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B.-Y. Chen's Inequality for Pointwise CR-Slant Warped Products in Cosymplectic Manifolds

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Abstract. Recently, pointwise CR-slant warped products introduced by Chen and Uddin in [14] for Kaehler manifolds. In the context of almost contact metric manifolds, in this paper, we study these submanifolds in cosymplectic manifolds. We investigate the geometry of such warped product and prove establish a lower bound relation between the second fundamental form and warping function. The equality case is also investigated.

1. Introduction

The notation of CR-submanifolds in the setting of Kaehler manifolds was introduced by Bejancu in [2], which is a generalization of totally real and holomorphic submanifolds of Kaehler manifolds. Then, CR-submanifolds have been investigated by many researchers in the different types of structures (see for instance [1, 3, 5]). Another generalization of such submanifolds is called slant submanifold which is introduced by B.-Y. Chen [7, 8]. Later, F. Etayo [15] defined the notion of pointwise slant submanifold under the name of quasi-slant submanifold. In [11], B.-Y. Chen and O.J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and they proved many nice new results on such submanifolds and introduce a method of constructions of new examples of such submanifolds.

On the other hand, Bishop and O'Neill [6] defined and studied the concept of warped product manifolds in order to study the manifolds with negative curvature. The idea of warped product submanifolds has been introduced by Chen in his series papers [9, 10]. He proved several fundamental results on the existence of CR-warped products in Kaehler manifolds. Later on, the geometric aspects of these manifolds have been studied by many researchers (see, for example, [12], [13], [20], [21], [25], [26], [27], [28], [30], [31], [36], [37], [38] and the reference therein). Specially, Sahin studied nonexistence of warped product semi-slant submanifolds in Kaehler manifolds [23], while he proved in [24] some results on warped product pointwise semi-slant submanifolds of Kaehler manifolds. Recently, Chen and Uddin introduced the notion of pointwise CR-slant warped products in Kaehler manifolds [14]. In the present paper we extend the result of [14] in cosymplectic manifolds.

The paper is organised as follows: we recall some basic formulas and definitions in Section 2, which are useful to the next section. In Section 3, we recall the definition of warped product and pointwise

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CR-slant warped products and provide some useful results. The last section is devoted to establish a sharp inequality for the squared norm of the second fundamental form for pointwise CR-slant warped products in cosymplectic manifolds. Also, the equality is also discussed in details.

2. Preliminaries

Let $\tilde{M}$ be a $(2n+1)$ dimensional C^∞ manifold, then $\tilde{M}$ is said to be *an almost contact metric* manifold if it equips with the structure tensors (φ, ξ, η, g) such that φ is a tensor field of type $(1,1)$, ξ a vector field, η is a $1-$form and g is a Riemannian metric on $\tilde{M}$ satisfying the following properties [4]

$$\varphi^2 X = -X + \eta(X)\xi, \ \ \varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ g(X, \xi) = \eta(X) \tag{2}$$

for all vector fields X, Y tangent to $\tilde{M}$. An almost contact metric manifold $\tilde{M}$ is said to be *cosymplectic* if [18]

$$(\tilde{\nabla}_X \varphi)Y = 0, \tag{3}$$

for all X, Y tangent to $\tilde{M}$.

Furthermore, on a cosymplectic manifold, we have

$$\tilde{\nabla}_X \xi = 0, \tag{4}$$

for any X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$ corresponding to the Riemannian metric g. It known that the covariant derivative φ of the tensor field is defined by

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y. \tag{5}$$

Let M be an m-dimensional submanifold of an almost contact metric manifold $\tilde{M}$ with the same Riemannian metric g induced on $\tilde{M}$. Denote by $\Gamma(TM)$ the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ be the set of all vector fields normal to M. Consider ∇ and $\nabla^\perp$ as the Levi-Civita connections on the tangent bundle TM and the normal bundle $T^\perp M$, respectively. Then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^\perp_X U \tag{7}$$

for each $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$, such that $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\tilde{M}$ and A_U is the shape operator of M corresponding to the normal vector U. Moreover, the relationship between them is given by

$$g(h(X, Y), U) = g(A_U X, Y) \tag{8}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$.

A submanifold for which the second fundamental form h is identically zero is said to be a *totally geodesic* submanifold, while it is called a *totally umbilical* submanifold if its second fundamental form h is given by

$$h(X, Y) = g(X, Y)H,$$

for each $X, Y \in \Gamma(TM)$, where

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i),$$

is the mean curvature vector, such that (e_i), $1 \leq i \leq m$ denotes a local orthonormal frame of the tangent space TM.
The squared norm of the second fundamental form h is defined by

$$\|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)), \tag{9}$$

and

$$h_{ij}^r = g(h(e_i, e_j), e_r)) \quad i, j = 1, 2..., m, \quad r = m + 1, ..., 2n + 1. \tag{10}$$

For a differentiable function f on an m-dimensional manifold M, the gradient $\vec{\nabla} f$ of f is defined as

$$g(\vec{\nabla} f, X) = X(f), \tag{11}$$

for any X tangent to M.
For $X \in \Gamma(TM)$, we can set

$$\varphi X = TX + FX, \tag{12}$$

where TX and FX are the tangential and normal components of φX, respectively. Also, for any $U \in \Gamma(T^{\perp}M)$, we may write

$$\varphi U = tU + fU, \tag{13}$$

where tU and fU are the tangential and normal components of φU, respectively. Furthermore, using (1) and (12), we get that

$$g(TX, Y) = -g(X, TY), \tag{14}$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold $\tilde{M}$ tangent to the structure vector field ξ is called an *invariant* if F is identically zero, that is $\varphi X \in \Gamma(TM)$, for any $X \in \Gamma(TM)$, while M is called an *anti-invariant* if T is identically zero, that is $\varphi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

In addition to invariant and anti-invariant submanifolds, there are many different classes of submanifolds of $\tilde{M}$, some of them are defined as below:

Definition 2.1. *[25] A submanifold M tangent to the structure vector field ξ of a an almost contact metric manifold $\tilde{M}$ is said to be a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions $(\mathcal{D}^{\mathcal{T}}, \mathcal{D}^{\perp})$ satisfying the following conditions:*

(i) $TM = \mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the 1-dimensional distribution spanned by ξ_p at $p \in M$.
(ii) The distribution $\mathcal{D}^{\mathcal{T}}$ is invariant under φ, i.e., $\varphi({\mathcal{D}^{\mathcal{T}}}_p) = {\mathcal{D}^{\mathcal{T}}}_p$ for all $p \in M$.
(iii) The distribution $\mathcal{D}^{\perp}$ is anti-invariant under φ, i.e., $\varphi(\mathcal{D}_p^{\perp}) \subset T_p^{\perp}(M)$ for all $p \in M$.

Definition 2.2. *[17] A submanifold M of a an almost contact metric manifold $\tilde{M}$ is said to be slant if for each $p \in M$ and a nonzero vector X tangent to M at p such that X is not proportional to $\langle \xi \rangle$ the Wirtinger angle $\theta(X) \in [0, \pi/2]$ between φX and T_pM is constant, i.e, it is independent of the choice of a non-zero vector $X \in T_pM$ and the choice of the point $p \in M$.*

Obviously if $\theta = 0$, then M becomes an invariant and if $\theta = \pi/2$, then M becomes an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

Definition 2.3. *[22], [19] A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be pointwise slant, if for each point $p \in M$, the Wirtinger angle $\theta(X)$ between φX and T_pM is independent of the choice of a non-zero vector $X \in T_pM$. The Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$ which is called the Wirtinger function or slant function of the pointwise slant submanifold.*

We note that a pointwise slant submanifold of $\tilde{M}$ is called *slant*, if its Wirtinger function θ is globally constant and also, it is called a *proper pointwise slant* if it is neither invariant nor anti-invariant nor θ is constant on M. ([8] and [24])

It was proved in [34] that a submanifold M tangent to the strutter vector field ξ is a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$ if and only if

$$T^2X = (\cos^2\theta)(-X + \eta(X)\xi), \tag{15}$$

for the slant function θ defined on M.

The following relations are natural results of (15)

$$g(TX, TY) = (\cos^2\theta)(g(X, Y) - \eta(Y)\eta(X)), \tag{16}$$

$$g(FX, FY) = (\sin^2\theta)(g(X, Y) - \eta(Y)\eta(X)), \tag{17}$$

for any $X, Y \in \Gamma(TM)$.

Another useful relation for pointwise slant submanifolds of $\tilde{M}$ comes from (1) and (15) given in [34] as follows

$$tFX = (\sin^2\theta)(-X + \eta(X)\xi), \qquad fFX = -FTX, \tag{18}$$

for any $X \in \Gamma(TM)$.

Definition 2.4. *Let M be a submanifold of a cosymplectic manifold $\tilde{M}$. Then, M is said to be a pointwise CR-slant submanifold if there exist three integrable distributions $\mathcal{D}^{\mathcal{T}}$, $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$ such that*

$$TM = \mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^{\perp} \oplus \mathcal{D}^{\theta} \oplus \langle\xi\rangle,$$

where $\mathcal{D}^{\mathcal{T}}$ is invariant distribution, $\mathcal{D}^{\perp}$ is anti-invariant distribution, $\mathcal{D}^{\theta}$ is a pointwise slant distribution, and $\langle\xi\rangle$ is the 1-dimensional distribution spanned by ξ_p at $p \in M$.

3. Pointwise CR-slant warped product submanifolds

In [6], Bishop and O'Neill defined the warped product manifolds as followes: Let M_1 and M_2 be two Riemannian manifolds with their Riemannian metrics g_1 and g_2, respectively, then their warped product manifold is denoted by $M = M_1 \times_f M_2$ with product structure

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (f \circ \pi)^2 g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(TM)$, where $\pi : M \to M_1$ and $\sigma : M \to M_2$ are the projections and $f : M_1 \to (0, \infty)$, a scalar function on M_1 and $*$ denotes the symbol for tangent maps. The function f is called the *warping function* of warped product. If the warping function f is constant, then the manifold M is said to be *trivial* or simply a *Riemannian product manifold*.

If $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, then from Lemma 7.3 of [6], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{19}$$

where ∇ is Levi-Civita connection on M. On a warped product manifold $M = M_1 \times_f M_2$, we have, M_1 is totally geodesic in M, and M_2 is totally umbilical in M.

Definition 3.1. *[14] A pointwise CR-slant submanifold $M = (M_T \times M_\perp)\times_f M_\theta$, where $M_T, M_\perp$, and M_θ are the integrable submanifolds of $\mathcal{D}^{\mathcal{T}}, \mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively, is called pointwise CR-slant warped product submanifold if it is equipped with the warped product metric*

$$g = g_{M_T \times M_\perp} + f^2 g_{M_\theta},$$

where $g_{M_T\times M_\perp}$ is the metric on $(M_T\times M_\perp)$, g_{M_θ} is the metric on M_θ, and f is a positive function depending only on $(M_T \times M_\perp)$.

It is known that if the slant function θ of $\mathcal{D}^\theta$ is constant, then M is called a *CR-slant warped product* [35] and if the warping function f is non constant, then $M = M_T\times M_\perp\times_f M_\theta$ is called a *proper pointwise CR-slant warped product submanifold* .

Remark 3.2. *If $M = (M_T\times M_\perp)\times_f M_\theta$ be a pointwise CR-slant warped product submanifold in $\tilde{M}$, then the tangent and the normal bundles of M are respectively decomposed as*

$$TM = \mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle, \tag{20}$$

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \nu, \tag{21}$$

where ν is the φ-invariant subbundle of the normal bundle $T^\perp M$.

Definition 3.3. *[14] A pointwise CR-slant warped product $M = (M_T \times M_\perp) \times_f M_\theta$ is said to be weakly $\mathcal{D}^\theta$-totally geodesic if $g(h(\mathcal{D}^\theta, \mathcal{D}^\theta), \varphi\mathcal{D}^\perp) = \{0\}$.*

Definition 3.4. *[14] A pointwise CR-slant warped product $M = (M_T \times M_\perp) \times_f M_\theta$ is said to be $\mathcal{D}^1 \oplus \mathcal{D}^2$-mixed totally geodesic if $h(\mathcal{D}^1, \mathcal{D}^2) = \{0\}$, such that $\mathcal{D}^1$ and $\mathcal{D}^2$ are distributions belong to $\{\mathcal{D}^{\mathcal{T}}, \mathcal{D}^\perp, \mathcal{D}^\theta\}$*

In this section, we discuss the geometry of the pointwise CR-slant warped product of the form $M = (M_T \times M_\perp) \times_f M_\theta$ in a cosymplectic manifold $\tilde{M}$, such that M_T, $M_\perp$ and M_θ are invariant, anti-invariant, and proper pintwise slant submanifolds of $\tilde{M}$, respectively. It is known that from [16] when the structure vector field ξ is tangent to the submanifold M_θ, then the warped product is trivial. Thus, we will not study this case for non existence of warped products. So, we consider ξ is tangent to $(M_T \times M_\perp)$, in this case either ξ is tangent to M_T or $M_\perp$ and we will discuss both cases.

First, we give the following useful results.

Lemma 3.5. *Let $M = (M_T \times M_\perp) \times_f M_\theta$ be a pointwise CR-slant warped product of a cosymplectic manifold $\tilde{M}$ such that ξ is tangent to $(M_T \times M_\perp)$. Then, we have the following:*

$$\xi(\ln f) = 0, \tag{22}$$

$$g(h(X, Y), FZ) = 0, \tag{23}$$

$$g(h(X, Z), \varphi U) = 0, \tag{24}$$

$$g(h(X, U), FZ) = 0, \tag{25}$$

where $X, Y \in \Gamma(\mathcal{D}^{\mathcal{T}})$, $U \in \Gamma(\mathcal{D}^\perp)$ and $Z \in \Gamma(\mathcal{D}^\theta)$.

Proof. First assertion was proved in [16], while the second assertion is directly and it can be obtained by using (6) and (19) with the orthogonality of vector fields. For the third part, if we consider any $X, Y \in \Gamma(\mathcal{D}^{\mathcal{T}})$, $U \in \Gamma(\mathcal{D}^{\perp})$ and $Z, \in \Gamma(\mathcal{D}^{\theta})$, by Gauss formula, we obtain

$$g(h(X,Z), \varphi U) = -g(\varphi\tilde{\nabla}_Z X, U).$$

Again from (6) with the cosymplectic character, we find

$$g(h(X,Z), \varphi U) = -g(\nabla_Z \varphi X, U).$$

So, by (19), we obtain

$$g(h(X,Z), \varphi U) = -\varphi X(\ln f) g(Z, U).$$

Thus, the orthogonality of vector fields gives (24). The last part also can be obtained by the similar technique. □

Lemma 3.6. *Let $M = (M_T \times M_\perp) \times {}_f M_\theta$ be a pointwise CR-slant warped product submanifold of a cosymplectic manifold $\tilde{M}$ such that ξ is tangent to $(M_T \times M_\perp)$. Then, the following is satisfied:*

$$g(h(U,Z), \varphi V) = g(h(U,V), FZ), \tag{26}$$

$$g(h(X,Z), FW) = -\varphi X(\ln f) g(Z, W) - X(\ln f) g(Z, TW), \tag{27}$$

$$g(h(Z,W), \varphi U) = U(\ln f) g(Z, TW) + g(h(U,Z), FW), \tag{28}$$

for any $X \in \Gamma(\mathcal{D}^{\mathcal{T}})$, $U, V \in \Gamma(\mathcal{D}^{\perp})$ and $Z, W \in \Gamma(\mathcal{D}^{\theta})$.

Proof. For any $X \in \Gamma(\mathcal{D}^{\mathcal{T}})$, $U, V \in \Gamma(\mathcal{D}^{\perp})$ and $Z, W \in \Gamma(\mathcal{D}^{\theta})$, from (8), we can write

$$g(h(U,Z), \varphi V) = g(A_{\varphi V} U, Z).$$

Now, using (7), we get

$$g(h(U,Z), \varphi V) = -g(\tilde{\nabla}_U \varphi V, Z),$$

Az $\tilde{M}$ is cosymplectic, by (5), we can write

$$g(h(U,Z), \varphi V) = g(\tilde{\nabla}_U V, \varphi Z).$$

Therefore, from (6) and (12), we arrive at

$$g(h(U,Z), \varphi V) = g(h(U,V), FZ) + g(\nabla_U V, TZ).$$

Thus, by the orthogonality of vector fields we prove (26). Next, we want to prove (27). Using (6) and (12), we have

$$g(h(X,Z), FW) = -g(\varphi\tilde{\nabla}_Z X, W) - g(\tilde{\nabla}_Z X, TW).$$

Since, $\tilde{M}$ has cosymplectic structure, then from (5) and (6), we derive

$$g(h(X,Z), FW) = -g(\nabla_Z \varphi X, W) - g(\nabla_Z X, TW).$$

Thus, (27) follows from the above relation together with (19). For the last assertion, by (6), we have

$$g(h(Z,W), \varphi U) = -g(\varphi\tilde{\nabla}_Z W, U) = -g(\tilde{\nabla}_Z \varphi W, U).$$

From (12), we find

$$g(h(Z,W),\varphi U) = g(\tilde{\nabla}_Z U, TW) - g(\tilde{\nabla}_Z FW, U).$$

Then using (2), (6) and (19), we derive

$$g(h(Z,W),\varphi U) = U(\ln f)g(Z,TW) - g(\varphi\tilde{\nabla}_Z FW, \varphi U) - \eta(\tilde{\nabla}_Z FW)\eta(U).$$

But, if we applying (2), then we get

$$g(h(Z,W),\varphi U) = U(\ln f)g(Z,TW) - g(\varphi\tilde{\nabla}_Z FW, \varphi U). \tag{29}$$

Furthermore, From (13), we have

$$g(\varphi\tilde{\nabla}_Z FW,\varphi U) = g(\tilde{\nabla}_Z tFW, \varphi U) + g(\tilde{\nabla}_Z fFW, \varphi U)$$

Using (18), we obtain

$$\begin{aligned} g(\varphi\tilde{\nabla}_Z FW,\varphi U) = &-(\sin^2\theta)g(\tilde{\nabla}_Z W,\varphi U) - 2(\sin\theta)(\cos\theta)(Z\theta)g(W,\varphi U) + (\sin^2\theta)g(\tilde{\nabla}_Z\eta(W)\xi,\varphi U)\\ &- 2(\sin\theta)(\cos\theta)(Z\theta)g(\eta(W)\xi,\varphi U) - g(\tilde{\nabla}_Z FTW,\varphi U).\end{aligned}$$

Hence, from (2), (6) and the orthogonality of vector fields, the above relation takes the form

$$g(\varphi\tilde{\nabla}_Z FW,\varphi U) = -(\sin^2\theta)g(h(Z,W),\varphi U) + g(\tilde{\nabla}_Z\varphi FTW, U).$$

Using (13) and (18), we can write

$$\begin{aligned} g(\varphi\tilde{\nabla}_Z FW,\varphi U) = &-(\sin^2\theta)g(h(Z,W),\varphi U) - (\sin^2\theta)g(\tilde{\nabla}_Z TW, U) - (\sin 2\theta)(Z\theta)g(TW,U)\\ &+ (\sin^2\theta)g(\tilde{\nabla}_Z\eta(TW)\xi, U) - (\sin 2\theta)(Z\theta)g(\eta(TW)\xi,U) - g(\tilde{\nabla}_Z FT^2W, U).\end{aligned}$$

From (2), (6), (15), and the orthogonality of vector field, we derive

$$g(\varphi\tilde{\nabla}_Z FW,\varphi U) = -(\sin^2\theta)g(h(Z,W),\varphi U) + (\sin^2\theta)g(\nabla_Z U, TW) + (\cos^2\theta)g(\tilde{\nabla}_Z FW, U).$$

Thus, by (7),(8) and (19), the above reduced to

$$g(\varphi\tilde{\nabla}_Z FW,\varphi U) = -(\sin^2\theta)g(h(Z,W),\varphi U) + U(\ln f)(\sin^2\theta)g(Z,TW) - (\cos^2\theta)g(h(Z,U)FW).$$

Therefore, (29) becomes

$$g(h(Z,W),\varphi U) = U(\ln f)g(Z,TW) + (\sin^2\theta)g(h(Z,W),\varphi U) - U(\ln f)(\sin^2\theta)g(Z,TW) + (\cos^2)g(h(Z,U)FW),$$

which is the required result. Hence, the lemma is proved completely. □

Theorem 3.7. *Let $M = (M_T \times M_\perp) \times {}_f M_\theta$ be a pointwise CR-slant warped product submanifold of a cosymplectic manifold $\tilde{M}$. Then M is locally trivial if M is both $\mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^\theta$ and $\mathcal{D}^\perp \oplus \mathcal{D}^\theta$-mixed totally geodesic such that ξ is tangent to $(M_T \times M_\perp)$.*

Proof. Let M be a $\mathcal{D}^{\mathcal{T}} \oplus \mathcal{D}^\theta$-mixed totally geodesic. Then, for any $X, Y \in \Gamma(\mathcal{D}^{\mathcal{T}})$, and $Z, W \in \Gamma(\mathcal{D}^\theta)$, from (27) we have

$$\varphi X(\ln f)g(Z,W) + X(\ln f)g(Z,TW) = 0. \tag{30}$$

Replacing X by φX in (30) by using (1) and (22), we obtain

$$-X(\ln f)g(Z,W) + \varphi X(\ln f)g(Z,TW) = 0.$$

Now, replacing W by TW in the above with using (2) and (15), we find

$$-X(\ln f)g(Z,TW) - \varphi X(\ln f)(\cos^2\theta)g(Z,W) = 0, \tag{31}$$

Adding equations (30) and (31), we reduce to

$$\varphi X(\sin^2\theta)(\ln f)g(Z,W) = 0.$$

Therefore, we deduce that the warping function f is constant. Thus, M is locally trivial. Also, if M is a $\mathcal{D}^\perp \oplus \mathcal{D}^\theta$-mixed totally geodesic, then for any $U, V \in \Gamma(\mathcal{D}^\perp)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, by (28) we have

$$g(h(Z,W),\varphi U) - U(\ln f)g(Z,TW) = 0. \tag{32}$$

Applying the polarization identity in (32), we can write

$$g(h(Z,W),\varphi U) + U(\ln f)g(Z,TW) = 0. \tag{33}$$

Subtracting (32) and (33), we derive

$$U(\ln f)g(Z,TW) = 0.$$

If we put $W = TW$ in the last equation with using (2) and (15),we find

$$U(\ln f)(\cos^2\theta)g(Z,W) = 0.$$

So, it follows that f is constant. Hence, M is locally trivial. $\square$

4. Inequality for pointwise CR-slant warped products

Let $M = (M_T \times M_\perp) \times {}_f M_\theta$ be a m-dimensional pointwise CR-slant warped product submanifold of a $(2n+1)$-dimensional cosymplectic manifold $\tilde{M}$ such that the structure vector field ξ tangent to $(M_T \times M_\perp)$, where M_T, $M_\perp$ and M_θ are invariant, anti-invariant, and proper pintwise slant submanifolds of $\tilde{M}$, respectively. Let us consider the $\dim M_T = 2p+1$, $\dim M_\perp = q$ and $\dim M_\theta = 2r$ and their corresponding tangent bundles are denoted by $\mathcal{D}^\mathcal{T}$, $\mathcal{D}^\perp$, and $\mathcal{D}^\theta$, respectively. We set the orthonormal frames of them as follows:

$$\mathcal{D}^\mathcal{T} \oplus \langle\xi\rangle = Span\{e_1, e_2, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots, e_{2p} = \varphi e_p, e_{2p+1} = \xi\}, \quad \mathcal{D}^\perp = Span\{e_{2p+2} = \hat{e}_1, \cdots, e_{2p+q+1} = \hat{e}_q\},$$

and

$$\mathcal{D}^\theta = Span\{e_{2p+q+2} = e_1^*, \cdots, e_{2p+q+r+1} = e_r^*, e_{2p+q+r+2} = e_{r+1}^* = \sec\theta\, Te_1^*, \cdots e_m = e_{2p+q+2r+1} = e_{2r}^* = \sec\theta\, Te_r^*\}.$$

Then the orthonormal frames of the normal subbundles $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$ and ν, respectively are given by

$$\varphi\mathcal{D}^\perp = Span\{e_{m+1} = \tilde{e}_1 = \varphi\hat{e}_1, \cdots, e_{m+q} = \tilde{e}_q = \varphi\hat{e}_q\}$$

$$\begin{aligned} F\mathcal{D}^\theta = Span\{&e_{m+q+1} = \tilde{e}_{q+1} = \csc\theta\, Fe_1^*, \cdots e_{m+q+r} = \tilde{e}_{q+r} = \csc\theta\, Fe_r^*, e_{m+q+r+1} = \tilde{e}_{q+r+1} = \csc\theta\sec\theta FTe_1^*, \\ &\cdots, e_{m+q+2r} = \tilde{e}_{q+2r} = \csc\theta\sec\theta FTe_r^*\}, \quad \nu = Span\{e_{m+q+2r+1} = \tilde{e}_{q+2r+1}, \cdots, e_{2n+1-m} = \tilde{e}_{2n-2p-2r-q}\}, \end{aligned}$$

where ν is the ϕ-invariant normal subbundle in $T^\perp M$.

The following theorem is the contact version of the main theorem given in [14] and this theorem generalise some others results.

Theorem 4.1. *Let $M = (M_T \times M_\perp) \times {}_f M_\theta$ be a weakly $\mathcal{D}^\theta-$totally geodesic pointwise CR-slant warped product submanifold of a cosymplectic manifold $\tilde{M}$, such that the structure vector field ξ tangent to $(M_T \times M_\perp)$. Then the squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq 4r\Big[\cot^2\theta||\nabla^\perp(\ln f)||^2 + (\csc^2\theta + \cot^2\theta)||\nabla^T(\ln f)||^2\Big], \tag{34}$$

where $r = \frac{1}{2}dim\ M_\theta$, $\nabla^T(\ln f)$ and $\nabla^\perp(\ln f)$ are the gradient components of the function $\ln f$ along M_T and $M_\perp$, respectively. The equality sign in (34) *holds identically if and only if the following are satisfied*

(i) M_T and $M_\perp$ are totally geodesic submanifolds of $\tilde{M}$.

(ii) M_θ is a totally umbilical submanifold in $\tilde{M}$.

Proof. From (9), we have

$$||h||^2 = \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m} g(h(e_i,e_j),e_k)^2.$$

Using the frame fields of $\mathcal{D}^{\mathcal{T}} \oplus \langle\xi\rangle$, $\mathcal{D}^\perp$, $\mathcal{D}^\theta$, $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$ and ν, we can write

$$\begin{aligned}||h||^2 = &\sum_{k=1}^{2n-2p-2r-q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_k)^2 + 2\sum_{k=1}^{2n-2p-2r-q}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,\hat{e}_j),\tilde{e}_k)^2 + \sum_{k=1}^{2n-2p-2r-q}\sum_{i,j=1}^{q} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_k)^2\\ &+ 2\sum_{k=1}^{2n-2p-2r-q}\sum_{i=1}^{q}\sum_{j=1}^{2s} g(h(\hat{e}_i,e^*_j),\tilde{e}_k)^2 + \sum_{k=1}^{2n-2p-2r-q}\sum_{i,j=1}^{2s} g(h(e^*_i,e^*_j),\tilde{e}_k)^2 + 2\sum_{k=1}^{2n-2p-2r-q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2r} g(h(e_i,e^*_j),\tilde{e}_k)^2.\end{aligned}$$

Applying the constructed frame fields, the above expression can be decomposed as

$$\begin{aligned}||h||^2 = &\sum_{k=1}^{q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\phi\hat{e}_k)^2 + (\csc^2\theta)\sum_{k=1}^{r}\sum_{i,j=1}^{2p+1}\Big[g(h(e_i,e_j),Fe_k{}^*)^2 + (\sec^2\theta)g(h(e_i,e_j),FTe_k{}^*)^2\Big] \\ &+ \sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i,j=1}^{2p+1} g(h(e_i,e_j),\tilde{e}_k)^2 + 2\sum_{k=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,\hat{e}_j),\phi\hat{e}_k)^2 + 2(\csc^2\theta)\sum_{k=1}^{r}\sum_{i=1}^{2p+1}\sum_{j=1}^{q}\Big[g(h(e_i,\hat{e}_j),Fe_k{}^*)^2\\ &+ (\sec^2\theta)g(h(e_i,\hat{e}_j),FTe_k{}^*)^2\Big] + \sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i,j=1}^{2p+1}\sum_{j=1}^{q} g(h(e_i,\hat{e}_j),\tilde{e}_k)^2 + \sum_{k=1}^{q}\sum_{i,j=1}^{q} g(h(\hat{e}_i,\hat{e}_j),\phi\hat{e}_k)^2\\ &+ (\csc^2\theta)\sum_{k=1}^{r}\sum_{i,j=1}^{q}\Big[g(h(\hat{e}_i,\hat{e}_j),Fe_k{}^*)^2 + (\sec^2\theta)g(h(\hat{e}_i,\hat{e}_j),FTe_k{}^*)^2\Big] + \sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i,j=1}^{q} g(h(\hat{e}_i,\hat{e}_j),\tilde{e}_k)^2\\ &+ 2\sum_{k=1}^{q}\sum_{i=1}^{q}\sum_{j=1}^{2r} g(h(\hat{e}_i,e^*_j),\phi\hat{e}_k)^2 + 2(\csc^2\theta)\sum_{k=1}^{r}\sum_{i=1}^{q}\sum_{j=1}^{2r}\Big[g(h(\hat{e}_i,e^*_j),Fe_k{}^*)^2 + (\sec^2\theta)g(h(\hat{e}_i,e^*_j),FTe_k{}^*)^2\Big]\\ &+ 2\sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i=1}^{q}\sum_{j=1}^{2r} g(h(\hat{e}_i,e^*_j),\tilde{e}_k)^2 + \sum_{k=1}^{q}\sum_{i,j=1}^{2r} g(h(e^*_i,e^*_j),\phi\hat{e}_k)^2 + (\csc^2\theta)\sum_{k=1}^{r}\sum_{i,j=1}^{2r}\Big[g(h(e^*_i,e^*_j),Fe_k{}^*)^2\\ &+ (\sec^2\theta)g(h(e^*_i,e^*_j),FTe_k{}^*)^2\Big] + \sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i,j=1}^{2r} g(h(e^*_i,e^*_j),\tilde{e}_k)^2 + 2\sum_{k=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2r} g(h(e_i,e_j{}^*),\phi\hat{e}_k)^2\\ &+ 2(\csc^2\theta)\sum_{k=1}^{q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2r}\Big[g(h(e_i,e_j{}^*),Fe_k{}^*)^2 + (\sec^2\theta)g(h(e_i,e_j{}^*),FTe_k{}^*)^2\Big] + 2\sum_{k=q+2r+1}^{2n-2p-2r-q}\sum_{i=1}^{2p+1}\sum_{j=1}^{2r} g(h(e_i,e_j{}^*),\tilde{e}_k)^2.\end{aligned} \tag{35}$$

We will leave the positive third, sixth, ninth, twelfth, fifteenth and eighteenth ν-component terms in (35). The second, fifth, thirteenth and sixteenth terms vanish identically by using Lemma (3.5) and for a weakly $\mathcal{D}^\theta-$totally geodesic warped product. Also, we could not find the relations for warped products of the first, fourth, seventh, and fourteenth in (35). Hence, we can leave these positive terms. On the other hand, from Lemma (3.6) with the constructed frame fields, the above expression can be simplified as

$$\begin{aligned}||h||^2 \geq\ & 2(\csc^2\theta)\sum_{k=1}^{r}\sum_{i=1}^{q}\sum_{j=1}^{2r}\Big\{[-e_i(\ln f)g(e_j^*, Te_k^*)]^2 + (\sec^2\theta)\,[-e_i(\ln f)g(e_j^*, T^2e_k^*)]^2\Big\}\\ &+ 2(\csc^2\theta)\sum_{k=1}^{r}\sum_{i=1}^{2p+1}\sum_{j=1}^{2r}\Big\{[-\varphi e_i(\ln f)g(e_j^*, e_k^*) - e_i(\ln f)g(e_j^*, Te_k^*)]^2\\ &+ (\sec^2\theta)\,[-\varphi e_i(\ln f)g(e_j^*, Te_k^*) - e_i(\ln f)g(e_j^*, T^2e_k^*)]^2\Big\}\end{aligned}$$

which gives

$$||h||^2 \geq 4r(\cot^2\theta)\sum_{i=1}^{q}(e_i(\ln f))^2 + 4r(\csc^2\theta)\sum_{i=1}^{2p+1}\Big[(\varphi e_i(\ln f))^2 + (e_i(\ln f))^2\Big] + 4r(\cot^2\theta)\sum_{i=1}^{2p+1}\Big[(\varphi e_i(\ln f))^2 + (e_i(\ln f))^2\Big].$$

Then, from the gradient definition, we get the required inequality (i). To prove the equality case of (34), we proceed as follows: from the leaving and vanishing terms in the right hand side of (35), we have

$$h(\mathcal{D}^\mathcal{T}, \mathcal{D}^\mathcal{T}) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0 \quad h(\mathcal{D}^\mathcal{T}, \mathcal{D}^\perp) = 0. \tag{36}$$

As M_T is totally geodesic in M [6, 8], by this fact with the first condition in (36), we conclude that M_T is totally geodesic submanifold of $\tilde{M}$. By a similar argument, we get $M_\perp$ is also totally geodesic submanifold of $\tilde{M}$. Thus, we prove assertion (i).
On the other hand, we have

$$h(\mathcal{D}^\mathcal{T}, \mathcal{D}^\theta) \subset F\mathcal{D}^\theta, \quad h(\mathcal{D}^\perp, \mathcal{D}^\theta) \subset F\mathcal{D}^\theta. \tag{37}$$

And also as M_θ is totally umbilical in M [6, 8], applying this fact with (37), we observe that M_θ is a totally umbilical in $\tilde{M}$. So, assertion (ii) follows. Hence, the theorem is proved completely. □

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