



Lyapunov Functions for Fractional Order h -Difference Systems

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Abstract. This paper presents some new propositions related to the fractional order h -difference operators, for the case of general quadratic forms and for the polynomial type, which allow proving the stability of fractional order h -difference systems, by means of the discrete fractional Lyapunov direct method, using general quadratic Lyapunov functions, and polynomial Lyapunov functions of any positive integer order, respectively. Some examples are given to illustrate these results.

1. Introduction

Fractional calculus, the study of integrals and differences of any order, is a topic of growing interest. Basic information on fractional calculus, concepts, ideas and their applications can be found e.g. in [17, 18, 27–29]. Dynamical systems are one of the most active areas because of their applications in various fields of science and engineering, and many authors have focused on the stability of the nonlinear fractional systems, see for instance [1, 4, 5, 9, 10, 19–21, 31, 33–36]. Due to the lack of geometrical interpretation of the fractional derivatives and the differences, it is difficult to find a valid tool to analyze the stability of the fractional equations, and to our knowledge there are few works on the stability of solutions for either fractional differential equations, see [19, 20, 34, 35] or fractional difference equations, see [1, 4, 5, 9, 10, 21, 31, 36].

This paper will focus on the stability of the fractional order h -difference systems. The fractional order h -difference operators are special operators of the discrete fractional calculus, which were initiated by Miller and Ross [23] in 1988. Since then there has been a great deal of interest in the discrete fractional calculus. The basic theory of the discrete fractional calculus can be found in [2, 3, 6, 8, 11–13, 15, 16, 24–26, 32] and other sources. The calculus of fractional h -differences was given for instance in [6, 15, 24–26, 32].

So, in order to prove the stability of the fractional order h -difference systems, it is not an easy task to directly extend the normal Lyapunov stability results to the fractional cases since the Leibniz law is complicated and does not hold generally. Matignon [22] proposes an explicit stability condition for a linear fractional differential systems. The articles [19, 20] present the fractional Lyapunov direct method to the fractional order differential systems, for the applications of this method, see [33–35]. However, it is a difficult task to find an appropriate Lyapunov function by means of this method. Some authors have

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proposed Lyapunov functions to prove the stability of the fractional order systems. For the application of this method, we refer to [1, 4, 5, 9, 10, 21, 31, 36].

In [21], we consider the stability of nabla (q, h) -fractional difference equations by Lyapunov direct method combining with the comparison lemma. In our present paper, we use a different method to show these comparison lemmas, that allow using general quadratic Lyapunov functions to analyse the stability of the fractional order h -difference systems. We also give some new propositions for the fractional order h -difference systems, which enable us to build polynomial Lyapunov functions of any order to determine the stability of such systems. As a consequence, we give a sufficient condition for these systems to be stable or asymptotically stable. Finally, some examples are given to illustrate our main results.

2. Preliminary Definitions

Let $(h\mathbb{N})_a := \{a, a + h, a + 2h, \dots\}$, where $h > 0, a \in \mathbb{R}$. We use the notation $\mathcal{F}_{(h\mathbb{N})_a}$ denotes the set of real valued functions defined on $(h\mathbb{N})_a$. Let $\sigma(t) = t + h$ for $t \in (h\mathbb{N})_a$. For the convenience of the readers, we will list some relevant results here.

Definition 2.1. (See [24, Definition 2.1]). For a function $y \in \mathcal{F}_{(h\mathbb{N})_a}$, the forward h -difference operator is defined as

$$(\Delta_h y)(t) = \frac{y(\sigma(t)) - y(t)}{h}, \quad t \in (h\mathbb{N})_a. \tag{1}$$

and the h -difference sum is given by

$$({}_a\Delta_h^{-1}y)(t) = \sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} y(sh)h, \quad t \in (h\mathbb{N})_a,$$

where, by convention, $({}_a\Delta_h^{-1}y)(a) = 0$.

Definition 2.2. (See [6, Definition 2.6]). For arbitrary $t, v \in \mathbb{R}$, the h -factorial function is defined by

$$t_h^{(v)} = h^v \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - v)}, \tag{2}$$

where Γ is the Euler gamma function with $\frac{t}{h} + 1 \notin \mathbb{Z}_- \cup \{0\}$, and we use the convention that $t_h^{(v)} = 0$, when $\frac{t}{h} + 1 - v$ is a nonpositive integer, and $\frac{t}{h} + 1$ is not a nonpositive integer.

Definition 2.3. (See [6, Definition 2.8]). For a function $y \in \mathcal{F}_{(h\mathbb{N})_a}$, the fractional h -sum of order $v > 0$ is given by

$$({}_a\Delta_h^{-v}y)(t) = \frac{h}{\Gamma(v)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-v} (t - \sigma(sh))_h^{(v-1)} y(sh), \quad t \in (h\mathbb{N})_{a+vh}, \tag{3}$$

and $({}_a\Delta_h^0y)(t) = y(t), \sigma(sh) = (s + 1)h$.

Definition 2.4. (See [24, Definition 2.6]). Let $v \in (n-1, n]$, and set $\mu = n-v$, where $n \in \mathbb{N}_1$. The Riemann-Liouville like fractional h -difference operator ${}_a\Delta_h^v$ of order v for a function $y \in \mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$({}_a\Delta_h^v y)(t) = (\Delta_h^n ({}_a\Delta_h^{-\mu} y))(t) = \frac{h}{\Gamma(\mu)} \Delta_h^n \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\mu} (t - \sigma(sh))_h^{(\mu-1)} y(sh), \quad t \in (h\mathbb{N})_{a+\mu h}. \tag{4}$$

Lemma 2.5. (See [24, Theorem 2.8]). Let $\nu \in (n - 1, n]$, and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The following formula is equivalent to (4):

$$({}_a\Delta_h^\nu y)(t) = \begin{cases} \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} y(sh), & \nu \in (n - 1, n), t \in (h\mathbb{N})_{a+\mu h}, \\ (\Delta_h^n y)(t), & \nu = n, t \in (h\mathbb{N})_a. \end{cases} \tag{5}$$

Lemma 2.6. (See [24, Definition 2.9]). Let $\nu \in (n - 1, n]$, and set $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The Caputo like h -difference operator ${}_a\Delta_{h,*}^\nu$ of order ν for a function $y \in \mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$({}_a\Delta_{h,*}^\nu y)(t) = ({}_a\Delta_h^{-\mu} (\Delta_h^n y))(t) = \frac{h}{\Gamma(\mu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\mu} (t - \sigma(sh))_h^{(\mu-1)} (\Delta_h^n y)(sh), \quad t \in (h\mathbb{N})_{a+\mu h}. \tag{6}$$

Lemma 2.7. (See [24, Proposition 2.11]). Let $\nu \in (n - 1, n]$, and $\mu = n - \nu$, where $n \in \mathbb{N}_1$. The following formula is equivalent to (6):

$$({}_a\Delta_{h,*}^\nu y)(t) = \begin{cases} \frac{h^{1-n}}{\Gamma(n - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(n-\nu)} (t - \sigma(sh))_h^{(n-\nu-1)} \sum_{r=0}^n (-1)^{r+1} \binom{n}{r} y((r + s)h), & \nu \in (n - 1, n), t \in (h\mathbb{N})_{a+\mu h}, \\ (\Delta_h^n y)(t), & \nu = n, t \in (h\mathbb{N})_a. \end{cases} \tag{7}$$

The following corollary appears in Mozyska et al [24, Corollary 4.2].

Corollary 2.8. (See [24, Corollary 4.2]). Let $\nu \in (0, 1]$. The following formula holds

$$({}_a\Delta_{h,*}^\nu y)(t) = ({}_a\Delta_h^\nu y)(t) - \frac{(t - a)_h^{(-\nu)} y(a)}{\Gamma(1 - \nu)}, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{8}$$

Lemma 2.9. The following properties are useful in this paper:

Delta difference of the h -falling factorial function (See [15, Lemma 3.2])

$${}_s\Delta_h(t - sh)_h^{(\nu)} = -\nu(t - \sigma(sh))_h^{(\nu-1)}, \tag{9}$$

where ${}_s\Delta_h(t - sh)_h^{(\nu)} = \frac{(t-sh-h)_h^{(\nu)} - (t-sh)_h^{(\nu)}}{h}$.

Summation by parts (See [4, Property 2.3])

$$\sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} x(sh + h)(\Delta_h y)(sh) = \frac{1}{h} (x(sh)y(sh)) \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} y(sh)(\Delta_h x)(sh), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{10}$$

3. General quadratic Lyapunov functions for stability

In this section, we will demonstrate stability of the fractional order h -difference systems by finding a general quadratic Lyapunov functions, using the discrete fractional Lyapunov direct method.

Consider the following nonlinear vector fractional order h -difference equations

$$\begin{cases} ({}_a\Delta_{h,*}^\nu x)(t) = f(t, x(t + \nu h)), & t \in (h\mathbb{N})_{a+(1-\nu)h}, \\ x(a) = x_0 \in \mathbb{R}^n, \end{cases} \tag{11}$$

and

$$\begin{cases} ({}_a\Delta_h^\nu x)(t) = f(t, x(t + \nu h)), & t \in (h\mathbb{N})_{a+(1-\nu)h}, \\ ({}_a\Delta_h^{\nu-1} x)(t)|_{t=a+(1-\nu)h} = h^{1-\nu} x_0 \in \mathbb{R}^n, \end{cases} \tag{12}$$

where $f : (h\mathbb{N})_{a+(1-\nu)h} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous with respect to x , $x : (h\mathbb{N})_a \rightarrow \mathbb{R}^n$, and $\nu \in (0, 1]$. By Schauder’s Fixed Point Theorem, it is easy to show that equations (11) and (12) has a solution which exists in $(h\mathbb{N})_a$, we could refer to [13, Theorem 6.11].

The constant vector x_{eq} is an *equilibrium point* of the nonlinear vector fractional order h -difference equation (11) (or (12)) if and only if $({}_a\Delta_{h,\nu}^\nu x_{eq})(t) = f(t, x_{eq}(t + \nu h)) = 0$ ($({}_a\Delta_{h,\nu}^\nu x_{eq})(t) = f(t, x_{eq}(t + \nu h))$) in the case of the Riemann-Liouville vector fractional h -difference equation) for all $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Assume that $f(t, 0) = 0$ so that the trivial solution $x \equiv 0$ is an equilibrium point of the fractional order h -difference system (11) (or (12)). Note that there is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables.

First, we present the following simple definitions and important facts.

Definition 3.1. *The equilibrium point $x = 0$ of the system (11) (or (12)) is said to be*

- (a) *stable, if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x(a)\| < \delta$ implies $\|x(a + kh)\| < \varepsilon$ for all $k \in \mathbb{N}_0$.*
- (b) *attractive, if there exists $\delta > 0$ such that $\|x(a)\| < \delta$ implies $\lim_{k \rightarrow \infty} x(a + kh) = 0$.*
- (c) *asymptotically stable, if it is stable and attractive.*

The fractional order h -difference system (11) (or (12)) is called *stable* (*asymptotically stable*) if their equilibrium point $x = 0$ is *stable* (*asymptotically stable*).

Definition 3.2. *Let $V : (h\mathbb{N})_a \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous scalar function. V is a Lyapunov function if it is a locally positive-definite function, i.e. $V(t, 0) = 0$, $V(t, x) > 0$, $\forall x \in U \setminus \{0\}$ with U being a neighborhood region around $x = 0$.*

Definition 3.3. (See [14, Definition 3.2]). *A function $\phi(r)$ is said to belong to the class \mathcal{K} if and only if $\phi \in C[[0, \rho], \mathbb{R}_+]$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotonically increasing in r .*

Definition 3.4. *A real valued function $V(t, x)$ defined on $(h\mathbb{N})_a \times S_\rho$, where $S_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, is said to be positive definite if and only if $V(t, 0) = 0$ for all $t \in (h\mathbb{N})_a$ and there exists $\phi(r) \in \mathcal{K}$ such that $\phi(r) \leq V(t, x)$, $\|x\| = r$, $(t, x) \in (h\mathbb{N})_a \times S_\rho$.*

Definition 3.5. *A real valued function $V(t, x)$ defined on $(h\mathbb{N})_a \times S_\rho$, where $S_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, is said to be decrescent if and only if $V(t, 0) = 0$ for all $t \in (h\mathbb{N})_a$ and there exists $\phi(r) \in \mathcal{K}$ such that $V(t, x) \leq \phi(r)$, $\|x\| = r$, $(t, x) \in (h\mathbb{N})_a \times S_\rho$.*

Now, let us recall that the \mathcal{Z} -transform of a sequence $\{y(n)\}_{n \in \mathbb{N}_0}$ is a complex function given by $Y(z) = \mathcal{Z}[y](z) = \sum_{k=0}^\infty \frac{y(k)}{z^k}$, where $z \in \mathbb{C}$ is a complex number for which this series converges absolutely. The \mathcal{Z} -transform of $\tilde{\varphi}_\alpha$ is defined as

$$\mathcal{Z}[\tilde{\varphi}_\alpha](z) = \sum_{k=0}^\infty \binom{k + \alpha - 1}{k} \frac{1}{z^k} = \sum_{k=0}^\infty (-1)^k \binom{-\alpha}{k} \frac{1}{z^k} = \left(\frac{z}{z-1}\right)^\alpha,$$

where

$$\tilde{\varphi}_\alpha(n) = \binom{n + \alpha - 1}{n} = (-1)^n \binom{-\alpha}{n}.$$

The convolution of $\tilde{\varphi}_\alpha$ and x is defined as

$$(\tilde{\varphi}_\alpha * x)(n) = \sum_{s=0}^n \binom{n-s+\alpha-1}{n-s} x(s).$$

The following relations hold for the \mathcal{Z} -transform, $\mathcal{Z}[x(n) * y(n)](z) = \mathcal{Z}[x(n)](z)\mathcal{Z}[y(n)](z)$, $\mathcal{Z}[y(n-1)](z) = \frac{1}{z}\mathcal{Z}[y(n)](z)$ for $|z| > R$, where R is the radius of convergence of $\mathcal{Z}[y(n)](z)$.

Proposition 3.6. (See [26, Proposition 16]). For $\nu \in (0, 1]$, $a \in \mathbb{R}$, let $y(n) := ({}_a\Delta_{h^*}^\nu x)(a + (1 - \nu)h + nh)$, where $n \in \mathbb{N}_0$. Then

$$\mathcal{Z}[y](z) = h^{-\nu} \left(\frac{z}{z-1} \right)^{1-\nu} [(z-1)X(z) - zx(a)], \tag{13}$$

where $X(z) = \mathcal{Z}[\bar{x}](z)$, and $\bar{x}(n) = x(a + nh)$.

Proposition 3.7. (See [26, Proposition 24]). For $\nu \in (0, 1]$, $a \in \mathbb{R}$, let $y(n) := ({}_a\Delta_h^\nu x)(a + (1 - \nu)h + nh)$, where $n \in \mathbb{N}_0$. Then

$$\mathcal{Z}[y](z) = z \left(\frac{hz}{z-1} \right)^{-\nu} X(z) - zh^{-\nu}x(a), \tag{14}$$

where $X(z) = \mathcal{Z}[\bar{x}](z)$, and $\bar{x}(n) = x(a + nh)$.

Lemma 3.8. (Diagonalization of a real symmetric matrix [7, p. 54]). Let $P \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then it may be transformed into a diagonal form by means of an orthogonal transformation, that is to say, there is an orthogonal matrix $B \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$P = B\Lambda B^T.$$

Now, we give the following lemmas for the Caputo fractional order h -difference, which will be useful for proving the stability of the system (11).

Lemma 3.9. Assume $\nu \in (0, 1]$, $x, y \in \mathbb{R}$, $({}_a\Delta_{h^*}^\nu x)(t) \geq ({}_a\Delta_{h^*}^\nu y)(t)$, $t \in (h\mathbb{N})_{a+(1-\nu)h}$. Then we have $x(t) - y(t) \geq x(a) - y(a)$ for $t \in (h\mathbb{N})_a$.

Proof. It follows from $({}_a\Delta_{h^*}^\nu x)(t) \geq ({}_a\Delta_{h^*}^\nu y)(t)$, that there exists a nonnegative function $M(\cdot)$ satisfying

$$({}_a\Delta_{h^*}^\nu x)(t) = ({}_a\Delta_{h^*}^\nu y)(t) + M(t). \tag{15}$$

Taking the \mathcal{Z} -transform of the equation (15), and using the Proposition 3.6, we get

$$h^{-\nu} \left(\frac{z}{z-1} \right)^{1-\nu} [(z-1)X(z) - zx(a)] = h^{-\nu} \left(\frac{z}{z-1} \right)^{1-\nu} [(z-1)Y(z) - zy(a)] + M(z),$$

where $X(z) = \mathcal{Z}[x](z)$, $Y(z) = \mathcal{Z}[y](z)$. It follows that

$$X(z) = Y(z) + h^\nu \frac{1}{z} \left(\frac{z}{z-1} \right)^\nu M(z) + \frac{z}{z-1} (x(a) - y(a)). \tag{16}$$

Applying the inverse \mathcal{Z} -transform to the equation (16) gives

$$x(t) = y(t) + h^\nu (\bar{\varphi}_\nu * M)(n-1) + (x(a) - y(a)),$$

where $\bar{\varphi}_\nu(n) = \binom{n+\nu-1}{n} = (-1)^n \binom{-\nu}{n}$. It follows from $M(t) \geq 0$ and $h^\nu (\bar{\varphi}_\nu * M)(n-1) \geq 0$ that $x(t) - y(t) \geq x(a) - y(a)$. The proof is complete. \square

Proposition 3.10. (See [4, Lemma 3.2]). For $\nu \in (0, 1]$, and $y \in \mathbb{R}$, the following inequality holds

$$({}_a\Delta_{h^*}^\nu y^2)(t) \leq 2y(t+vh)({}_a\Delta_{h^*}^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{17}$$

Proposition 3.11. For $\nu \in (0, 1]$, and $y \in \mathbb{R}^n$, the following relationship holds

$$\frac{1}{2} {}_a\Delta_{h^*}^\nu (y^T(t)Py(t)) \leq y^T(t+vh)P({}_a\Delta_{h^*}^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{18}$$

where $P \in \mathbb{R}^n \times \mathbb{R}^n$ is a constant, square, symmetric, and positive definite matrix.

Proof. Since the matrix P is symmetric, using Lemma 3.8, there exists an orthogonal matrix $B \in \mathbb{R}^{n \times n}$, and a diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$\frac{1}{2}y^T(t)Py(t) = \frac{1}{2}y^T(t)B\Lambda B^T y(t) = \frac{1}{2}(B^T y(t))^T \Lambda (B^T y(t)).$$

Set $x(t) := B^T y(t)$, then we have

$$\frac{1}{2} {}_a\Delta_{h,*}^v (x^T(t)\Lambda x(t)) = \frac{1}{2} ({}_a\Delta_{h,*}^v \sum_{i=1}^n \lambda_{ii} x_i^2)(t) = \sum_{i=1}^n \lambda_{ii} \frac{1}{2} ({}_a\Delta_{h,*}^v x_i^2)(t), \tag{19}$$

Applying Proposition 3.10 to the equality (19), and remembering the positivity hypothesis of the matrix P we have that $\lambda_{ii} > 0$, then

$$\frac{1}{2} {}_a\Delta_{h,*}^v (x^T(t)\Lambda x(t)) \leq \sum_{i=1}^n \lambda_{ii} x_i(t + vh) ({}_a\Delta_{h,*}^v x_i)(t),$$

that is,

$$\frac{1}{2} {}_a\Delta_{h,*}^v (x^T(t)\Lambda x(t)) \leq x^T(t + vh)\Lambda({}_a\Delta_{h,*}^v x)(t). \tag{20}$$

Then, replacing $x(t) = B^T y(t)$ in the inequality (20), we have

$$\frac{1}{2} {}_a\Delta_{h,*}^v ((B^T y(t))^T \Lambda (B^T y(t))) \leq (B^T y(t + vh))^T \Lambda {}_a\Delta_{h,*}^v (B^T y(t)). \tag{21}$$

Rearranging and using $B\Lambda B^T = P$ in the inequality (21), we get

$$\frac{1}{2} {}_a\Delta_{h,*}^v (y^T(t)Py(t)) \leq y^T(t + vh)P({}_a\Delta_{h,*}^v y)(t).$$

The proof is complete. \square

Lemma 3.12. *Let $x = 0$ be an equilibrium point of the system (11). If there exists a positive definite and decrescent scalar function $V(t, x)$, class- \mathcal{K} functions γ_1, γ_2 , and γ_3 such that*

$$\gamma_1(\|x(t)\|) \leq V(t, x(t)) \leq \gamma_2(\|x(t)\|), \quad t \in (h\mathbb{N})_a, \tag{22}$$

and

$$({}_a\Delta_{h,*}^v V)(t, x(t)) \leq -\gamma_3(\|x(t + vh)\|). \tag{23}$$

Then the system (11) is asymptotically stable.

Proof. From the equations (22) and (23), we have

$$({}_a\Delta_{h,*}^v V)(t, x(t)) \leq -\gamma_3(\gamma_2^{-1}(V(t + vh, x(t + vh))),$$

where γ_2^{-1} denotes the inverse of γ_2 . It is evident that $\gamma_3 \circ \gamma_2^{-1}$ is a discrete class \mathcal{K} function.

Considering a fractional difference equation

$$({}_a\Delta_{h,*}^v U)(t, x(t)) = -\gamma_3(\gamma_2^{-1}(U(t + vh, x(t + vh))).$$

Similar to the proof of [4, Theorem 2.10], we could show that $V(t, x(t)) \leq U(t, x(t))$. Then, as the proof of [4, Theorem 2.9], we could obtain $\lim_{t \rightarrow \infty} U(t, x(t)) = 0$. Since γ_1 is a class \mathcal{K} function, and $\gamma_1(\|x(t)\|) \leq V(t, x(t))$, it follows that $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is complete. \square

Theorem 3.13. Assume $x = 0$ is an equilibrium point of the system (11), if the following condition is satisfied

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (11) is stable. Also, if

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x \neq 0,$$

then the system (11) is asymptotically stable.

Proof. Let us propose the following Lyapunov function, which is positive definite

$$V(t, x(t)) = \frac{1}{2}x^T(t)Px(t).$$

Using Proposition 3.11, we obtain

$$({}_a\Delta_{h,*}^\nu V)(t) \leq x^T(t + \nu h)P({}_a\Delta_{h,*}^\nu x)(t) = x^T(t + \nu h)Pf(t, x(t + \nu h)) \leq 0, \quad (24)$$

by Lemma 3.9, we have

$$V(t, x(t)) \leq V(a, x(a)),$$

that is,

$$\frac{1}{2}x^T(t)Px(t) \leq \frac{1}{2}x^T(a)Px(a).$$

Since $B\Lambda B^T = P$, we obtain

$$\frac{1}{2}x^T(t)B\Lambda B^T x(t) \leq \frac{1}{2}x^T(a)B\Lambda B^T x(a).$$

Set $y(t) := B^T x(t)$, then we have

$$\frac{1}{2}y^T(t)\Lambda y(t) \leq \frac{1}{2}y^T(a)\Lambda y(a).$$

Since Λ is a diagonal matrix, it follows that

$$\sum_{i=1}^n \lambda_{ii} y_i^2(t) \leq \sum_{i=1}^n \lambda_{ii} y_i^2(a).$$

Hence, we have

$$\lambda_{\min} \|y(t)\|^2 \leq \lambda_{\max} \|y(a)\|^2,$$

where $\lambda_{\min} = \min\{|\lambda_{ii}| : 1 \leq i \leq n\}$, $\lambda_{\max} = \max\{|\lambda_{ii}| : 1 \leq i \leq n\}$.

Since B is an orthogonal matrix, and $y(t) = B^T x(t)$, we have

$$\lambda_{\min} \|x(t)\|^2 \leq \lambda_{\max} \|x(a)\|^2.$$

According to the definition of stability in the sense of Lyapunov, we see that the system (11) is stable in the sense of Lyapunov.

If

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x \neq 0,$$

similar to the above step, we can show that the system (11) is stable. Using Proposition 3.11, we have $({}_a\Delta_{h,*}^\nu V)(t, x(t)) \leq x^T(t + \nu h)P({}_a\Delta_{h,*}^\nu x)(t) < 0$, that is, the fractional order h -difference of V function is negative definite. Given the relationship between positive definite functions and class- \mathcal{K} functions in [30]. As a result, the system (11) is asymptotically stable from Lemma 3.12. The proof is complete. \square

In what follows, we will present results concerning the Riemann-Liouville fractional order h -difference, which are important to prove the stability of the system (12).

Lemma 3.14. Assume $\nu \in (0, 1]$, $x, y \in \mathbb{R}$, $({}_a\Delta_h^\nu x)(t) \geq ({}_a\Delta_h^\nu y)(t)$, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, and $x(a) \leq y(a)$. Then we have $x(t) - y(t) \geq x(a) - y(a)$ for $t \in (h\mathbb{N})_a$.

Proof. It follows from $({}_a\Delta_h^\nu x)(t) \geq ({}_a\Delta_h^\nu y)(t)$ that there exists a nonnegative function $M(\cdot)$ satisfying

$$({}_a\Delta_h^\nu x)(t) = ({}_a\Delta_h^\nu y)(t) + M(t). \tag{25}$$

Taking the \mathcal{Z} -transform of the equation (25), and using the Proposition 3.7, we get

$$z\left(\frac{hz}{z-1}\right)^{-\nu} X(z) - zh^{-\nu}x(a) = z\left(\frac{hz}{z-1}\right)^{-\nu} Y(z) - zh^{-\nu}y(a) + M(z).$$

It follows that

$$X(z) = Y(z) + \frac{1}{z}\left(\frac{hz}{z-1}\right)^\nu M(z) + \left(\frac{z}{z-1}\right)^\nu (x(a) - y(a)). \tag{26}$$

Applying the inverse \mathcal{Z} -transform to the equation (26) gives

$$x(t) = y(t) + h^\nu(\tilde{\varphi}_\nu * M)(n-1) + \tilde{\varphi}_\nu(n)(x(a) - y(a)),$$

where $\tilde{\varphi}_\nu(n) = \binom{n+\nu-1}{n} = (-1)^n \binom{-\nu}{n} \leq 1$. It follows from $\tilde{\varphi}_\nu(n) \geq 0$, $x(a) \leq y(a)$, $M(t) \geq 0$, and $h^\nu(\tilde{\varphi}_\nu * M)(n-1) \geq 0$ that $x(t) - y(t) \geq x(a) - y(a)$. The proof is complete. \square

Similar to the proof of [31, Lemma 2.10], we have the following Proposition 3.15.

Proposition 3.15. For $\nu \in (0, 1]$, and $y \in \mathbb{R}$, the following inequality holds

$$({}_a\Delta_h^\nu y^2)(t) \leq 2y(t+\nu h)({}_a\Delta_h^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{27}$$

Proof. If $\nu = 1$, we have

$$({}_a\Delta_h^\nu y^2)(t) = (\Delta_h y^2)(t) = \frac{y^2(t+h) - y^2(t)}{h},$$

and

$$2y(t+\nu h)({}_a\Delta_h^\nu y)(t) = 2y(t+h)(\Delta_h y)(t) = \frac{2y^2(t+h) - 2y(t+h)y(t)}{h} \geq \frac{y^2(t+h) - y^2(t)}{h}.$$

So, the inequality (27) holds for $\nu = 1$.

If $\nu \in (0, 1)$, it follows from Corollary 2.8 that

$$({}_a\Delta_{h,*}^\nu y)(t) = ({}_a\Delta_h^\nu y)(t) - \frac{(t-a)_h^{(-\nu)}y(a)}{\Gamma(1-\nu)}.$$

Then, we have

$$y(t+\nu h)({}_a\Delta_{h,*}^\nu y)(t) = y(t+\nu h)({}_a\Delta_h^\nu y)(t) - y(t+\nu h)\frac{(t-a)_h^{(-\nu)}y(a)}{\Gamma(1-\nu)},$$

and

$$({}_a\Delta_{h,*}^\nu y^2)(t) = ({}_a\Delta_h^\nu y^2)(t) - \frac{(t-a)_h^{(-\nu)}y^2(a)}{\Gamma(1-\nu)}.$$

Since ${}_s\Delta_h y^2(t + \nu h) = 0$, and using the summation by parts formula, we obtain

$$\begin{aligned} & 2y(t + \nu h)({}_a\Delta_{h,\nu}^{\nu} y)(t) - ({}_a\Delta_{h,\nu}^{\nu} y^2)(t) \\ &= -\frac{h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu)} {}_s\Delta_h (y(t + \nu h) - y(sh))^2 \\ &= -\frac{(t - sh)_h^{(-\nu)}}{\Gamma(1 - \nu)} (y(t + \nu h) - y(sh))^2 \Big|_{s=\frac{a}{h}}^{\frac{t}{h} + \nu} + \frac{h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (y(t + \nu h) - y(sh))^2 {}_s\Delta_h (t - sh)_h^{(-\nu)} \\ &= \frac{(t - a)_h^{(-\nu)}}{\Gamma(1 - \nu)} (y(t + \nu h) - y(a))^2 + \frac{h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} (y(t + \nu h) - y(sh))^2. \end{aligned}$$

Then, we arrive at

$$\begin{aligned} & 2y(t + \nu h)({}_a\Delta_h^{\nu} y)(t) - ({}_a\Delta_h^{\nu} y^2)(t) \\ &= 2y(t + \nu h)({}_a\Delta_{h,\nu}^{\nu} y)(t) + 2y(t + \nu h) \frac{(t - a)_h^{(-\nu)} y(a)}{\Gamma(1 - \nu)} - \left[({}_a\Delta_{h,\nu}^{\nu} y^2)(t) + \frac{(t - a)_h^{(-\nu)} y^2(a)}{\Gamma(1 - \nu)} \right] \\ &= 2y(t + \nu h) \frac{(t - a)_h^{(-\nu)} y(a)}{\Gamma(1 - \nu)} - \frac{(t - a)_h^{(-\nu)} y^2(a)}{\Gamma(1 - \nu)} + \frac{(t - a)_h^{(-\nu)}}{\Gamma(1 - \nu)} (y(t + \nu h) - y(a))^2 \\ &\quad + \frac{h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} (y(t + \nu h) - y(sh))^2 \\ &= \frac{(t - a)_h^{(-\nu)} y^2(t + \nu h)}{\Gamma(1 - \nu)} + \frac{h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} (y(t + \nu h) - y(sh))^2 \\ &\geq 0. \end{aligned}$$

So, we have $({}_a\Delta_h^{\nu} y^2)(t) \leq 2y(t + \nu h)({}_a\Delta_h^{\nu} y)(t)$. The proof is complete. \square

Proposition 3.16. For $\nu \in (0, 1]$, and $y \in \mathbb{R}^n$, the following relationship holds

$$\frac{1}{2} {}_a\Delta_h^{\nu} (y^T(t) P y(t)) \leq y^T(t + \nu h) P ({}_a\Delta_h^{\nu} y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{28}$$

where $P \in \mathbb{R}^n \times \mathbb{R}^n$ is a constant, square, symmetric, and positive definite matrix.

Proof. The proof is similar to Proposition 3.11, so we omit the details. \square

Lemma 3.17. Let $x = 0$ be an equilibrium point of the system (12). Assume there exists a positive definite and decrescent scalar function $V(t, x)$, class- \mathcal{K} functions γ_1, γ_2 , and γ_3 such that

$$\gamma_1(\|x(t)\|) \leq V(t, x(t)) \leq \gamma_2(\|x(t)\|), \quad t \in (h\mathbb{N})_a, \tag{29}$$

and

$$({}_a\Delta_h^{\nu} V)(t, x(t)) \leq -\gamma_3(\|x(t + \nu h)\|). \tag{30}$$

Then the system (12) is asymptotically stable.

Proof. The proof is similar to Lemma 3.12, so we omit the details. \square

Theorem 3.18. Assume $x = 0$ is an equilibrium point of the system (12), if the following condition is satisfied

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (12) is stable. Also, if

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x \neq 0,$$

then the system (12) is asymptotically stable.

Proof. Let us propose the following Lyapunov function, which is positive definite

$$V(t, x(t)) = \frac{1}{2}x^T(t)Px(t).$$

Using Proposition 3.16, we have

$$({}_a\Delta_h^\nu V)(t) \leq x^T(t + \nu h)P({}_a\Delta_h^\nu x)(t) = x^T(t + \nu h)Pf(t, x(t + \nu h)) \leq 0, \tag{31}$$

and by Lemma 3.14, we have

$$V(t, x(t)) \leq V(a, x(a)),$$

that is,

$$\frac{1}{2}x^T(t)Px(t) \leq \frac{1}{2}x^T(a)Px(a).$$

Similar to the proof of Theorem 3.13, we have

$$\lambda_{\min}\|x(t)\|^2 \leq \lambda_{\max}\|x(a)\|^2.$$

According to the definition of stability in the sense of Lyapunov, we obtain the system (12) is stable in the sense of Lyapunov.

If

$$x^T(t + \nu h)Pf(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x \neq 0,$$

similar to the above step, we can show that the system (12) is stable. Using Proposition 3.16, we have $({}_a\Delta_h^\nu V)(t, x(t)) \leq x^T(t + \nu h)P({}_a\Delta_h^\nu x)(t) < 0$, that is, the fractional order h -difference of V function is negative definite. Given the relationship between positive definite functions and class- \mathcal{K} functions in [30]. Then, from Lemma 3.17, the system (12) is asymptotically stable. The proof is complete. \square

4. Polynomial Lyapunov functions for stability

In this section, we will introduce several propositions, which generalize the Lemma 2.10 in [31] and Lemma 3.2 in [4], which are very important to show the Lyapunov stability for the fractional order h -difference systems.

Proposition 4.1. For $\nu \in (0, 1]$, $y \in \mathbb{R}$, $y(t) \geq 0$, $t \in (h\mathbb{N})_a$, and $l \in \{2k + 1, k \in \mathbb{N}_1\}$, the following inequality holds

$$({}_a\Delta_{h,*}^\nu y^l)(t) \leq ly^{l-1}(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{32}$$

Proof. We need to equivalently prove

$$({}_a\Delta_{h,\nu}^{\nu} y^l)(t) - ly^{l-1}(t + \nu h)({}_a\Delta_{h,\nu}^{\nu} y)(t) \leq 0. \tag{33}$$

For $\nu = 1$, the inequality (33) can be written as

$$(\Delta_h y^l)(t) - ly^{l-1}(t + h)(\Delta_h y)(t) \leq 0. \tag{34}$$

We will show the inequality (34) holds by induction. When $l = 3$, we have

$$\begin{aligned} 3y^2(t + h)(\Delta_h y)(t) - (\Delta_h y^3)(t) &= 3y^2(t + h) \frac{y(t + h) - y(t)}{h} - \frac{y^3(t + h) - y^3(t)}{h} \\ &= \frac{2y^3(t + h) - 3y^2(t + h)y(t) + y^3(t)}{h}. \end{aligned}$$

According to Proposition 3.10, we have

$$(\Delta_h y^2)(t) - 2y(t + h)(\Delta_h y)(t) \leq 0,$$

that is,

$$\frac{2y^3(t + h) - 3y^2(t + h)y(t)}{h} \geq \frac{y^2(t + h)y(t) - 2y^2(t)y(t + h)}{h}. \tag{35}$$

Using the inequality (35), and $y(t) \geq 0$, we have

$$3y^2(t + h)(\Delta_h y)(t) - (\Delta_h y^3)(t) \geq \frac{y^3(t) - 2y^2(t)y(t + h) + y^2(t + h)y(t)}{h} = \frac{y(t)(y(t + h) - y(t))^2}{h} \geq 0.$$

So, we obtain

$$(\Delta_h y^3)(t) \leq 3y^2(t + h)(\Delta_h y)(t).$$

Assume the inequality

$$(\Delta_h y^k)(t) - ky^{k-1}(t + h)(\Delta_h y)(t) \leq 0$$

is true for $k = 3, 5, \dots, l$, we will show the following case $k = l + 2$ is true, that is,

$$(\Delta_h y^{l+2})(t) \leq (l + 2)y^{l+1}(t + h)(\Delta_h y)(t).$$

Then, we have

$$\begin{aligned} &(l + 2)y^{l+1}(t + h)(\Delta_h y)(t) - (\Delta_h y^{l+2})(t) \\ &= (l + 2)y^{l+1}(t + h) \frac{y(t + h) - y(t)}{h} - \frac{y^{l+2}(t + h) - y^{l+2}(t)}{h} \\ &= \frac{(l + 1)y^{l+2}(t + h) - (l + 2)y^{l+1}(t + h)y(t) + y^{l+2}(t)}{h}. \end{aligned}$$

From the induction assumption, we have

$$(\Delta_h y^{\frac{l+1}{2}+1})(t) \leq \left(\frac{l+1}{2} + 1\right)y^{\frac{l+1}{2}}(t + h)(\Delta_h y)(t),$$

that is,

$$\frac{(l + 1)y^{l+2}(t + h) - (l + 2)y^{l+1}(t + h)y(t)}{h} \geq \frac{y^{l+1}(t + h)y(t) - 2y^{\frac{l+1}{2}}(t + h)y^{\frac{l+1}{2}+1}(t)}{h}. \tag{36}$$

Using the inequality (36), and $y(t) \geq 0$, we have

$$\begin{aligned} (l + 2)y^{l+1}(t + h)(\Delta_h y)(t) - (\Delta_h y^{l+2})(t) &\geq \frac{y^{l+1}(t + h)y(t) - 2y^{\frac{l+1}{2}}(t + h)y^{\frac{l+1}{2}+1}(t) + y^{l+2}(t)}{h} \\ &= \frac{y(t)\left(y^{\frac{l+1}{2}}(t + h) - y^{\frac{l+1}{2}}(t)\right)^2}{h} \geq 0. \end{aligned}$$

So, we obtain

$$(\Delta_h y^l)(t) - ly^{l-1}(t + \nu h)(\Delta_h y)(t) \leq 0, \quad l \in \{2k + 1, k \in \mathbb{N}_1\}.$$

For $\nu \in (0, 1)$, we prove the inequality (33) by induction. First, we show the case $l = 3$ holds. Using a summation by parts formula, we have

$$\begin{aligned} &3y^2(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^3)(t) \\ &= \frac{3y^2(t + \nu h)h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y)(sh) - \frac{h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y^3)(sh) \\ &= \frac{3y^2(t + \nu h)h}{\Gamma(1 - \nu)} \left[\frac{(t - sh)_h^{(-\nu)} y(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \right] \\ &\quad - \frac{h}{\Gamma(1 - \nu)} \left[\frac{(t - sh)_h^{(-\nu)} y^3(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y^3(sh) \right] \\ &= 2h^{-\nu} y^3(t + \nu h) - \frac{3(t - a)_h^{(-\nu)} y^2(t + \nu h)y(a)}{\Gamma(1 - \nu)} - \frac{3y^2(t + \nu h)h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \\ &\quad + \frac{(t - a)_h^{(-\nu)} y^3(a)}{\Gamma(1 - \nu)} + \frac{h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y^3(sh). \end{aligned}$$

It follows from Proposition 3.10 that

$$({}_a\Delta_{h,*}^\nu y^2)(t) \leq 2y(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t).$$

Applying a summation by parts formula, we have

$$\begin{aligned} &2y(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^2)(t) \\ &= \frac{2y(t + \nu h)h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y)(sh) - \frac{h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y^2)(sh) \\ &= \frac{2y(t + \nu h)h}{\Gamma(1 - \nu)} \left[\frac{(t - sh)_h^{(-\nu)} y(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \right] \\ &\quad - \frac{h}{\Gamma(1 - \nu)} \left[\frac{(t - sh)_h^{(-\nu)} y^2(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y^2(sh) \right] \\ &= h^{-\nu} y^2(t + \nu h) - \frac{2(t - a)_h^{(-\nu)} y(t + \nu h)y(a)}{\Gamma(1 - \nu)} - \frac{2y(t + \nu h)h\nu}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \end{aligned}$$

$$+ \frac{(t - a)_h^{(-\nu)} y^2(a)}{\Gamma(1 - \nu)} + \frac{hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y^2(sh) \geq 0,$$

that is,

$$\begin{aligned} & 2h^{-\nu} y^3(t + \nu h) \\ & \geq \frac{4(t - a)_h^{(-\nu)} y^2(t + \nu h)y(a)}{\Gamma(1 - \nu)} + \frac{4y^2(t + \nu h)hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \\ & \quad - \frac{2(t - a)_h^{(-\nu)} y(t + \nu h)y^2(a)}{\Gamma(1 - \nu)} - \frac{2y(t + \nu h)hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y^2(sh). \end{aligned} \tag{37}$$

Using the inequality (37), and $y(t) \geq 0$, we have

$$\begin{aligned} & 3y^2(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^3)(t) \\ & \geq \frac{(t - a)_h^{(-\nu)} y^2(t + \nu h)y(a)}{\Gamma(1 - \nu)} + \frac{y^2(t + \nu h)hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \\ & \quad - \frac{2(t - a)_h^{(-\nu)} y(t + \nu h)y^2(a)}{\Gamma(1 - \nu)} - \frac{2y(t + \nu h)hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y^2(sh) \\ & \quad + \frac{(t - a)_h^{(-\nu)} y^3(a)}{\Gamma(1 - \nu)} + \frac{hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y^3(sh) \\ & = \frac{(t - a)_h^{(-\nu)} y(a)}{\Gamma(1 - \nu)} (y(t + \nu h) - y(a))^2 + \frac{hv}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) (y(t + \nu h) - y(sh))^2 \geq 0. \end{aligned}$$

So, we obtain

$$({}_a\Delta_{h,*}^\nu y^3)(t) - 3y^2(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t) \leq 0.$$

Now, we assume the inequality

$$({}_a\Delta_{h,*}^\nu y^k)(t) \leq ky^{k-1}(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t)$$

is true for $k = 3, 5, \dots, l$, we will show the following $k = l + 2$ case is true, that is,

$$({}_a\Delta_{h,*}^\nu y^{l+2})(t) \leq (l + 2)y^{l+1}(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t).$$

Using summation by parts formula, we have

$$\begin{aligned} & (l + 2)y^{l+1}(t + \nu h)({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^{l+2})(t) \\ & = \frac{(l + 2)y^{l+1}(t + \nu h)h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y)(sh) - \frac{h}{\Gamma(1 - \nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu)} (\Delta_h y^{l+2})(sh) \\ & = \frac{(l + 2)y^{l+1}(t + \nu h)h}{\Gamma(1 - \nu)} \left[\frac{(t - sh)_h^{(-\nu)} y(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h} + \nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h} + \nu - 1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \right] \end{aligned}$$

$$\begin{aligned}
 & - \frac{h}{\Gamma(1-\nu)} \left[\frac{(t-sh)_h^{(-\nu)} y^{l+2}(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{l+2}(sh) \right] \\
 & = (l+1)h^{-\nu} y^{l+2}(t+\nu h) - \frac{(l+2)(t-a)_h^{(-\nu)} y^{l+1}(t+\nu h)y(a)}{\Gamma(1-\nu)} + \frac{(t-a)_h^{(-\nu)} y^{l+2}(a)}{\Gamma(1-\nu)} \\
 & - \frac{(l+2)y^{l+1}(t+\nu h)h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) + \frac{h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{l+2}(sh).
 \end{aligned}$$

From the induction assumption, we have

$$({}_a\Delta_{h,*}^\nu y^{\frac{l+1}{2}+1})(t) \leq \left(\frac{l+1}{2} + 1\right) y^{\frac{l+1}{2}}(t+\nu h) ({}_a\Delta_{h,*}^\nu y)(t).$$

Using a summation by parts formula, we get

$$\begin{aligned}
 & \left(\frac{l+1}{2} + 1\right) y^{\frac{l+1}{2}}(t+\nu h) ({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^{\frac{l+1}{2}+1})(t) \\
 & = \frac{(\frac{l+1}{2} + 1) y^{\frac{l+1}{2}}(t+\nu h) h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu)} (\Delta_h y)(sh) - \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu)} (\Delta_h y^{\frac{l+1}{2}+1})(sh) \\
 & = \frac{(\frac{l+1}{2} + 1) y^{\frac{l+1}{2}}(t+\nu h) h}{\Gamma(1-\nu)} \left[\frac{(t-sh)_h^{(-\nu)} y(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) \right] \\
 & - \frac{h}{\Gamma(1-\nu)} \left[\frac{(t-sh)_h^{(-\nu)} y^{\frac{l+1}{2}+1}(sh)}{h} \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \nu \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{\frac{l+1}{2}+1}(sh) \right] \\
 & = \frac{l+1}{2} h^{-\nu} y^{\frac{l+1}{2}+1}(t+\nu h) - \frac{(\frac{l+1}{2} + 1)(t-a)_h^{(-\nu)} y^{\frac{l+1}{2}}(t+\nu h)y(a)}{\Gamma(1-\nu)} \\
 & - \frac{(\frac{l+1}{2} + 1) y^{\frac{l+1}{2}}(t+\nu h) h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) + \frac{(t-a)_h^{(-\nu)} y^{\frac{l+1}{2}+1}(a)}{\Gamma(1-\nu)} \\
 & + \frac{h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{\frac{l+1}{2}+1}(sh) \geq 0,
 \end{aligned}$$

that is,

$$\begin{aligned}
 & (l+1)h^{-\nu} y^{l+2}(t+\nu h) \\
 & \geq \frac{(l+3)(t-a)_h^{(-\nu)} y^{l+1}(t+\nu h)y(a)}{\Gamma(1-\nu)} + \frac{(l+3)y^{l+1}(t+\nu h)h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) \\
 & - \frac{2(t-a)_h^{(-\nu)} y^{\frac{l+1}{2}}(t+\nu h)y^{\frac{l+1}{2}+1}(a)}{\Gamma(1-\nu)} - \frac{2y^{\frac{l+1}{2}}(t+\nu h)h\nu}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{\frac{l+1}{2}+1}(sh).
 \end{aligned} \tag{38}$$

Using the inequality (38), and $y(t) \geq 0$, we obtain

$$(l+2)y^{l+1}(t+\nu h) ({}_a\Delta_{h,*}^\nu y)(t) - ({}_a\Delta_{h,*}^\nu y^{l+2})(t)$$

$$\begin{aligned}
 &\geq \frac{(t-a)_h^{(-\nu)} y^{l+1}(t+vh)y(a)}{\Gamma(1-\nu)} + \frac{y^{l+1}(t+vh)hv}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) \\
 &\quad - \frac{2(t-a)_h^{(-\nu)} y^{\frac{l+1}{2}}(t+vh)y^{\frac{l+1}{2}+1}(a)}{\Gamma(1-\nu)} - \frac{2y^{\frac{l+1}{2}}(t+vh)hv}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{\frac{l+1}{2}+1}(sh) \\
 &\quad + \frac{(t-a)_h^{(-\nu)} y^{l+2}(a)}{\Gamma(1-\nu)} + \frac{hv}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y^{l+2}(sh) \\
 &= \frac{(t-a)_h^{(-\nu)} y(a)}{\Gamma(1-\nu)} \left(y^{\frac{l+1}{2}}(t+vh) - y^{\frac{l+1}{2}}(a) \right)^2 \\
 &\quad + \frac{hv}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu-1)} y(sh) \left(y^{\frac{l+1}{2}}(t+vh) - y^{\frac{l+1}{2}}(sh) \right)^2 \geq 0.
 \end{aligned}$$

So, we obtain

$$({}_a\Delta_{h,*}^\nu y^l)(t) - ly^{l-1}(t+vh)({}_a\Delta_{h,*}^\nu y)(t) \leq 0, \quad l \in \{2k+1, k \in \mathbb{N}_1\}.$$

The proof is complete. \square

Proposition 4.2. For $\nu \in (0, 1]$, $y \in \mathbb{R}$, and $m \in \mathbb{N}_1$, the following inequality holds

$$({}_a\Delta_{h,*}^\nu y^{2^m})(t) \leq 2^m y^{(2^m-1)}(t+vh)({}_a\Delta_{h,*}^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{39}$$

Proof. To prove the inequality (39), we start by iterating m times, and using Proposition 3.10, which results in

$$\begin{aligned}
 ({}_a\Delta_{h,*}^\nu y^{2^m})(t) &\leq 2y^{2^{m-1}}(t+vh)({}_a\Delta_{h,*}^\nu y^{2^{m-1}})(t) \\
 &\leq 2^2 y^{2^{m-2}}(t+vh)y^{2^{m-2}}(t+vh)({}_a\Delta_{h,*}^\nu y^{2^{m-2}})(t) \leq \dots \\
 &\leq 2^m y^{2^{m-1}}(t+vh)y^{2^{m-2}}(t+vh) \dots y^{2^0}(t+vh)({}_a\Delta_{h,*}^\nu y^{2^0})(t),
 \end{aligned}$$

which is equivalent to

$$({}_a\Delta_{h,*}^\nu y^{2^m})(t) \leq 2^m y^{(2^m-1)}(t+vh)({}_a\Delta_{h,*}^\nu y)(t).$$

The proof is complete. \square

Theorem 4.3. Assume $x = 0$ is an equilibrium point of the system (11). Then the following statements are satisfied:

(i) For $x_i(t) \geq 0$ ($i = 1, 2, \dots, n$), $t \in (h\mathbb{N})_a$, and $l \in \{2k+1, k \in \mathbb{N}_1\}$, if the following condition is satisfied

$$x_i^{l-1}(t+vh)f_i(t, x(t+vh)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (11) is stable. Also, if

$$x_i^{l-1}(t+vh)f_i(t, x(t+vh)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x_i \neq 0,$$

then the system (11) is asymptotically stable.

(ii) For $m \in \mathbb{N}_1$, if the following condition is satisfied

$$x_i^{(2^m-1)}(t+vh)f_i(t, x(t+vh)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (11) is stable. Also, if

$$x_i^{(2^m-1)}(t+vh)f_i(t, x(t+vh)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x_i \neq 0,$$

then the system (11) is asymptotically stable.

Proof. (i) Let us propose the following Lyapunov function, which is positive definite

$$V(t, x(t)) = \sum_{i=1}^n \frac{x_i^l(t)}{l}.$$

Using Proposition 4.1, we obtain

$$({}_a\Delta_{h,*}^v V)(t) \leq \sum_{i=1}^n x_i^{l-1}(t + \nu h)({}_a\Delta_{h,*}^v x_i)(t) = \sum_{i=1}^n x_i^{l-1}(t + \nu h) f_i(t, x(t + \nu h)) \leq 0.$$

Hence, by Lemma 3.9, we have

$$V(t, x(t)) \leq V(a, x(a)),$$

that is,

$$\sum_{i=1}^n \frac{x_i^l(t)}{l} \leq \sum_{i=1}^n \frac{x_i^l(a)}{l}.$$

According to the definition of stability in the sense of Lyapunov, we obtain the system (11) is stable in the sense of Lyapunov.

If

$$x_i^{l-1}(t + \nu h) f_i(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x \neq 0,$$

similar to the above step, we can show that the system (11) is stable. Using Proposition 4.1, we have $({}_a\Delta_{h,*}^v V)(t, x(t)) \leq \sum_{i=1}^n x_i^{l-1}(t + \nu h)({}_a\Delta_{h,*}^v x_i)(t) < 0$, that is, the fractional order h -difference of V function is negative definite. Given the relationship between positive definite functions and class- \mathcal{K} functions in [30]. It follows from Lemma 3.12 that the system (11) is asymptotically stable.

(ii) The proof is similar to the previous one, by Proposition 4.2 and the positive definite Lyapunov function:

$$V(t, x(t)) = \sum_{i=1}^n \frac{x_i^{2m}(t)}{2^m}.$$

The proof is complete. \square

Proposition 4.4. For $\nu \in (0, 1]$, $y \in \mathbb{R}$, $y(t) \geq 0$, $t \in (h\mathbb{N})_a$, and $l \in \{2k + 1, k \in \mathbb{N}_1\}$, the following inequality holds

$$({}_a\Delta_{h,*}^v y^l)(t) \leq l y^{l-1}(t + \nu h)({}_a\Delta_{h,*}^v y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{40}$$

Proof. We need to equivalently prove

$$({}_a\Delta_{h,*}^v y^l)(t) - l y^{l-1}(t + \nu h)({}_a\Delta_{h,*}^v y)(t) \leq 0. \tag{41}$$

For $\nu = 1$, we can show as in the proof of Proposition 4.1. For $\nu \in (0, 1)$, we show the inequality (41) by induction. When $l = 3$, using Lemma 2.5, we have

$$\begin{aligned} & 3y^2(t + \nu h)({}_a\Delta_{h,*}^v y)(t) - ({}_a\Delta_{h,*}^v y^3)(t) \\ &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{t}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} (3y^2(t + \nu h)y(sh) - y^3(sh)) \end{aligned}$$

$$= 2h^{-\nu}y^3(t + \nu h) + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} (3y^2(t + \nu h)y(sh) - y^3(sh)).$$

It follows from Proposition 3.15 that

$$({}_a\Delta_h^\nu y^2)(t) \leq 2y(t + \nu h)({}_a\Delta_h^\nu y)(t).$$

Using Lemma 2.5, we have

$$\begin{aligned} & 2y(t + \nu h)({}_a\Delta_h^\nu y)(t) - ({}_a\Delta_h^\nu y^2)(t) \\ &= \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} (2y(t + \nu h)y(sh) - y^2(sh)) \\ &= h^{-\nu}y^2(t + \nu h) + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} (2y(t + \nu h)y(sh) - y^2(sh)) \geq 0, \end{aligned}$$

that is,

$$2h^{-\nu}y^3(t + \nu h) \geq \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} (2y(t + \nu h)y^2(sh) - 4y^2(t + \nu h)y(sh)). \tag{42}$$

Using the inequality (42), and $y(t) \geq 0$, we have

$$\begin{aligned} & 3y^2(t + \nu h)({}_a\Delta_h^\nu y)(t) - ({}_a\Delta_h^\nu y^3)(t) \\ & \geq \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} (2y(t + \nu h)y^2(sh) - 4y^2(t + \nu h)y(sh)) \\ & \quad + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} (3y^2(t + \nu h)y(sh) - y^3(sh)) \\ & = -\frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) (y^2(t + \nu h) + y^2(sh) - 2y(t + \nu h)y(sh)) \\ & = -\frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) (y(t + \nu h) + y(sh))^2 \geq 0. \end{aligned}$$

So, we obtain

$$({}_a\Delta_h^\nu y^3)(t) - 3y^2(t + \nu h)({}_a\Delta_h^\nu y)(t) \leq 0.$$

Now, we assume the inequality

$$({}_a\Delta_h^\nu y^k)(t) \leq ky^{k-1}(t + \nu h)({}_a\Delta_h^\nu y)(t)$$

is true for $k = 3, 5, \dots, l$, we will show the following $k = l + 2$ case is true, that is,

$$({}_a\Delta_h^\nu y^{l+2})(t) \leq (l + 2)y^{l+1}(t + \nu h)({}_a\Delta_h^\nu y)(t).$$

Using Lemma 2.5, we have

$$\begin{aligned} & (l + 2)y^{l+1}(t + \nu h)({}_a\Delta_h^\nu y)(t) - ({}_a\Delta_h^\nu y^{l+2})(t) \\ &= \frac{(l + 2)y^{l+1}(t + \nu h)h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) - \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} y^{l+2}(sh) \\ &= (l + 1)h^{-\nu} y^{l+2}(t + \nu h) + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} [(l + 2)y^{l+1}(t + \nu h)y(sh) - y^{l+2}(sh)]. \end{aligned}$$

From the induction assumption, we have

$$({}_a\Delta_h^\nu y^{\frac{l+1}{2}+1})(t) \leq \left(\frac{l + 1}{2} + 1\right) y^{\frac{l+1}{2}}(t + \nu h)({}_a\Delta_h^\nu y)(t).$$

By Lemma 2.5, we have

$$\begin{aligned} & \left(\frac{l + 1}{2} + 1\right) y^{\frac{l+1}{2}}(t + \nu h)({}_a\Delta_h^\nu y)(t) - ({}_a\Delta_h^\nu y^{\frac{l+1}{2}+1})(t) \\ &= \frac{\left(\frac{l+1}{2} + 1\right) y^{\frac{l+1}{2}}(t + \nu h)h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) - \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} (t - \sigma(sh))_h^{(-\nu-1)} y^{\frac{l+1}{2}+1}(sh) \\ &= \frac{l + 1}{2} h^{-\nu} y^{\frac{l+1}{2}+1}(t + \nu h) + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} \left[\left(\frac{l + 1}{2} + 1\right) y^{\frac{l+1}{2}}(t + \nu h)y(sh) - y^{\frac{l+1}{2}+1}(sh)\right] \geq 0, \end{aligned}$$

that is,

$$lh^{-\nu} y^{l+2}(t + \nu h) \geq \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} \left[-(l + 3)y^{l+1}(t + \nu h)y(sh) + 2y^{\frac{l+1}{2}}(t + \nu h)y^{\frac{l+1}{2}+1}(sh)\right]. \quad (43)$$

Using the inequality (43), and $y(t) \geq 0$, we have

$$\begin{aligned} & (l + 2)y^{l+1}(t + \nu h)({}_a\Delta_h^\nu y)(t) - ({}_a\Delta_h^\nu y^{l+2})(t) \\ &\geq \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} \left[-(l + 3)y^{l+1}(t + \nu h)y(sh) + 2y^{\frac{l+1}{2}}(t + \nu h)y^{\frac{l+1}{2}+1}(sh)\right] \\ &\quad + \frac{h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} [(l + 2)y^{l+1}(t + \nu h)y(sh) - y^{l+2}(sh)] \\ &= \frac{-h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \left(y^{l+1}(t + \nu h) + y^{l+1}(sh) - 2y^{\frac{l+1}{2}}(t + \nu h)y^{\frac{l+1}{2}}(sh)\right) \\ &= \frac{-h}{\Gamma(-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} y(sh) \left(y^{\frac{l+1}{2}}(t + \nu h) - y^{\frac{l+1}{2}}(sh)\right)^2 \geq 0, \end{aligned}$$

So, we obtain

$$({}_a\Delta_h^\nu y^l)(t) - ly^{l-1}(t + \nu h)({}_a\Delta_h^\nu y)(t) \leq 0, \quad l \in \{2k + 1, k \in \mathbb{N}_1\}.$$

The proof is complete. \square

Proposition 4.5. For $\nu \in (0, 1]$, $y \in \mathbb{R}$, and $m \in \mathbb{N}_1$, the following inequality holds

$$({}_a\Delta_h^\nu y^{2^m})(t) \leq 2^m y^{2^{m-1}}(t + \nu h)({}_a\Delta_h^\nu y)(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{44}$$

Proof. To prove the inequality (44), we start by iterating m times, and using Proposition 3.15, which results in

$$\begin{aligned} ({}_a\Delta_h^\nu y^{2^m})(t) &\leq 2y^{2^{m-1}}(t + \nu h)({}_a\Delta_h^\nu y^{2^{m-1}})(t) \\ &\leq 2^2 y^{2^{m-2}}(t + \nu h)y^{2^{m-2}}(t + \nu h)({}_a\Delta_h^\nu y^{2^{m-2}})(t) \leq \dots \\ &\leq 2^m y^{2^{m-1}}(t + \nu h)y^{2^{m-2}}(t + \nu h) \dots y^{2^0}(t + \nu h)({}_a\Delta_h^\nu y^{2^0})(t), \end{aligned}$$

which is equivalent to

$$({}_a\Delta_h^\nu y^{2^m})(t) \leq 2^m y^{2^{m-1}}(t + \nu h)({}_a\Delta_h^\nu y)(t).$$

The proof is complete. \square

Theorem 4.6. Assume $x = 0$ is an equilibrium point of the system (12). Then the following statements are satisfied:

(i) For $x_i(t) \geq 0$ ($i = 1, 2, \dots, n$), $t \in (h\mathbb{N})_a$, and $l \in \{2k + 1, k \in \mathbb{N}_1\}$, if the following condition is satisfied

$$x_i^{l-1}(t + \nu h)f_i(t, x(t + \nu h)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (12) is stable. Also, if

$$x_i^{l-1}(t + \nu h)f_i(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x_i \neq 0,$$

then the system (12) is asymptotically stable.

(ii) For $m \in \mathbb{N}_1$, if the following condition is satisfied

$$x_i^{(2^m-1)}(t + \nu h)f_i(t, x(t + \nu h)) \leq 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h},$$

then the system (12) is stable. And if

$$x_i^{(2^m-1)}(t + \nu h)f_i(t, x(t + \nu h)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x_i \neq 0,$$

then the system (12) is asymptotically stable.

Proof. (i) Let us propose the following Lyapunov function, which is positive definite

$$V(t, x(t)) = \sum_{i=1}^n \frac{x_i^l(t)}{l}.$$

Using Proposition 4.4 gives us

$$({}_a\Delta_h^\nu V)(t) \leq \sum_{i=1}^n x_i^{l-1}(t + \nu h)({}_a\Delta_h^\nu x_i)(t) = \sum_{i=1}^n x_i^{l-1}(t + \nu h)f_i(t, x(t + \nu h)) \leq 0.$$

By Lemma 3.14, we have

$$V(t, x(t)) \leq V(a, x(a)),$$

that is,

$$\sum_{i=1}^n \frac{x_i^l(t)}{l} \leq \sum_{i=1}^n \frac{x_i^l(a)}{l}.$$

According to the definition of stability in the sense of Lyapunov, we obtain the system (12) is stable in the sense of Lyapunov.

If

$$x_i^{l-1}(t+vh)f_i(t, x(t+vh)) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}, \forall x_i \neq 0,$$

similar to the above step, we can show that the system (12) is stable. By Proposition 4.4, we have $({}_a\Delta_h^\nu V)(t, x(t)) \leq \sum_{i=1}^n x_i^{l-1}(t+vh)({}_a\Delta_h^\nu x_i)(t) < 0$, that is, the fractional order h -difference of V function is negative definite. Given the relationship between positive definite functions and class- \mathcal{K} functions in [30]. It can be concluded from Lemma 3.17 that the system (12) is asymptotically stable.

(ii) The proof is similar to the previous one, by Proposition 4.5 and the positive definite Lyapunov function:

$$V(t, x(t)) = \sum_{i=1}^n \frac{x_i^{2m}(t)}{2^m}.$$

The proof is complete. \square

5. Numerical Results

Now, we give some numerical examples to illustrate the application of the results established in the previous sections.

Example 5.1. Consider the following fractional order h -difference system

$$\begin{cases} ({}_a\Delta_{h,*}^\nu x_1)(t) = -x_1(t+vh), & x_1(a) = 0.1, \\ ({}_a\Delta_{h,*}^\nu x_2)(t) = -x_2(t+vh), & x_2(a) = 0.2, \end{cases} \quad (45)$$

where $\nu = 0.5, a = 0, h = 1, t \in (h\mathbb{N})_{a+(1-\nu)h}$, and this difference system has a trivial solution $x(t) = (x_1(t), x_2(t))^T = 0$. We can see that

$$x^T(t+vh)Pf(t, x(t+vh)) = (x_1(t+vh), x_2(t+vh))P \begin{bmatrix} -x_1(t+vh) \\ -x_2(t+vh) \end{bmatrix} = -(x_1(t+vh) + x_2(t+vh))^2 \leq 0,$$

$$\text{where } P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus, from Theorem 3.13, the origin of the system (45) is stable, as it can be seen from Figures 1 and 2.

Example 5.2. Consider the following fractional order h -difference system

$$\begin{cases} ({}_a\Delta_h^\nu x_1)(t) = -\frac{1}{2}x_2^{16}(t+vh)x_1(t+vh), & x_1(a) = 0.1, \\ ({}_a\Delta_h^\nu x_2)(t) = -\frac{1}{2}x_1^2(t+vh)x_2(t+vh), & x_2(a) = 0.2, \end{cases} \quad (46)$$

where $\nu = 0.5, a = 0, h = 1, t \in (h\mathbb{N})_{a+(1-\nu)h}$, and this difference system has a trivial solution $x(t) = (x_1(t), x_2(t))^T = 0$. We can see that

$$\begin{aligned} x^T(t+vh)Pf(t, x(t+vh)) &= (x_1(t+vh), x_2(t+vh))P \begin{bmatrix} -\frac{1}{2}x_2^{16}(t+vh)x_1(t+vh) \\ -\frac{1}{2}x_1^2(t+vh)x_2(t+vh) \end{bmatrix} \\ &= -\frac{1}{2}x_2^{16}(t+vh)x_1^2(t+vh) - \frac{1}{2}x_2^2(t+vh)x_1^2(t+vh) \leq 0, \end{aligned}$$

$$\text{where } P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, from Theorem 3.18, the origin of the system (46) is stable, as it can be seen from Figures 3 and 4.

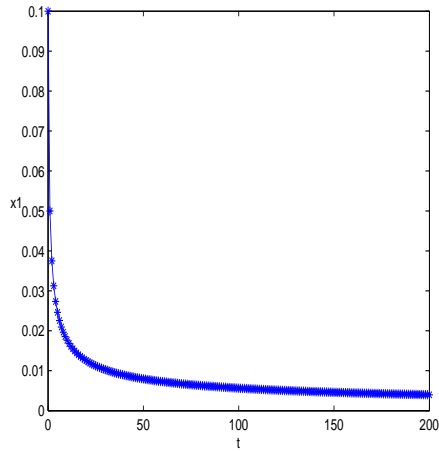


Figure 1: Stability of x_1 for $\nu = 0.5$.

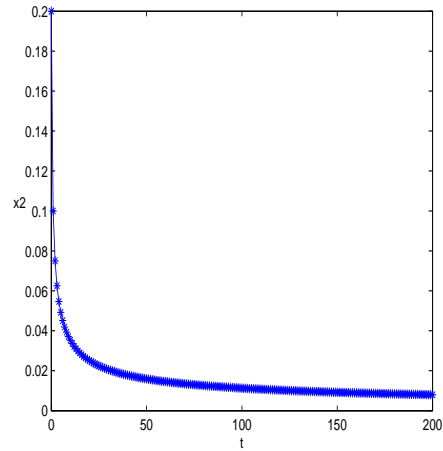


Figure 2: Stability of x_2 for $\nu = 0.5$.

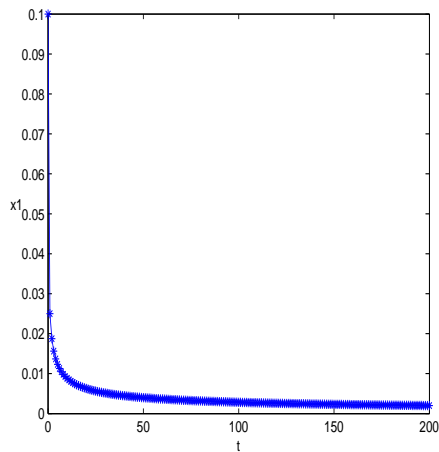


Figure 3: Stability of x_1 for $\nu = 0.5$.

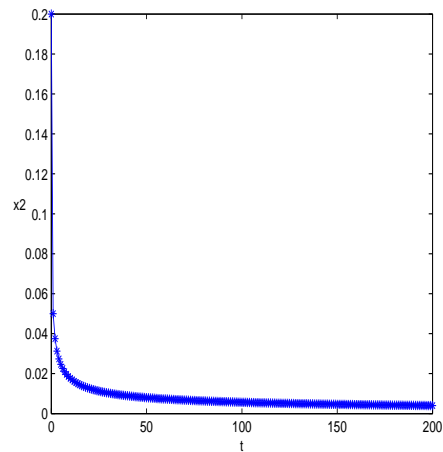


Figure 4: Stability of x_2 for $\nu = 0.5$.

Example 5.3. Consider the following fractional order h -difference system

$$\begin{cases} ({}_a\Delta_{h,*}^\nu x_1)(t) = -x_1^3(t + \nu h), & x_1(a) = 0.4, \\ ({}_a\Delta_{h,*}^\nu x_2)(t) = -x_1^2(t + \nu h) - x_2(t + \nu h), & x_2(a) = 0.2, \end{cases} \quad (47)$$

where $\nu = 0.5$, $x_i(t) \geq 0$ ($i = 1, 2$), $a = 0$, $h = 1$, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, and this difference system has a trivial solution $x(t) = (x_1(t), x_2(t))^T = 0$.

We can see that

$$x_1^2(t + \nu h)({}_a\Delta_{h,*}^\nu x_1)(t) = x_1^2(t + \nu h)(-x_1^3(t + \nu h)) = -x_1^5(t + \nu h) \leq 0,$$

$$x_2^2(t + \nu h)({}_a\Delta_{h,*}^\nu x_2)(t) = x_2^2(t + \nu h)(-x_1^2(t + \nu h) - x_2(t + \nu h)) = -x_1^2(t + \nu h)x_2^2(t + \nu h) - x_2^3(t + \nu h) \leq 0.$$

Thus, from Theorem 4.3 (i), the origin of the system (47) is stable, as it can be seen from Figures 5 and 6.

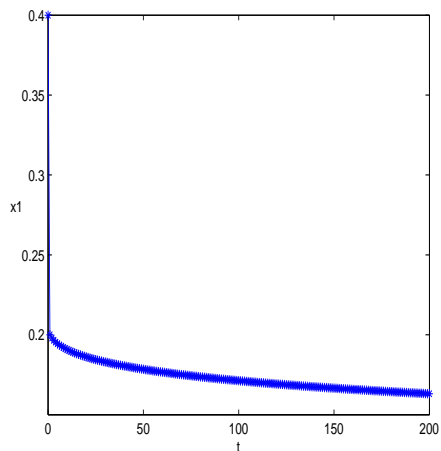


Figure 5: Stability of x for $\nu = 0.5$.

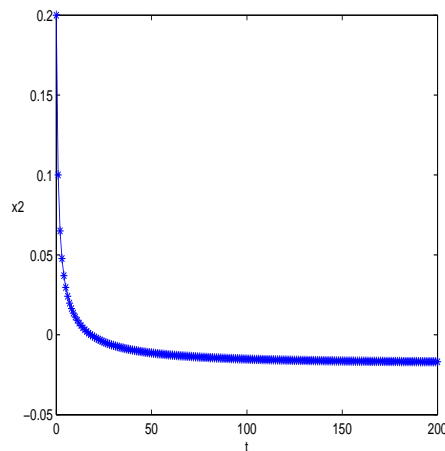


Figure 6: Stability of x for $\nu = 0.5$.

Example 5.4. Consider the following fractional order h -difference equation

$$\begin{cases} ({}_a\Delta_h^\nu x_1)(t) = -x_1(t + \nu h) - x_2^3(t + \nu h), & ({}_a\Delta_h^{\nu-1} x_1)(t)|_{t=a+(1-\nu)h} = h^{1-\nu} 0.4, \\ ({}_a\Delta_h^\nu x_2)(t) = -x_1^2(t + \nu h), & ({}_a\Delta_h^{\nu-1} x_2)(t)|_{t=a+(1-\nu)h} = h^{1-\nu} 0.2, \end{cases} \quad (48)$$

where $\nu = 0.5$, $x_i(t) \geq 0$ ($i = 1, 2$), $a = 0$, $h = 1$, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, and this difference equation has a trivial solution $x(t) = (x_1(t), x_2(t))^T = 0$.

We can see that

$$x_1^2(t + \nu h)({}_a\Delta_h^\nu x_1)(t) = x_1^2(t + \nu h)(-x_1(t + \nu h) - x_2^3(t + \nu h)) = -x_1^3(t + \nu h) - x_1^2(t + \nu h)x_2^3(t + \nu h) \leq 0,$$

$$x_2^2(t + \nu h)({}_a\Delta_h^\nu x_2)(t) = x_2^2(t + \nu h)(-x_1^2(t + \nu h)) = -x_1^2(t + \nu h)x_2^2(t + \nu h) \leq 0.$$

Thus, from Theorem 4.6 (i), the origin of the equation (48) is stable, as can be seen from Figures 7 and 8.

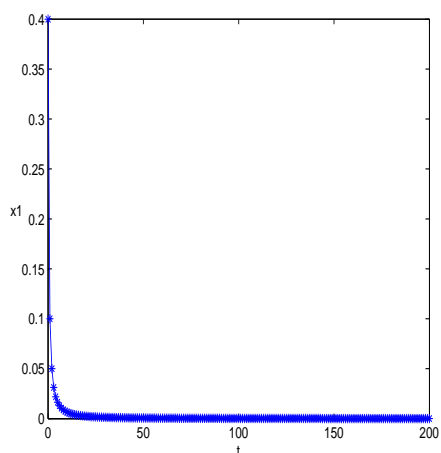


Figure 7: Stability of x for $\nu = 0.5$.

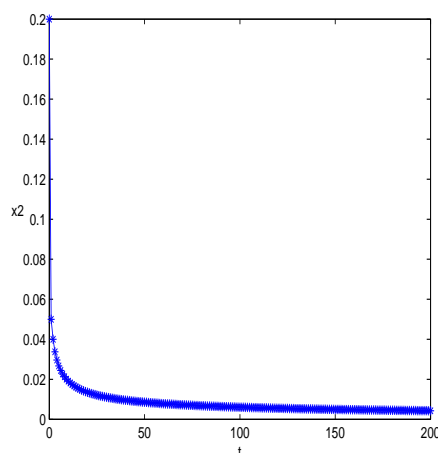


Figure 8: Stability of x for $\nu = 0.5$.

6. Conclusion

This paper presents some new propositions, which allow the application of general quadratic Lyapunov functions to the stability analysis of the fractional order h -difference systems by means of the discrete fractional Lyapunov direct method. In addition, this work gives a generalization of Lemma 2.10 in [31] and Lemma 3.2 in [4], that allows establishing a broader family of Lyapunov functions to determine the stability of the fractional order h -difference systems. As a result, we give the sufficient conditions for these systems to be stable or asymptotically stable. In addition, some examples are given to show the established results.

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References

- [1] N. AGUILA-CAMACHO, M. A. DUARTE-MERMOUD, J. A. GALLEGOS, Lyapunov functions for fractional order systems, *Commun. Nonlinear Sci. Numer. Simul.* 19 (2014) 2951-2957.
- [2] F. M. ATICI, F. WU, Existence of solutions for nonlinear fractional difference equations with initial conditions, *Dynam. Systems Appl.* 23 (2-3) (2014), 265-276.
- [3] F. M. ATICI, P. W. ELOE, Two-point boundary value problems for finite fractional difference equations, *J. Difference Equ. Appl.* 17 (4) (2011) 445-456.
- [4] D. BALEANU, G. C. WU, Y. R. BAI, F. L. CHEN, Stability analysis of Caputo-like discrete fractional systems, *Commun. Nonlinear Sci. Numer. Simul.* 48 (2017) 520-530.
- [5] J. BARANOWSKI, M. ZAGOROWSKA, W. BAUER, T. DZIWIŃSKI, P. PIATEK, Applications of direct Lyapunov method in Caputo non-integer order systems, *Elektron Elektrotech* 21 (2) (2015) 10-13.
- [6] N. R. O. BASTOS, R. A. C. FERREIRA, D. F. M. TORRES, Discrete time fractional variational problems, *Signal Processing* 91 (2011) 513-524.
- [7] R. BELLMAN, *Introduction to Matrix Analysis*, Society for Industrial and Applied Mathematics (SIAM), New York, 1960.
- [8] J. F. CHENG, *Fractional Difference Equations*, Xiamen University Press, Xiamen, 2010.
- [9] M. A. DUARTE-MERMOUD, N. AGUILA-CAMACHO, J. A. GALLEGOS, R. CASTRO-LINARES, Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, *Commun. Nonlinear Sci. Numer. Simul.* 22 (2015) 650-659.
- [10] G. FERNANDEZ-ANAYA, G. NAVA-ANTONIO, J. JAMOUS-GALANTE, R. MUÑOZ-VEGA, E. G. HERNÁNDEZ-MARTÍNEZ, Lyapunov functions for a class of nonlinear systems using Caputo derivative, *Commun. Nonlinear Sci. Numer. Simul.* 43 (2017) 91-99.
- [11] C. S. GOODRICH, Some new existence results for fractional difference equations, *Int. J. Dyn. Syst. Differ. Equ.* 3 (1-2) (2011) 145-162.
- [12] C. S. GOODRICH, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Comput. Math. Appl.* 61 (2011) 191-202.
- [13] C. S. GOODRICH, A. C. PETERSON, *Discrete Fractional Calculus*, Springer, New York, 2015.
- [14] F. JARAD, T. ABDELJAWAD, D. BALEANU, K. BIÇEN, On the stability of some discrete fractional nonautonomous systems, *Abstr. Appl. Anal.* 2012 (2012) 1-9.
- [15] B. G. JIA, X. LIU, F. F. DU, M. WANG, The solution of a new Caputo-like fractional h -difference equation, *Rocky Mountain J. Math.* 48 (5) (2018) 1607-1630.
- [16] J. JONNALAGADDA, Solutions of perturbed linear nabla fractional difference equations, *Differ. Equ. Dyn. Syst.* 22 (3) (2014) 281-292.
- [17] T. KACZOREK, *Selected Problems of Fractional Systems Theory*, Springer-Verlag, Berlin, Heidelberg, 2011.
- [18] A. KILBAS, H. SRIVASTAVA, J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, 2016.
- [19] Y. LI, Y. Q. CHEN, I. PODLUBNY, Mittag-Leffler stability of fractional order nonlinear dynamic systems, *Automatica* 45 (2009) 1965-1969.
- [20] Y. LI, Y. Q. CHEN, I. PODLUBNY, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.* 59 (2010) 1810-1821.
- [21] X. LIU, B. G. JIA, L. H. ERBE, A. C. PETERSON, Stability analysis for a class of nabla (q, h) -fractional difference equations, *Turk. J. Math.* 43 (2019) 664-687.
- [22] D. MATIGNON, Stability properties for generalized fractional differential systems, *ESAIM Proc.* 5 (1998) 145-158.
- [23] K. S. MILLER, B. ROSS, Fractional difference calculus, In: *Proc. International Symposium on Univalent Functions, Fractional Calculus and their Applications*, Nihon University, Kōriyama, 1988, 139-152.
- [24] D. MOZYRSKA, E. GIREJKO, Overview of fractional h -difference operators, In: *Advances in Harmonic Analysis and Operator Theory*, Birkhäuser, Basel, 2013, 253-268.
- [25] D. MOZYRSKA, E. GIREJKO, M. WYRWAS, Comparison of h -difference fractional operators, In: *Advances in the Theory and Applications of Non-integer Order Systems*, Springer, Heidelberg, 2013, 191-197.

- [26] D. MOZYRSKA, M. WYRWAS, The \mathcal{Z} -transform method and delta type fractional difference operators, *Discret. Dyn. Nat. Soc.* 2015 (2015) 1-12.
- [27] I. PODLUBNY, *Fractional Differential Equations, Mathematics in Sciences and Engineering*, Vol. 198, Academic Press, San Diego, 1999.
- [28] S. POLIDORO, M. A. RAGUSA, Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term, *Rev. Mat. Iberoamericana* 24 (3) (2008) 1011-1046.
- [29] Y. SHI, Q. MA, X. H. DING, Dynamical behaviors in a discrete fractional-order predator-prey system, *Filomat* 32 (17) (2018) 5857-5874.
- [30] J.-J. E. SLOTINE, W. LI, *Applied Nonlinear Control*, Englewood Cliffs, NJ: Prentice Hall, 1991.
- [31] G. C. WU, D. BALEANU, W. H. LUO, Lyapunov functions for Riemann-Liouville-like fractional difference equations, *Appl. Math. Comput.* 314 (2017) 228-236.
- [32] M. WYRWAS, E. PAWLUSZEWICZ, E. GIREJKO, Stability of nonlinear h -difference systems with n fractional orders, *Kybernetika* 51 (1) (2015) 112-136.
- [33] M. WYRWAS, D. MOZYRSKA, On Mittag-Leffler stability of fractional order difference systems, In: *Advances in Modeling and Control of Non-integer order systems*, Springer, Cham, 2015, 209-220.
- [34] J. M. YU, H. HU, S. B. ZHOU, X. R. LIN, Generalized Mittag-Leffler stability of multi-variables fractional order nonlinear systems, *Automatica* 49 (2013) 1798-1803.
- [35] F. R. ZHANG, C. P. LI, Y. Q. CHEN, Asymptotical stability of nonlinear fractional differential systems with Caputo derivative, *Internat. J. Differ. Equ.* 2011 (2011) 1-12.
- [36] X. F. ZHOU, L. G. HU, S. LIU, W. JIANG, Stability criterion for a class of nonlinear fractional differential systems, *Appl. Math. Lett.* 28 (2014) 25-29.