a(1-b) \leq \dfrac{1}{2}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that*

1. $\limsup_{n\to\infty} d(x_n,x) \leq r$,
2. $\limsup_{n\to\infty} d(y_n,x) \leq r$,
3. $\limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Definition 2.3. *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X,d).*

1. *The asymptotic radius* $r(\{x_n\})$ *of* $\{x_n\}$ *is given by*

$$r(\{x_n\}) := \inf_{x\in X}\{r(x,\{x_n\})\},$$

where $r(x,\{x_n\}) := \limsup_{n\to\infty} d(x_n, x)$.

2. *The asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ *is the set*

$$A(\{x_n\}) := \{x \in X : \ r(x,\{x_n\}) = r(\{x_n\})\}.$$

In 2006, Dhompongsa et al proved that for each bounded sequence $\{x_n\}$ in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see Proposition 7 in [10]). This motivates the following notion of $\triangle$-convergence which is regarded as some sort of weak convergence in CAT(0) spaces; of course this analogy is by no means complete.

Definition 2.4. *[18] Let* (X, d) *be a CAT(0) space. A sequence* $\{x_n\}$ *in* X *is said to* $\triangle$*-converge to* $x \in X$ *if and only if* x *is the unique asymptotic center of all subsequences of* $\{x_n\}$*. In this case, we write* $\triangle - \lim_{n\to\infty} x_n = x$ *and call* x *the* $\triangle$*-limit of* $\{x_n\}$.

We recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

Proposition 2.5. *([21], Proposition 3.12). Let* $\{x_n\}$ *be a bounded sequence in a CAT(0) space* (X, d) *and let* $C \subset X$ *be a closed convex subset which contains* $\{x_n\}$*. Then,*

(i) $\triangle - \lim_{n\to\infty} x_n = x$ *implies* $\{x_n\} \rightharpoonup x$;

(ii) *if* $\{x_n\}$ *is regular, then* $\{x_n\} \rightharpoonup x$ *implies* $\triangle - \lim_{n\to\infty} x_n = x$.

Lemma 2.6. *The following assertions in a CAT(0) space hold:*

(i) *[19] Every bounded sequence in a complete* $CAT(0)$ *space has a* $\triangle$*-convergent subsequence.*

(ii) *[9] If* $\{x_n\}$ *is a bounded sequence in a closed convex subset* C *of a complete CAT(0) space* (X, d)*, then the asymptotic center of* $\{x_n\}$ *is in* C.

(iii) *[19] If* $\{x_n\}$ *is a bounded sequence in a complete CAT(0) space* (X, d) *with* $A(\{x_n\}) = \{p\}$, $\{v_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{v_n\}) = \{v\}$*, and the sequence* $\{d(x_n, v)\}$ *converges, then* $p = v$.

3. A $\triangle$-Convergence Theorem

We begin this section by recalling the notion of mean nonexpansive mappings; we then compare this latter with the class of $\alpha - \psi$-contractive mappings.

Definition 3.1. *Let* C *be a nonempty subset of* (X, d)*. A mapping* $T : C \to C$ *is said to be mean nonexpansive if*

$$d(Tx, Ty) \le a\, d(x, y) + b\, d(x, Ty), \quad \forall x, y \in C,$$

where a *and* b *are two nonnegative real numbers such that* $a + b \le 1$.

In 2012 Samet, C. Vetro, P. Vetro [25] introduced the notion of $\alpha - \psi$–contractive type operator and proved some fixed point results. Then let us introduce this notion in CAT(0) spaces.
Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 3.2. *[25] For every function* $\psi : [0, +\infty) \to [0, +\infty)$ *the following holds:*
if ψ *is nondecreasing, then for each* $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ *implies* $\psi(t) < t$.

Definition 3.3. *Let (X, d) be a complete CAT(0) space and C be a nonempty bounded closed convex subset of X and $T : C \to C$ be a given mapping. We say that T is an $\alpha - \psi-$contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(T(x), T(y)) \leq \psi(d(x, y)), \text{ for all } x, y \in X.$$

Definition 3.4. *Let $f : X \to X$ and $\alpha : X \times X \to [0; 1)$. We say that f is α-admissible if*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(f(x), f(y)) \geq 1, \quad \forall x, y \in X.$$

Theorem 3.5. *([14]) Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi-$contractive type mapping satisfying the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.
Then, T has a fixed point.

Further let us give a fixed point result concerning $\alpha - \psi-$contractive mapping in CAT(0) space.

Theorem 3.6. *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a $\alpha - \psi-$contractive type mapping with $\alpha(x, y) \geq 1$, and let $\{x_n\} \subset X$ be an approximate fixed point sequence (i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$) and $\{x_n\} \rightharpoonup \omega$. Then $T(\omega) = \omega$.*

Proof. Since $\{x_n\}$ is an approximate fixed point sequence, we define:

$$\Phi(x) = \limsup_{n\to\infty} d(T^m x_n, x) \quad \forall m \geq 1. \tag{2}$$

We claim that $\Phi(Tx) \leq \Phi(x)$ holds for each $x \in C$. In fact, if $m = 1$, by the definition of $\alpha - \psi-$contractive type mappings whit $\alpha(x, y) \geq 1$ and (2),

$$\begin{aligned}\Phi(Tx) &= \limsup_{n\to\infty} d(Tx_n, Tx) \\ &\leq \limsup_{n\to\infty} \alpha(x_n, x)d(Tx_n, Tx) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &= \Phi(x).\end{aligned}$$

By continuing this process, we conclude that $\Phi(T^m x) \leq \Phi(x)$ holds for any positive integer m. In particular, we have

$$\lim_{n\to\infty} \Phi(T^m \omega) \leq \Phi(\omega), \tag{3}$$

Assume by contradiction that $\{T^m \omega\}$ contains no norm-convergent subsequence, we can assume that there exists $\epsilon_0 > 0$ such that

$$d(T^n \omega, T^m \omega) \geq \epsilon_0, \quad n \neq m. \tag{4}$$

For the above ϵ_0, we can take $\theta > 0$ such that

$$(\Phi(\omega) + \theta)^2 < \Phi(\omega)^2 + \frac{{\epsilon_0}^2}{4}. \tag{5}$$

By the definition of Φ and (3), there exists $N, M \in \mathbb{N}$ such that for any $m \geq M$

$$d(T^m \omega, x_n) < \Phi(\omega) + \theta \quad \forall n \geq N.$$

Hence, the (CN) inequality, (4) and (5) imply that

$$\begin{aligned}
d\Big(\frac{T^{m_1}\omega \oplus T^{m_2}\omega}{2}, x_n\Big)^2 &\leq \frac{1}{2}d(T^{m_1}\omega, x_n)^2 + \frac{1}{2}d(T^{m_2}\omega, x_n)^2 - \frac{1}{4}d(T^{m_1}\omega \oplus T^{m_2}\omega)^2 \\
&\leq \frac{1}{2}(\Phi(\omega) + \theta)^2 + \frac{1}{2}(\Phi(\omega) + \theta)^2 - \frac{1}{4}{\epsilon_0}^2 \\
&< \Phi(\omega)^2
\end{aligned}$$

holds for any $m_1, m_2 \geq M$. Let $z = \frac{T^{m_1}\omega \oplus T^{m_2}\omega}{2}$, then $z \in C$ and $z \neq \omega$, we have got a contradiction with $\Phi(\omega) = \inf_{x\in C} \Phi(x)$. So $\{T^m\omega\}$ contains norm-convergent subsequence, denoted by $\{T^{m_i}\omega\}$. We may assume that

$T^{m_i}\omega \to \omega'$, then

$$\limsup_{n\to\infty} d(\omega', x_n) = \limsup_{n\to\infty} \lim_{n\to\infty} d(T^{m_i}\omega, x_n) = \lim_{n\to\infty} \Phi(T^{m_i}\omega) \leq \Phi(\omega).$$

Since $\Phi(\omega) = \inf_{x\in C} \Phi(x)$, hence $\omega' = \omega$. So $T^{m_i}\omega \to \omega$. Again using the definition of $\alpha - \psi-$contractive type mappings whit $\alpha(x, y) \geq 1$, we have

$$\begin{aligned}
d(T^{m_i}\omega, T\omega) &\leq \alpha(T^{m_{i-1}}\omega, \omega)d(T^{m_i}\omega, T\omega) \\
&\leq \psi(d(T^{m_{i-1}}\omega, \omega)) \\
&\leq d(T^{m_{i-1}}\omega, \omega)
\end{aligned}$$

Taking the limit of both sides, then $d(\omega, T\omega) \leq d(\omega, \omega)$. So we obtain $\omega = T\omega$, thus ω is a fixed point of T, i.e. $T\omega = \omega$. □

By using Theorem 3.6 and Proposition 2.5 we conclude the following theorem:

Theorem 3.7. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $T : C \to C$ be a $\alpha-\psi-$contractive type mapping. If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\triangle-\lim_{n\to\infty} x_n = p$, then $T(p) = p$.*

Next, let us give the main result of this section-a $\triangle$-convergence theorem for $\alpha-\psi-$contractive mappings in CAT(0) spaces. Then, we introduce a new iterative algorithm to approximate the fixed point of our mapping.

Theorem 3.8. *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of (X, d). Let $T : C \to C$ be a $\alpha-\psi-$contractive type mapping with $\alpha(x, y) \geq 1$ for all $x, y \in X$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\eta_n\}_{n=1}^{\infty}$ $\{\mu_n\}_{n=1}^{\infty}$,$\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$ also $\{\alpha_n\}$ be a sequence in a closed interval $[c, d]$ with $0 < c \leq d < 1$ and $0 < c(1-d) \leq \frac{1}{2}$. Then $\{x_n\}_{n=1}^{\infty}$ defined by*

$$\begin{cases}
x_1 \in C, \\
z_n = (1-\alpha_n)x_n \oplus \alpha_n T(x_n), \\
y_n = (1-\eta_n-\mu_n)x_n \oplus \eta_n T(x_n) \oplus \mu_n T(z_n), \\
x_{n+1} = (1-\beta_n-\gamma_n)T(x_n) \oplus \beta_n T(z_n) \oplus \gamma_n T(y_n),
\end{cases} \tag{6}$$

is $\triangle$-convergent to some point $p \in Fix(T)$.

Proof. By using Theorem 3.5, we conclude that $Fix(T) \neq \emptyset$. Now, we will divide the proof into three steps.

Step 1. First, we will prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in Fix(T)$, where $\{x_n\}$ is defined by (6). For this purpose, let $p \in Fix(T)$. By Lemma 2.1 and using the fact that $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ we obtain

$$\begin{aligned} d(z_n, p) &= d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(T(x_n), p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n \alpha(x_n, p) d(T(x_n), p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\le d(x_n, p), \end{aligned} \tag{7}$$

for all $n \in \mathbb{N}$. This together with the fact that $\{\beta_n\}_{n=1}^{\infty} \subset (0, 1)$ yields

$$\begin{aligned} d(y_n, p) &= d((1-\eta_n-\mu_n)x_n \oplus \eta_n T(x_n) \oplus \mu_n T(z_n), p) \\ &\le (1-\eta_n-\mu_n)d(x_n, p) + \eta_n d(T(x_n), p) + \mu_n d(T(z_n), p) \\ &\le (1-\eta_n-\mu_n)d(x_n, p) + \eta_n \alpha(x_n, p) d(T(x_n), p) + \mu_n \alpha(z_n, p) d(T(z_n), p) \\ &\le (1-\eta_n-\mu_n)d(x_n, p) + \eta_n \psi(d(x_n, p)) + \mu_n \psi(d(z_n, p)) \\ &\le (1-\eta_n-\mu_n)d(x_n, p) + \eta_n d(x_n, p) + \mu_n d(z_n, p) \\ &\le (1-\eta_n-\mu_n)d(x_n, p) + \eta_n d(x_n, p) + \mu_n d(x_n, p) \\ &\le d(x_n, p), \end{aligned} \tag{8}$$

for each $n \in \mathbb{N}$. It now follows that

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\beta_n-\gamma_n)T(x_n) \oplus \beta_n T(z_n) \oplus \gamma_n T(y_n), p) \\ &\le (1-\beta_n-\gamma_n)d(T(x_n), p) + \beta_n d(T(z_n), p) + \gamma_n d(T(y_n), p) \\ &\le (1-\beta_n-\gamma_n)\alpha(x_n, p)d(T(x_n), p) + \beta_n \alpha(z_n, p) d(T(z_n), p) + \gamma_n \alpha(y_n, p) d(T(y_n), p) \\ &\le (1-\beta_n-\gamma_n)\psi(d(x_n, p)) + \beta_n \psi(d(z_n, p)) + \gamma_n \psi(d(y_n, p)) \\ &\le (1-\beta_n-\gamma_n)d(x_n, p) + \beta_n d(z_n, p) + \gamma_n d(y_n, p) \\ &\le (1-\beta_n-\gamma_n)d(x_n, p) + \beta_n d(x_n, p) + \gamma_n d(x_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{9}$$

Consequently, we have $d(x_{n+1}, p) \le d(x_n, p)$ for all $n \ge 1$. This implies that $\{d(x_n, p)\}$ is a decreasing sequence of real numbers. Since this sequence is bounded below, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists. Thus, $\{x_n\}$ is bounded.

Step 2. We will prove that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Without loss of generality, we may assume that

$$\mathbf{r} := \lim_{n\to\infty} d(x_n, p). \tag{10}$$

Therefore

$$\begin{aligned} \limsup_{n\to\infty} d(T(x_n), p) &\le \limsup_{n\to\infty} \alpha(x_n, p) d(T(x_n), p) \\ &\le \limsup_{n\to\infty} \psi(d(x_n, p)) \\ &\le d(x_n, p) = \mathbf{r}. \end{aligned} \tag{11}$$

According to (7), we also have

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, p) &= \limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) \\
&\leq (1-\alpha_n)\limsup_{n\to\infty} d(x_n, p) + \alpha_n \limsup_{n\to\infty} d(T(x_n), p) \\
&\leq (1-\alpha_n)\limsup_{n\to\infty} d(x_n, p) + \alpha_n \limsup_{n\to\infty} \alpha(x_n, p) d(T(x_n), p) \\
&\leq (1-\alpha_n)\limsup_{n\to\infty} d(x_n, p) + \alpha_n \limsup_{n\to\infty} d(x_n, p) \\
&\leq \limsup_{n\to\infty} d(x_n, p) = \mathbf{r}. \qquad (12)
\end{aligned}$$

On the other hand, using (8) we can write

$$\begin{aligned}
\mathbf{r} = \limsup_{n\to\infty} d(x_{n+1}, p) &= \limsup_{n\to\infty} d((1-\beta_n-\gamma_n)T(x_n) \oplus \beta_n T(z_n) \oplus \gamma_n T(y_n), p) \\
&\leq (1-\beta_n-\gamma_n) d(T(x_n), p) + \beta_n d(T(z_n), p) + \gamma_n d(T(y_n), p) \\
&\leq (1-\beta_n-\gamma_n)\alpha(x_n, p) d(T(x_n), p) + \beta_n \alpha(z_n, p) d(T(z_n), p) + \gamma_n \alpha(y_n, p) d(T(y_n), p) \\
&\leq (1-\beta_n-\gamma_n)\psi(d(x_n, p)) + \beta_n \psi(d(z_n, p)) + \gamma_n \psi(d(y_n, p)) \\
&\leq (1-\beta_n-\gamma_n) d(x_n, p) + \beta_n d(z_n, p) + \gamma_n d(y_n, p) \\
&\leq (1-\beta_n-\gamma_n) d(x_n, p) + \beta_n d(z_n, p) + \gamma_n d(x_n, p) \\
&\leq (1-\beta_n) d(x_n, p) + \beta_n d(z_n, p) \\
&\leq (1-\beta_n)\mathbf{r} + \beta_n d(z_n, p)
\end{aligned}$$

which implies that

$$\mathbf{r} = \limsup_{n\to\infty} d(z_n, p). \qquad (13)$$

Therefore,

$$\mathbf{r} = \limsup_{n\to\infty} d(z_n, p) = \limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p). \qquad (14)$$

By using Lemma 2.2 together with (10), (11) and (14), we have

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \qquad (15)$$

Therefore, we are done.

Step 3. Define

$$\Omega_\triangle(x_n) := \bigcup_{\{v_n\}\subseteq\{x_n\}} A(\{v_n\}) \subseteq Fix(T).$$

We claim that the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T and that $\Omega_\triangle(x_n)$ consists of exactly one point. To this end, we assume that $v \in \Omega_\triangle(x_n)$. It follows from the definition of $\Omega_\triangle(x_n)$ that there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. Now, use the assertion (**i**) in Lemma 2.6 to obtain a subsequence $\{\rho_n\}$ of $\{v_n\}$ such that

$$\triangle - \lim_{n\to\infty} \rho_n = \rho \in C.$$

It now follows from Theorem 3.7 that $\rho \in Fix(T)$. Since the sequence $\{d(v_n, \rho)\}$ is convergent, it follows from the assertion (**ii**) in Lemma 2.6 that $v = \rho$. Therefore $\Omega_\triangle(x_n) \subseteq Fix(T)$. Finally, we show that $\Omega_\triangle(x_n)$ consists of exactly one point. Let $\{v_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $v = \rho \in Fix(T)$. Since $\{d(x_n, \rho)\}$ converges, by assertion (**iii**) in Lemma 2.6, we have $x = \rho \in Fix(T)$, that is, $\Omega_\triangle(x_n) = x$. This completes the proof. □

4. Numerical Results

In the following, we supply a numerical example of a $\alpha - \psi-$contractive type Mapping satisfying the conditions of Theorem 3.8, and some numerical experiment results to explain the conclusion of our algorithm.
Obviously, every $\alpha-\psi-$contractive type mapping satisfies the Banach contraction principle, (with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$). Note that a $\alpha - \psi-$contractive type mapping is not necessarily contraction.
Example 2. Suppose that $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2} - \frac{1}{4} & x > 1; \\ \frac{x}{4} & 0 \leq x \leq 1; \\ 0 & x < 0. \end{cases}$$

Now, we define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1.5 & x, y \in [0, 1]; \\ 1 & otherwise. \end{cases}$$

Clearly T is an $\alpha - \psi-$contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$.
Moreover, Clearly T is α-admissible.
Now, all the hypotheses of Theorem 3.8 are satisfied. Consequently, T has a fixed point. In this example, 0 is fixed points of T. Put

$$\alpha_n = \beta_n = \eta_n = \mu_n = \gamma_n = \frac{1}{n + 100}.$$

By using MATHEMATICA, we computed the iterates of algorithm (6) for $x_1 = 0.5 \in [-1, 1]$. Finally, by the numerical experiments we compared Mann iteration and Ishikawa iteration process with our algorithm (6) (see Table 1). Moreover, the convergence behaviors of these algorithms for five different initial points $x_1 = 0.5, -1.5, 0, 1, 1.5 \in [-1, 1]$ are shown in Figure 1. We conclude that x_n converges to 0.

Numerical Results			
Step	Our Algorithm	Mann Algorithm	Ishikawa Algorithm
1	0.5	0.5	0.5
2	0.124972	0.496287	0.492602
3	0.0312363	0.492638	0.485384
4	0.00780743	0.489051	0.478341
5	0.00195145	0.485524	0.471467
6	0.000487763	0.482056	0.464756
7	0.000121916	0.478645	0.458202
8	0.0000304731	0.47529	0.451802
9	7.6168×10^{-6}	0.47199	0.445548
10	1.90384×10^{-6}	0.468742	0.439438
11	4.75871×10^{-7}	0.465546	0.433466
12	1.18946×10^{-76}	0.4624	0.427628
13	2.97312×10^{-8}	0.459304	0.42192
14	7.43148×10^{-9}	0.456255	0.416338
15	1.85755×10^{-9}	0.453254	0.410878
16	4.64308×10^{-10}	0.450298	0.405536
17	1.16058×10^{-10}	0.447386	0.400309
18	2.90096×10^{-11}	0.444519	0.395193
19	7.25124×10^{-12}	0.441693	0.390186
20	1.81252×10^{-12}	0.438909	0.385283
21	4.53059×10^{-12}	0.436166	0.380482
22	1.13247×10^{-12}	0.433463	0.37578
23	2.83076×10^{-14}	0.430798	0.371174
24	7.07584×10^{-15}	0.428171	0.366661
25	1.7687×10^{-15}	0.425581	0.362239
26	4.42111×10^{-16}	0.423028	0.357905
27	1.10512×10^{-16}	0.42051	0.353657
28	2.76242×10^{-17}	0.418027	0.349492
29	6.9051×10^{-18}	0.415577	0.345409
30	1.72604×10^{-18}	0.413161	0.341404

Table 1: Numerical results corresponding to $x_1 = \dfrac{1}{2}$ for 30 steps.

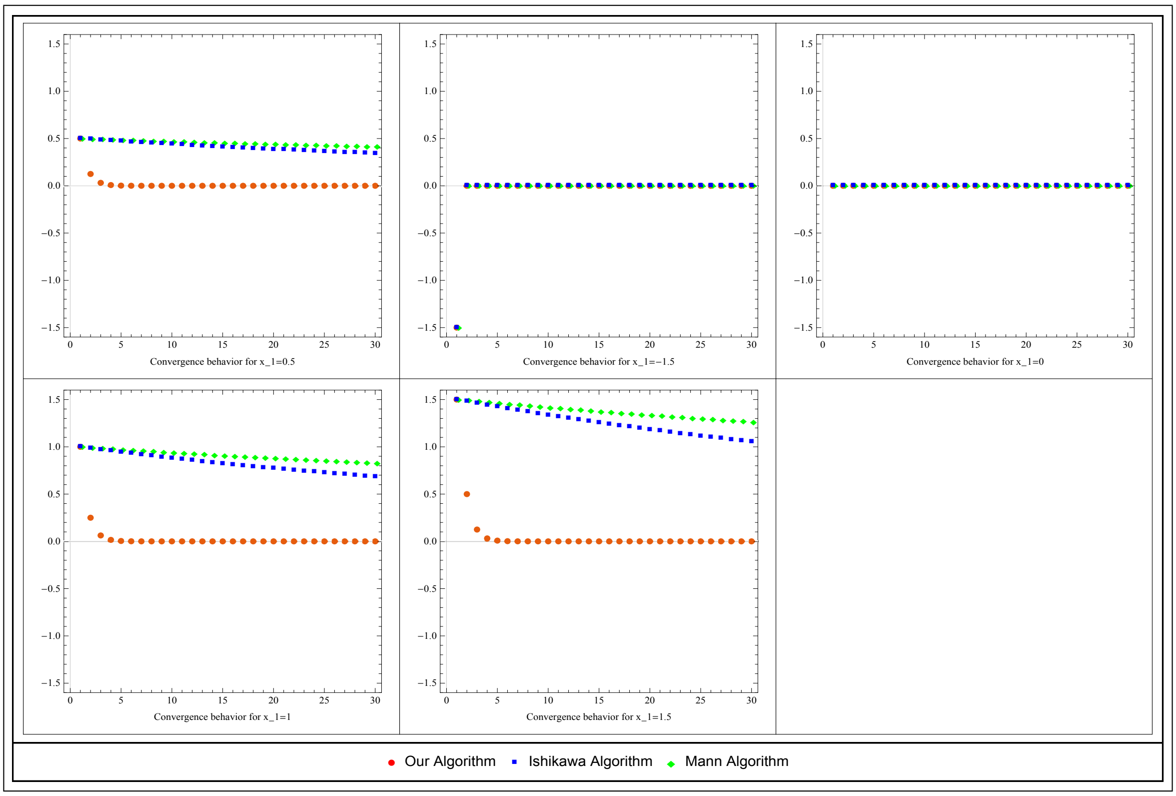

Figure 1: Convergence behaviors corresponding to $x_1 = 0.5,\ -1.5,\ 0,\ 1,\ 1.5$ for 30 steps.

Example 3. Consider $X = \mathbb{R}^2$ equipped with the Euclidean norm. Let $x = (x_1, x_2) \in \mathbb{R}^2$, then the squared distance of x from the origin, O, is

$$||x||^2 = x_1^2 + x_2^2.$$

Consider $C = [-1, 1] \times [-1, 1]$ which is a bounded, closed, and convex subset of X. We define the mapping $\mathrm{K} : C \longrightarrow C$ by

$$\mathrm{K}(x_1, x_2) := (\frac{1}{3}x_1, \frac{1}{3}x_2)$$

K is a $\alpha - \psi$−contractive type Mapping with $\alpha(x, y) = 1$ and $\psi(x) = \frac{x}{2}$. Clearly, zero is the only fixed point of the mapping K . In this case, our algorithm is the following:

$$\begin{cases} x_{(1)} = (x_{(1)_1}, x_{(1)_2}) \in C, \\ (z_{(n)_1}, z_{(n)_2}) = (1 - \alpha_n)(x_{(n)_1}, x_{(n)_2}) + \alpha_n K((x_{(n)_1}, x_{(n)_2})), \\ (y_{(n)_1}, y_{(n)_2}) = (1 - \eta_n - \mu_n)(x_{(n)_1}, x_{(n)_2}) + \eta_n K((x_{(n)_1}, x_{(n)_2})) + \mu_n K((z_{(n)_1}, z_{(n)_2})), \\ (x_{(n+1)_1}, x_{(n+1)_2}) = (1 - \beta_n - \gamma_n)K((x_{(n)_1}, x_{(n)_2})) + \beta_n K((z_{(n)_1}, z_{(n)_2})) + \gamma_n K((y_{(n)_1}, y_{(n)_2})). \end{cases} \tag{16}$$

Put $\alpha_n = \beta_n = \eta_n = \mu_n = \gamma_n = \dfrac{1}{n + 100}$.

Using MATHEMATICA, we have computed the iterates of the algorithm (16) for $x_{(1)} = (\frac{1}{2}, \frac{1}{2}) \in C$ for 500 steps. Finally, by the numerical experiments we compared Ishikawa iteration process with our algorithm (16). The convergence behaviors of these algorithms are shown in Figure 2. The conclusion is that x_n converges to $(0, 0)$.

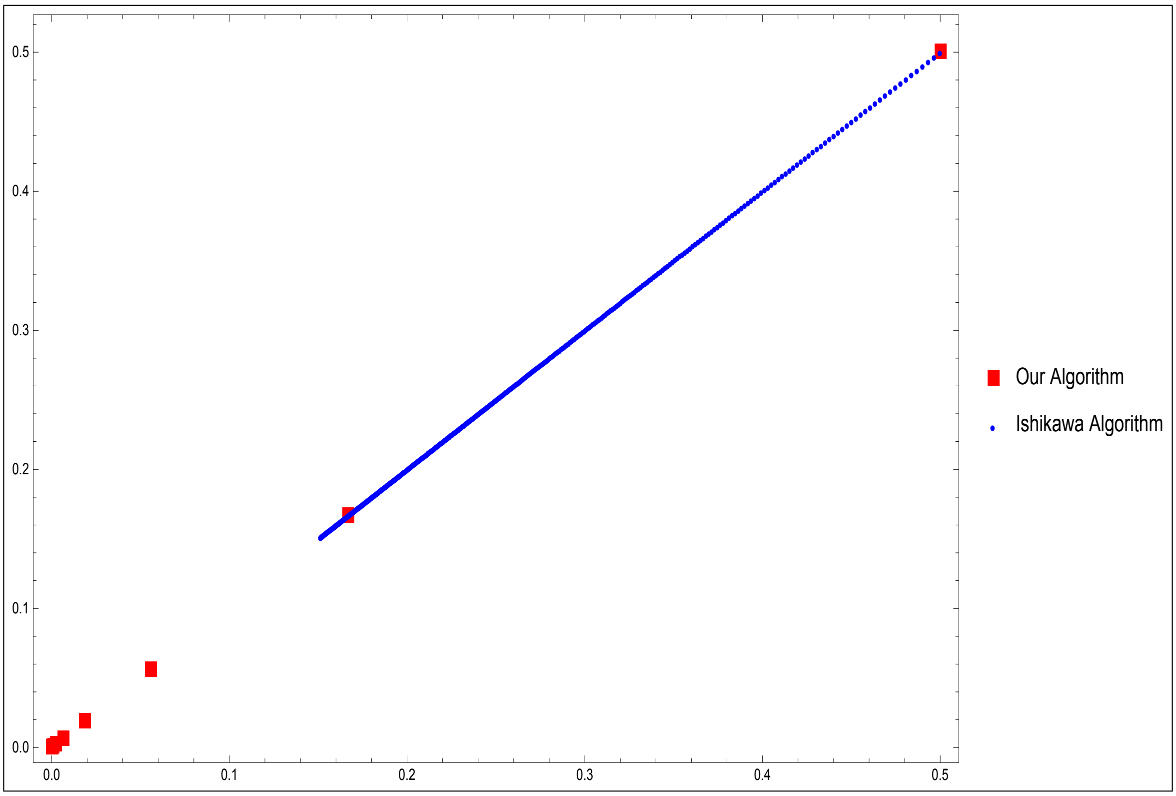

Figure 2: Convergence behaviors corresponding to $x_{(1)} = (\frac{1}{2}, \frac{1}{2})$ for 500 steps.

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