Gaussian Pell and Gaussian Pell-Lucas Quaternions

Hasan Arslan*

Abstract. The main aim of this work is to introduce the Gaussian Pell quaternion $Q_{Gp}$ and Gaussian Pell-Lucas quaternion $Q_{Gq}$, where the components of $Q_{Gp}$ and $Q_{Gq}$ are Pell numbers $p_n$ and Pell-Lucas numbers $q_n$, respectively. Firstly, we obtain the recurrence relations and Binet formulas for $Q_{Gp}$ and $Q_{Gq}$. We use Binet formulas to prove Cassini’s identity for these quaternions. Furthermore, we give some basic identities for $Q_{Gp}$ and $Q_{Gq}$ such as some summation formulas, the terms with negative indices and the generating functions for these complex quaternions.

1. PRELIMINARIES AND INTRODUCTION

The quaternions, which are a members of a noncommutative division algebra, were first invented by W. R. Hamilton in 1843 as an extension of the set of complex numbers. The set of real quaternions is denoted by $H$. A quaternion $k$ is represented in the form:

$$k = k_0 e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3 = (k_0, k_1, k_2, k_3),$$

where $k_0, k_1, k_2, k_3$ are real numbers and $e_0, e_1, e_2, e_3$ are the fundamental quaternionic units such that

$$e_0^2 = 1, e_0 e_1 = e_1 e_0 = e_1, i = 1, 2, 3, e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1.$$ (1)

Quaternions find uses in pure and applied mathematics, quantum physics, the special theory of relativity and analysis, see for example [1], [8], [9], [15], [16]. In the literature, it can be found many researchers working on the structure of Fibonacci sequences and their generalizations see [2], [3], [6], [10], [17], [21], [25], [26], [27], [28]. Due to [27], the generalized Gaussian Fibonacci sequence $Gf_n(p, q; a, b)$ is defined by in the following way:

$$Gf_{n+1} = pGf_n + qGf_{n-1}, Gf_0 = a, Gf_1 = b$$ (2)

where $a$ and $b$ are initial values. If we take $p = 2, q = 1, a = i, b = 1$ in the equation (2) then we get the Gaussian Pell sequence

$$\{Gp_n\} = \{i, 1, 2 + i, 5 + 2i, \cdots\}.$$
In the equation (2) if we substitute \( p = 2, q = 1, a = 1 - i, b = 1 + i \) then we obtain the Gaussian Pell-Lucas sequence
\[
\{G_{n}\} = \{1 - i, 1 + i, 3 + i, 7 + 3i, \ldots \}.
\]
Also we have \( G_{p_{n}} = p_{n} + ip_{n-1} \) and \( G_{q_{n}} = q_{n} + iq_{n-1} \), where \( p_{n} \) and \( q_{n} \) are \( nth \) Pell and Pell-Lucas numbers, respectively.

In [11], Djordjevic and Srivastava introduced the generalized incomplete Fibonacci polynomials and the generalized incomplete Lucas polynomials in a systematic way and also they investigated their structures. In [12], Djordjevic and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers and derived their generating functions. In [13], Djordjevic and Srivastava established two different sequences of numbers, which are generalizations of the classical Fibonacci numbers, and obtained many important combinatorial properties of these general sequences of numbers. In [14], Srivastava, Tuglu and Cetin defined new families of the q-Fibonacci and q-Lucas polynomials providing the q-analogues of the incomplete Fibonacci and Lucas numbers, respectively. They proved some properties of these q-polynomial sequences such as recurrence relations, summation formulas and generating functions. In [22], Raina and Srivastava constructed a new class of numbers involving the familiar Lucas sequences and they deduced a number of results for this class of numbers such as hypergeometric representations, recurrence relations, generating functions and summation formulas.

Horadam [3] introduced the quaternions with the \( nth \) Fibonacci and Lucas numbers coefficients as follows:
\[
Q_{n} = f_{n}e_{0} + f_{n+1}e_{1} + f_{n+2}e_{2} + f_{n+3}e_{3},
\]
\[
K_{n} = l_{n}e_{0} + l_{n+1}e_{1} + l_{n+2}e_{2} + l_{n+3}e_{3},
\]
respectively, where \( f_{n} \) and \( l_{n} \) are the usual \( nth \) Fibonacci and Lucas numbers.

Quaternions whose coefficients consist of Fibonacci-like numbers have been studying by many researchers, recently, see [3], [5], [7], [19], [20], [23], [24]. In [23], Halici dealt with the Fibonacci quaternions and obtained some combinatorial properties. In [4], Horadam studied on the Pell and Pell-Lucas sequences and he presented some identities for them as follows:
\[
p_{n+1}p_{n-1} - p_{n}^{2} = (-1)^{n} \quad \text{(Cassini-like formula)}
\]
\[
p_{n}(p_{n+1} - p_{n-1}) = p_{2n}
\]
\[
p_{r}p_{n+1} - p_{r-1}p_{n} = p_{n+r}
\]
\[
p_{n}^{2} + p_{n+1}^{2} = p_{2n+1}
\]
\[
p_{2n+1}p_{2n} = 2p_{2n}^{2} - 2p_{2n}^{2} - (-1)^{n}
\]
\[
(-1)^{p}p_{d}p_{b} = p_{n+d}p_{n+b} - p_{n+d}p_{n+b}
\]
\[
p_{n} = (-1)^{n+1}p_{n}.
\]

It was firstly suggested by Horadam in [5] the idea to consider Pell quaternions. In [7], Cimen and Ipek introduced the Pell and Pell-Lucas quaternions, respectively, as follows:
\[
Q_{p_{n}} = p_{n}e_{0} + p_{n+1}e_{1} + p_{n+2}e_{2} + p_{n+3}e_{3}, \quad (3)
\]
\[
Q_{q_{n}} = q_{n}e_{0} + q_{n+1}e_{1} + q_{n+2}e_{2} + q_{n+3}e_{3}, \quad (4)
\]

They then investigated the structures of Pell and Pell-Lucas quaternions by the methods which depend more on the properties of Pell and Pell-Lucas numbers in [7].

In this paper, we introduce Gaussian Pell and Gaussian Pell-Lucas quaternions and derive their some combinatorial properties such as Binet formulas, Cassini identities, negatively subscripted terms and the generating functions.
2. GAUSSIAN PELL AND GAUSSIAN PELL-LUCAS QUATERNIONS

Any complex quaternion $\Psi$ is defined in the following form

$$\Psi = \Psi_0 + \Psi_1 e_1 + \Psi_2 e_2 + \Psi_3 e_3,$$

where each $\Psi_i$, $i = 0, 1, 2, 3$ is a complex number and $e_0, e_1, e_2, e_3$ are defined as in (1). The set of all complex quaternions is denoted by $\mathbb{H}_C$. We can rewrite the complex quaternion $\Psi$ as

$$\Psi = k + ik', i^2 = -1$$

where $k$ and $k'$ are real quaternions.

Halici [24] introduced the complex Fibonacci quaternions and gave some properties. In the similar way, we can define Gaussian Pell and Gaussian Pell-Lucas quaternions as follows:

$$QGp_n = Gp_n e_0 + Gp_{n+1} e_1 + Gp_{n+2} e_2 + Gp_{n+3} e_3$$

(5)

$$QGq_n = Gq_n e_0 + Gq_{n+1} e_1 + Gq_{n+2} e_2 + Gq_{n+3} e_3$$

(6)

where $Gp_n$ and $Gq_n$ stand for $n$th Gaussian Pell and Gaussian Pell-Lucas numbers. Since $Gp_n = p_n + ip_{n-1}$, we get $QGp_n = Qp_n + iQp_{n-1}$, where $Qp_n$ and $Qp_{n-1}$ are $n$th and $(n-1)$th Pell quaternions as in (3). Similarly, because of $Gq_n = q_n + iq_{n-1}$, we have $QGq_n = Qq_n + iQq_{n-1}$, where $Qq_n$ and $Qq_{n-1}$ are $n$th and $(n-1)$th Pell-Lucas quaternions as in (4).

Basic operations on Gaussian Pell quaternions such as addition, subtraction, multiplication are defined just as in real quaternions. The quaternion conjugate of $QGp_n$ is defined as

$$QGp'_n = Gp_n e_0 - Gp_{n+1} e_1 - Gp_{n+2} e_2 - Gp_{n+3} e_3.$$

The complex conjugate of $QGp_n$ is given by

$$\overline{QGp_n} = Gp_n e_0 + Gp_{n+1} e_1 + Gp_{n+2} e_2 + Gp_{n+3} e_3.$$

For any complex quaternion $\Psi = \Psi_0 e_0 + \Psi_1 e_1 + \Psi_2 e_2 + \Psi_3 e_3$, the quaternion norm of $\Psi$ is defined by $\|\Psi\| = \Psi_0^2 + \Psi_1^2 + \Psi_2^2 + \Psi_3^2$. Since each component of $\Psi$ is a complex number, then the norm of $\Psi$ is a complex number. Thus, we give the norms of the Gaussian Pell quaternion $QGp_n$ and the Gaussian Pell-Lucas quaternion $QGq_n$ in the following form:

$$N_{QGp_n} = \overline{QGp_n} QGp'_n = p_n^2 + p_{n+1}^2 + p_{n+2}^2 + p_{n+3}^2$$

and

$$N_{QGq_n} = \overline{QGq_n} QGq'_n = q_n^2 + q_{n+1}^2 + q_{n+2}^2 + q_{n+3}^2,$$

respectively.

**Proposition 2.1.** For the Gaussian Pell quaternion $QGp_n$ and the Gaussian Pell-Lucas quaternion $QGq_n$, we have the following identities:

$$N_{QGp_n} = 12(1 + i)p_{2n+2}.$$  

$$N_{QGq_n} = 24(1 + i)p_{2n+2}.$$  

**Proof.** From the definition of the norm of a Gaussian Pell quaternion, we can write

$$N_{QGp_n} = \overline{QGp_n} QGp'_n = p_n^2 + p_{n+1}^2 + p_{n+2}^2 + p_{n+3}^2$$

$$= (p_n + ip_{n-1})^2 + (p_{n+1} + ip_n)^2 + (p_{n+2} + ip_{n+1})^2 + (p_{n+3} + ip_{n+2})^2$$

$$= p_{n+3}^2 - p_{n-1}^2 + 2(p_n(p_{n-1} + p_{n+1}) + p_{n+2}(p_{n+1} + p_{n+3})).$$
Since \( p_{n+1} + p_{n+2} = 2q_n \) and \( 2p_nq_n = p_{2n} \), then we have \( N_{\text{GQp}_n} = p_{n+3}^2 - p_{n+1}^2 + 12p_{2n+2} \). Moreover, from Page 148 of Koshy [29], we get \( p_{n+2}^2 - p_{n+1}^2 = 12p_{2n+2} \). Therefore, we get \( N_{\text{GQp}_n} = 12(1 + i)p_{2n+2} \).

As for the norm of \( \text{QGp}_n \), since the well-known identities \( q_{n+1} + q_{n-1} = 4p_n, \) \( 2p_nq_n = p_{2n}, p_{n+2} + p_{n-2} = 6p_n \) and \( q_{n+3}^2 - q_{n-1}^2 = 24p_{2n+2} \), then we obtain

\[
N_{\text{QGp}_n} = q_n^2 + q_{n+1}^2 + q_{n+2}^2 + q_{n+3}^2
= q_{n+3}^2 - q_{n-1}^2 + 2(q_{n}(q_{n-1} + q_{n+1}) + q_{n+2}(q_{n+1} + q_{n+3}))
= 24(1 + i)p_{2n+2}.
\]

Hence the proof is completed. \( \square \)

The inverse of any complex quaternion \( \Psi \) is given by \( \Psi^{-1} = \frac{\Psi^*}{N_\Psi}, N_\Psi \neq 0 \), see [18]. The next corollary is clearly seen by definition of Gaussian Pell quaternion.

**Corollary 2.2.** For the \( \text{QGp}_n, \text{QGp}_n^* \) and \( \text{QQp}_n \), we have the following identities:

\[
\text{QGp}_n + \text{QGp}_n^* = 2\text{Gp}_n e_0.
\]

\[
\text{QGp}_n + \text{QQp}_n = 2\text{Qp}_n.
\]

\[
\text{QGp}_n^2 + \text{QGp}_n \text{QGp}_n^* = 2\text{QGp}_n \text{Gp}_n.
\]

In the following lemma, we give the second-order linear recurrence relations for Gaussian Pell and Gaussian Pell-Lucas quaternions.

**Lemma 2.3.** Let \( n \) be a positive integer. Then we have the following identities:

\[
\text{QGp}_n + 2\text{QGp}_{n+1} = \text{QGp}_{n+2}, \tag{7}
\]

\[
\text{QGq}_n + 2\text{QGq}_{n+1} = \text{QGq}_{n+2}, \tag{8}
\]

\[
\text{QGp}_n - \text{QGp}_{n+1} e_1 - \text{QGp}_{n+2} e_2 - \text{QGp}_{n+3} e_3 = 12\text{Qq}_{n+3}.
\]

**Proof.** Using the equation \( \text{QGp}_n = \text{Qp}_n + i\text{Qp}_{n-1} \) and the relation \( \text{Qp}_n = 2\text{Qp}_{n-1} + \text{Qp}_{n-2} \) given in Proposition 2 of [7], we conclude that

\[
\text{QGp}_n + 2\text{QGp}_{n+1} = \text{Qp}_n + i\text{Qp}_{n-1} + 2(\text{Qp}_{n+1} + i\text{Qp}_n)
= (\text{Qp}_n + 2\text{Qp}_{n+1}) + i(\text{Qp}_{n-1} + 2\text{Qp}_n)
= \text{Qp}_{n+2} + i\text{Qp}_{n+1} = \text{QGp}_{n+2}.
\]

The second-order recurrence relation for Gaussian Pell-Lucas quaternions is obtained in the similar way. To prove the last assertion, we need the relations \( p_{n+1} + p_{n-1} = 2q_n \) and \( q_{n+2} + q_{n-2} = 6q_n \) given in [29]. Thus we get

\[
\text{QGp}_n - \text{QGp}_{n+1} e_1 - \text{QGp}_{n+2} e_2 - \text{QGp}_{n+3} e_3 = \text{Gp}_n + \text{Gp}_{n+2} + \text{Gp}_{n+4} + \text{Gp}_{n+6}
= 2\text{Gq}_{n+1} + 2\text{Gq}_{n+5}
= 12\text{Gq}_{n+3}.
\]

We hence complete the proof. \( \square \)

The next corollary immediately follows from the definitions (5) and (6) and the identities \( q_{n+1} = p_{n+1} + p_n, \) \( q_n = p_{n+1} - p_n, p_{n+1} + p_{n-1} = 2q_n \) and \( 2p_n + q_n = q_{n+1} \) given in [29].
Corollary 2.4. Let $n$ be a positive integer. Then we have

$$
QGp_n + QGp_{n+1} = QGq_{n+1},
$$

$$
QGp_{n+1} - QGp_n = QGq_n,
$$

$$
QGp_{n+1} + QGp_{n-1} = 2QGq_n,
$$

$$
2QGp_n + QGq_n = QGq_{n+1},
$$

Taking into the relations (7) and (8) account, we deduce the explicit formulas for Gaussian Pell and Gaussian Pell-Lucas quaternions. From [29], Binet formulas for the Pell and Pell-Lucas numbers are

$$
p_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad q_n = \frac{\alpha^n + \beta^n}{2},
$$

respectively, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Here we note that $\alpha + \beta = 2$, $\alpha - \beta = 2 \sqrt{2}$ and $\alpha \beta = -1$.

Before we prove Binet formulas for Gaussian Pell and Gaussian Pell-Lucas quaternions we will give the following useful lemma.

Lemma 2.5. For $n \geq 1$ we have

$$
aQGp_n + QGp_{n-1} = \alpha^n A \quad \text{and} \quad q_n = \frac{\alpha^n + \beta^n}{2},
$$

where $A = \sum_{s=0}^{3} (\alpha^{s+1} e_s + \beta^{s+1} e_{-s})$. In a similar way to the equation (10), by considering the identity $\beta^n = \beta p_n + p_{n-1}$, we get

$$
\beta QGp_n + QGp_{n-1} = \beta^n B,
$$

where $B = \sum_{s=0}^{3} (\beta^{s+1} e_s + i \beta^{s+1} e_{-s})$. Therefore, we get

$$
aQGp_n + QGp_{n-1} = \alpha^n A, \quad \beta QGp_n + QGp_{n-1} = \beta^n B,
$$

where $A = \sum_{s=0}^{3} (\alpha^{s+1} e_s + i \alpha^{s+1} e_{-s})$. In a similar way to the equation (10), by considering the identity $\beta^n = \beta p_n + p_{n-1}$, we get

$$
bQGp_n + QGp_{n-1} = \beta^n B.
$$

Now we are in a position to give the Binet formula for Gaussian Pell and Gaussian Pell-Lucas quaternions.
Theorem 2.6 (Binet Formula). For any positive integer \( n \), the Binet formula for the Gaussian Pell quaternion \( Q_{Gp}^n \) is
\[
Q_{Gp}^n = \frac{\alpha^n A - \beta^n B}{\alpha - \beta},
\]
and for the Gaussian Pell-Lucas quaternion \( Q_{Gq}^n \) is
\[
Q_{Gq}^n = \frac{\alpha^n A + \beta^n B}{2},
\]
where \( A = \sum_{s=0}^{3} (\alpha^s + i\alpha^{s-1})e_s \) and \( B = \sum_{s=0}^{3} (\beta^s + i\beta^{s-1})e_s \).

Proof. By subtracting the equation (12) from the equation (11), we obtain
\[
Q_{Gp}^n = \frac{\alpha^n A - \beta^n B}{\alpha - \beta}.
\]
By adding the equation (11) to the equation (12), we have
\[
\alpha^n A + \beta^n B = (\alpha + \beta)Q_{Gp}^n + 2Q_{Gp}^{n-1}.
\]
Taking into account \( \alpha + \beta = 2 \) and the identity \( Q_{Gp}^n + Q_{Gp}^{n-1} = Q_{Gq}^n \) from the equation (9), we get
\[
Q_{Gq}^n = \frac{\alpha^n A + \beta^n B}{2}.
\]

\[\square\]

Theorem 2.7 (Cassini identities). For any positive integer \( n \), the following identities are hold:
\[
Q_{Gp}^{n+1}Q_{Gp}^{n-1} - Q_{Gp}^n = (-1)^n \frac{(\alpha^2 + 2)AB + \beta^2 BA}{(\alpha - \beta)^2},
\]
and
\[
Q_{Gq}^{n+1}Q_{Gq}^{n-1} - Q_{Gq}^n = (-1)^n \frac{(\alpha^2 + 2)AB + \beta^2 BA}{4},
\]
where \( A = \sum_{s=0}^{3} (\alpha^s + i\alpha^{s-1})e_s \) and \( B = \sum_{s=0}^{3} (\beta^s + i\beta^{s-1})e_s \).

Proof. The proof follows immediately from the Theorem 2.6. \[\square\]

The following corollary analogous to Theorem 8 in [7] is obtained by using Binet formulas for Gaussian Pell and Gaussian Pell-Lucas quaternions.

Corollary 2.8. For \( n \geq 0 \), the following equality hold:
\[
Q_{Gq}^n = 2Q_{Gq}^{n+1}Q_{Gq}^{n-1} - (-1)^n AB,
\]
where \( A = \sum_{s=0}^{3} (\alpha^s + i\alpha^{s-1})e_s \) and \( B = \sum_{s=0}^{3} (\beta^s + i\beta^{s-1})e_s \).

We will give the next lemma analogous to the identity \( p_{m+n} = p_mp_{n+1} + p_{m-1}p_n \) given in [4].

Lemma 2.9. For \( m, n \geq 0 \), we have
\[
P_{m+n} = p_mE_{m+1} + p_{m-1}E_n.
\]
Proof. By the definition of Gaussian Pell number and the identity \( p_{m+n} = p_mp_{n+1} + p_{m-1}p_n \), we get

\[
G_{p_{m+n}} = p_{m+n} + ip_{m+n-1}
\]

\[
= p_mp_{n+1} + p_{m-1}p_n + i(p_mp_n + p_{m-1}p_{n-1})
\]

\[
= p_n(p_{n+1} + ip_n) + p_{m-1}(p_n + ip_{n-1})
\]

\[
= p_nG_{p_{n+1}} + p_{m-1}G_{p_n}.
\]

\[
\]

As a result of Lemma 2.9, we can restate \( G_{p_n} \) as \( G_{p_n} = p_{n-1}G_{p_2} + p_{n-2}G_{p_1} \) by putting \( n-1 \) and \( 1 \) instead of \( m \) and \( n \), respectively. By using the identity \( p_{-n} = (-1)^{n+1}p_n \) and \( G_{p_n} = p_{n-1}G_{p_2} + p_{n-2}G_{p_1} \), we define the negatively subscripted terms for \( G_{p_n} \) as

\[
G_{p_{-n}} = (-1)^n(p_{n+1}G_{p_2} - p_{n+2}G_{p_1})
\]

Before obtaining the negatively subscripted terms of Gaussian Pell quaternions, we will give the following theorem.

**Theorem 2.10.** For \( m, n \geq 0 \), we have the following identity:

\[
QG_{p_{m+n}} = p_mQG_{p_{n+1}} + p_{m-1}QG_{p_n}.
\]

**Proof.** Using Lemma 2.9 and the definition of Gaussian Pell quaternions, then we get

\[
QG_{p_{m+n}} = G_{p_{m+n}}e_0 + G_{p_{m+n+1}}e_1 + G_{p_{m+n+2}}e_2 + G_{p_{m+n+3}}e_3
\]

\[
= (p_{m+n}G_{p_n} + p_{m-1}G_{p_{n+2}})e_0 + (p_nG_{p_{n+2}} + p_{m-1}G_{p_{n+1}})e_1 +
\]

\[
(p_mG_{p_{n+3}} + p_{m-1}G_{p_{n+2}})e_2 + (p_mG_{p_{n+4}} + p_{m-1}G_{p_{n+3}})e_3
\]

\[
= p_m(G_{p_{n+1}}e_0 + G_{p_{n+2}}e_1 + G_{p_{n+3}}e_2 + G_{p_{n+4}}e_3) +
\]

\[
p_{m-1}(G_{p_n}e_0 + G_{p_{n+1}}e_1 + G_{p_{n+2}}e_2 + G_{p_{n+3}}e_3)
\]

\[
= p_mQG_{p_{n+1}} + p_{m-1}QG_{p_n}.
\]

\[
\]

If we take \( m \rightarrow n-1 \) and \( n \rightarrow 1 \), then we deduce alternative formula for \( QG_{p_n} \) as follows:

\[
QG_{p_n} = p_{n-1}QG_{p_2} + p_{n-2}QG_{p_1}.
\]

(15)

In the same way, we write another formula for \( QG_{q_n} \) as

\[
QG_{q_n} = p_{n-1}QG_{q_2} + p_{n-2}QG_{q_1}.
\]

(16)

Therefore, from equation (15) and the identity \( p_{-n} = (-1)^{n+1}p_n \) we can define Gaussian Pell quaternion with negative indices in the following way:

\[
QG_{p_{-n}} = (-1)^n(p_{n+1}QG_{p_2} - p_{n+2}QG_{p_1}).
\]

Similarly, by means of the equation (16) one can obtain Gaussian Pell-Lucas quaternion with negative indices

\[
QG_{q_{-n}} = (-1)^n(p_{n+1}QG_{q_2} - p_{n+2}QG_{q_1}).
\]

**Theorem 2.11.** For \( n \geq 0 \), we have the following summation formulas:

\[
\sum_{i=0}^{n} \binom{n}{i} 2^i QG_{p_i} = QG_{2n}.
\]

\[
\sum_{i=0}^{n} (-1)^i QG_{p_i} = \left( \frac{\alpha - \beta}{2} \right)^{n+3}[B - (-1)^nA].
\]
Proof. From the Binet Formula of $QGp_n$, we get
\[
\sum_{i=0}^{n} \binom{n}{i} 2^i QGp_i = \sum_{i=0}^{n} \binom{n}{i} 2^i \left( \frac{\alpha^i A - \beta^i B}{\alpha - \beta} \right)
\]
\[
= \frac{A}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (2\alpha)^i - \frac{B}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (2\beta)^i
\]
\[
= \frac{A}{\alpha - \beta} [(1 + 2\alpha)^n] - \frac{B}{\alpha - \beta} [(1 + 2\beta)^n]
\]
\[
= \frac{A\alpha^{2n} - B\beta^{2n}}{\alpha - \beta} = QGp_{2n},
\]
and
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i QGp_i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left( \frac{\alpha^i A - \beta^i B}{\alpha - \beta} \right)
\]
\[
= \frac{A}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-\alpha)^i - \frac{B}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-\beta)^i
\]
\[
= \frac{A}{\alpha - \beta} [(1 - \alpha)^n] - \frac{B}{\alpha - \beta} [(1 - \beta)^n]
\]
\[
= \frac{A\alpha^{2n} - B\beta^{2n}}{\alpha - \beta} = QGp_{2n},
\]
where $A = QGp_1 - \beta QGp_0$ and $B = QGp_1 - \alpha QGp_0$ from the equations (13) and (14). \qed

The proof of the next corollary is easily seen by using the equations (7) and (9), the relation $QGp_n = Qp_n + iQp_{n-1}$ and Theorem 6 in [7] and Theorem 5 in [25].

Corollary 2.12. For the Gaussian Pell quaternion $QGp_n$, the following identities hold:
\[
\sum_{i=1}^{n} QGp_i = \frac{1}{2} [QGp_{n+1} - QGp_1].
\]
\[
\sum_{i=1}^{n} QGp_{2i} = \frac{1}{2} [QGp_{2n+1} - QGp_1].
\]
\[
\sum_{i=1}^{n} QGp_{2i-1} = \frac{1}{2} [QGp_{2n} - QGp_1].
\]

Since Gaussian Pell and Gaussian Pell-Lucas quaternions also satisfy second-order linear recurrence relation, then we can derive the generating functions for these quaternions. Thus we can give the following theorem.

Theorem 2.13. The generating function for the nth Gaussian Pell quaternion $QGp_n$ is
\[
G(x, t) = \frac{(1 - 2t)QGp_0 + QGp_1 t}{1 - 2t - t^2},
\]
and the generating function for the nth Gaussian Pell-Lucas quaternion $QGq_n$ is
\[
H(x, t) = \frac{(1 - 2t)QGq_0 + QGq_1 t}{1 - 2t - t^2}.
\]
Proof. Let $G(x,t) = \sum_{n=0}^{\infty} QGp_n(x) t^n = QGp_0 + QGp_1 t + QGp_2 t^2 + \cdots + QGp_n t^n + \cdots$ be the generating function for the $n$th Gaussian Pell quaternion $QGp_n$. Then we derive

$$tG(x,t) = QGp_0 t + QGp_1 t^2 + QGp_2 t^3 + \cdots + QGp_n t^{n+1} + \cdots$$

and

$$t^2 G(x,t) = QGp_0 t^2 + QGp_1 t^3 + QGp_2 t^4 + QGp_3 t^5 + \cdots + QGp_n t^{n+2} + \cdots.$$ 

After simple computations, we get $G(x,t) = \frac{(1-2t)QGp_0 + QGp_1}{1 - 2t - t^2}$ due to the fact that $QGp_n = 2QGp_{n-1} + QGp_{n-2}$. In an as similar manner, we obtain the generating function of Gaussian Pell-Lucas quaternions. $\square$

References