



Proximity Inductive Dimension and Brouwer Dimension Agree on Compact Hausdorff Spaces

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Abstract. We show that the proximity inductive dimension defined by Isbell agrees with the Brouwer dimension originally described by Brouwer (for Polish spaces without isolated points) on the class of compact Hausdorff spaces. This shows that Fedorchuk's example of a compact Hausdorff space whose Brouwer dimension exceeds its Lebesgue covering dimension is an example of a space whose proximity inductive dimension exceeds its proximity dimension as defined by Smirnov. This answers Isbell's question of whether or not proximity inductive dimension and proximity dimension coincide.

1. Introduction

Proximity spaces in their modern form were described during the early 1950's by Efremovič, [1, 2]. Variations of the original definition can be found in [9]. The proximity structure is meant to capture the notion of what it means for two subsets of a space to be "close." Proximity spaces play a sort of middle role between uniform spaces and topological spaces. Specifically, every uniform structure on a set induces a proximity structure on the same set. Likewise, every proximity structure on a set induces a completely regular topology on that set. In the case where a proximity structure is induced by a uniform structure, the topology induced by the proximity structure coincides with the topology induced by the uniform structure. Neither the correspondence between uniform structures and proximity structures, nor the correspondence between proximity structures and topological structures are injective. More precisely, there are distinct proximity structures that induce the same topology [9, Remark 2.18], and there are distinct uniform structures that induce the same proximity structure [9, Chapter 3, Section 12].

The dimension theory of proximity spaces began when Smirnov defined the proximity dimension δd of proximity spaces using δ -coverings, [10]. The proximity dimension is an invariant in the category of proximity spaces and serves as an analog of the covering dimension dim . In the case of compact Hausdorff spaces, whose topology is induced by a unique proximity, the dimensions δd and dim coincide. An inductive dimension for proximity spaces was defined by Isbell in [5]. This definition used the notion of a "freeing set." Isbell posed the problem of whether or not δInd and δd coincide in [6] and [7].

The purpose of this paper is to show that the Brouwer dimension Dg and δInd coincide on the class of compact Hausdorff spaces. We begin in Sections 2 and 3 by reviewing definitions and concepts related to proximity spaces, dimensions of proximity spaces, and the dimension function Dg . In Section 4 we prove

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our main result, the equality of δInd and Dg on compact Hausdorff spaces, by showing the equivalence of the notions of cut and freeing set in that class. From this equivalence, the coincidence of the two dimension function follows quickly. As a corollary, we solve Isbell’s problem regarding the coincidence of δInd and δd by showing that Fedorchuk’s space B constructed in [4] satisfies $\delta Ind(B) = 3$ and $\delta d(B) = 2$. Throughout this paper we use the notation \bar{A} and $int(A)$ for the closure and interior of a subset A within a topological space X .

2. Proximity Spaces and Their Dimensions

In this section, we will review the necessary definitions and results related to proximity spaces. The citations are not necessarily where the corresponding definitions or results first appeared, but where they can be easily found. The initial results in this section about proximity spaces can be found in [9].

Definition 2.1. Let X be a set and δ a binary relation on 2^X . The relation δ is said to be a **proximity relation**, or simply a **proximity** on X , if the following axioms are satisfied for all $A, B, C \subseteq X$:

1. $A\delta B$ if and only if $B\delta A$.
2. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.
3. $A\delta B$ implies that $A \neq \emptyset$ and $B \neq \emptyset$.
4. $A \cap B \neq \emptyset$ implies that $A\delta B$.
5. $A\bar{\delta}B$ implies that there is an $E \subseteq X$ such that $A\delta E$ and $(X \setminus E)\bar{\delta}B$.

We use $A\bar{\delta}B$ to denote the negation of $A\delta B$. A pair (X, δ) where X is a set and δ is a proximity on X is called a **proximity space**. If for a proximity space (X, δ) the relation δ satisfies the additional axiom that for all $x, y \in X$, $\{x\}\delta\{y\}$ if and only if $x = y$, we call the proximity δ and the proximity space (X, δ) **separated**.

As mentioned in the introduction, every proximity space has a natural topological structure. This topology is defined in the following way:

Proposition 2.2. Let (X, δ) be a proximity space. Then the function $cl : 2^X \rightarrow 2^X$ defined by

$$cl(A) = \{x \in X \mid \{x\}\delta A\}$$

is a closure operator on X . Moreover, the corresponding topology is Hausdorff if and only if (X, δ) is separated.

We will call the topology on a proximity space (X, δ) described above the **topology induced by the proximity** δ . A set $U \subseteq X$ is open in the induced topology if and only if for all $x \in U$ one has that $\{x\}\bar{\delta}(X \setminus U)$. This topology is always completely regular, [9]. From this point forward all proximity spaces are assumed to be separated.

Proposition 2.3. Let (X, δ) be a proximity space and let $Y \subseteq X$ be a nonempty set. The relation δ_Y on all subsets $A, B \subseteq Y$ defined by

$$A\delta_Y B \iff A\delta B$$

is a proximity on Y whose induced topology coincides with the topology Y inherits as a subspace of X .

The proximity relation above is called the **subspace proximity**. In later sections we won’t denote subspaces proximities differently from the proximities they are defined from.

Proposition 2.4. Let (X, δ) be a proximity space. Then for all $A, B \subseteq X$

$$A\delta B \iff \bar{A}\bar{\delta}\bar{B}$$

Definition 2.5. Let X be a topological space. A proximity relation δ on X is said to be **compatible** with the topology on X if the topology induced by δ is the original topology on X .

Proposition 2.6. If X is a compact Hausdorff space, then there is a unique proximity on X that is compatible with the topology on X . It is defined by:

$$A\delta B \iff \overline{A} \cap \overline{B} \neq \emptyset$$

Definition 2.7. If A and B are subsets of a proximity space (X, δ) , then we say that B is a δ -**neighbourhood** of A if $A\delta(X \setminus B)$. We denote this relationship by writing $A \ll B$.

A useful fact about δ -neighbourhoods in proximity spaces is the following, adapted from a similar result in [9, Lemma 3.2]:

Lemma 2.8. Let (X, δ) be a proximity space. Then:

1. $A \ll B \iff \overline{A} \ll B$
2. $A \ll B \iff A \ll \text{int}(B)$

With these initial basic definitions and results in hand, we proceed to the definition of the proximity dimension and the theorem relating the proximity dimension and the covering dimension of a compact Hausdorff space. Even though we only utilize Theorem 2.11 in the later part of the paper, we include the definition of the proximity dimension for completeness. The definition and theorem can be found in [10].

Definition 2.9. Let (X, δ) be a proximity space. A finite cover A_1, \dots, A_n of X is called a δ -cover if there is another finite cover B_1, \dots, B_n of X such that $B_i \ll A_i$ for $i = 1, \dots, n$.

Definition 2.10. Let (X, δ) be a proximity space. The **proximity dimension** of X , denoted $\delta d(X)$, is defined in the following way:

1. $\delta d(X) = -1$ if and only if $X = \emptyset$.
2. For $n \geq 0$, $\delta d(X) \leq n$ if and only if every δ -cover \mathcal{U} can be refined by a δ -cover of order at most $n + 1$.
3. $\delta d(X)$ is the least integer n such that $\delta d(X) \leq n$. If there is no such integer, then $\delta d(X) = \infty$.

Theorem 2.11. If X is a compact Hausdorff space, then $\delta d(X) = \dim(X)$.

Note that there is no ambiguity in Theorem 2.11 as Proposition 2.6 grants that there is only one possible interpretation of δd on compact Hausdorff spaces.

Next we proceed to Isbell’s proximity inductive dimension. These definitions and results can be found in [7].

Definition 2.12. Let (X, δ) be a proximity space and let $A, B \subseteq X$ be such that $A\delta\overline{B}$. A subset $D \subseteq X$ is said to δ -**separate** A and B , or is a δ -**separator** between A and B , if $X \setminus D = U \cup V$ where $A \subseteq U$, $B \subseteq V$, and $U\delta\overline{V}$. A subset $H \subseteq X$ is said to **free** A and B , or be a **freeing set** for A and B , if $H\delta\overline{(A \cup B)}$ and every δ -neighbourhood of H that is disjoint from $A \cup B$ is a δ -separator between A and B .

Definition 2.13. Let (X, δ) be a proximity space. The **proximity inductive dimension** of X , denoted $\delta \text{Ind}(X)$, is defined inductively:

1. $\delta \text{Ind}(X) = -1$ if and only if $X = \emptyset$.
2. For $n \geq 0$, $\delta \text{Ind}(X) \leq n$ if and only if for every pair of subsets $A, B \subseteq X$ such that $A\delta\overline{B}$ there is a set $H \subseteq X$ that frees A and B and is such that $\delta \text{Ind}(H) \leq n - 1$.
3. $\delta \text{Ind}(X)$ is the least integer n such that $\delta \text{Ind}(X) \leq n$. If there is no such n , then $\delta \text{Ind}(X) = \infty$.

Proposition 2.14. If (Y, δ) is a proximity space and $X \subseteq Y$ is a dense subspace, then $\delta \text{Ind}(X) \geq \delta \text{Ind}(Y)$.

We note that Proposition 2.14 implies that in Definition 2.13 we could take A, B , and H to be closed without altering the value of δInd .

Proposition 2.15. *For every proximity space (X, δ) , $\delta Ind(X) \geq \delta d(X)$.*

Definition 2.16. Let X be a topological space and $A, B \subseteq X$ disjoint closed subsets. A closed set $C \subseteq X$ is called a **separator** in X between A and B if there are disjoint open sets $U, V \subseteq X$ such that $X \setminus C = U \cup V$, with $A \subseteq U$ and $B \subseteq V$.

The following result is a specific case of a more general result found in [5, Lemma 2].

Proposition 2.17. *Let X be a compact Hausdorff space and $A, B \subseteq X$ disjoint closed subsets. If $C \subseteq X$ is a separator in X between A and B , then C frees A and B .*

The converse to the above result is not true.

Example 2.18. Let X be the “Topologist’s Sine Curve.” That is X is the closure of the set

$$S = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x \in (0, 1]\}$$

in \mathbb{R}^2 equipped with the subspace topology. If we define $A = \{(0, -1)\}$, $B = \{(1, \sin(1))\}$, and $C = \{(0, 1)\}$ then C frees A and B , but is not a separator between them. To see that C is not a separator note that if $X \setminus C = U \cup V$ where U and V are disjoint open subsets of X containing A and B , respectively, then by the connectivity of S we must have that $S \subseteq V$. However, $X \setminus (S \cup C)$ intersects the closure of S , so we must also have that $X \setminus (S \cup C)$ is also in V , creating a contradiction. To see that C is a freeing set for A and B we observe that any open set D containing C and not containing A or B necessarily contains a metric ball centered at a point of S . Thus $S \setminus D$ is not connected. Then it is not hard to see that $X \setminus D = U \cup V$ where U and V are disjoint open sets in X with the component of $S \setminus D$ containing B being contained in U and the complement of this component in $X \setminus D$ being contained in V . Moreover these open sets can be chosen so that $\overline{U} \cap \overline{V} = \emptyset$ in X . This means that $U \delta V$ where δ is the unique proximity the induces the topology on X as described in Proposition 2.6. By Lemma 2.8, any δ -neighbourhood of C that is not close to $A \cup B$ contains an open neighbourhood of C that is disjoint from A and B . Therefore, C is a freeing set for A and B .

3. Brouwer Dimension

In this section, we will review the basic definitions and results related to the Brouwer dimension. For a brief history of the invariant, see [3] or [4].

Definition 3.1. A **continuum** is a nonempty compact connected Hausdorff space.

In some places in the literature, such as [8], the word “compactum” is used for nonempty compact connected Hausdorff spaces.

Definition 3.2. Let X be a topological space and let $A, B \subseteq X$ be disjoint closed subsets. A closed subset $C \subseteq X$ that is disjoint from $A \cup B$ is called a **cut** between A and B if every continuum $K \subseteq X$ such that $K \cap A \neq \emptyset$ and $K \cap B \neq \emptyset$ also satisfies $K \cap C \neq \emptyset$.

It is an easy exercise to show that every separator in a topological space is also a cut. In Example 2.18 the set C was shown to not be a separator. It is however, a cut. To see this, note that a continuum in X that contains B must be of the form \overline{S}_t where

$$S_t = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x \in (t, 1]\}$$

for some $t \in [0, 1]$. The only such set that also contains A is \overline{S}_0 which is equal to X itself. In other words, the only continuum in X that contains both A and B is X itself, which contains C as well. Therefore, C is a cut between A and B .

Definition 3.3. Let X be a normal Hausdorff topological space. The **Brouwer dimension** of X , denoted $Dg(X)$, is defined inductively:

1. $Dg(X) = -1$ if and only if $X = \emptyset$.
2. For $n \geq 0$, $Dg(X) \leq n$ if and only if for every pair of disjoint closed sets $A, B \subseteq X$ there is a cut $C \subseteq X$ between A and B such that $Dg(C) \leq n - 1$.
3. $Dg(X)$ is the least integer n such that $Dg(X) \leq n$. If there is no such integer, then $Dg(X) = \infty$.

The next two lemmas will be used in the next section.

Lemma 3.4. Let X be a compact Hausdorff space. Let $A, B \subseteq X$ be disjoint closed sets and let $C \subseteq X$ be a cut between A and B . If $D \subseteq X$ is an open (or closed) neighbourhood of C that is disjoint from A and B , then there is no connected $K \subseteq X \setminus D$ such that $K \cap A \neq \emptyset$ and $K \cap B \neq \emptyset$.

Proof. Let A, B , and C be as described in the statement of the lemma. Let $D \subseteq X$ be an open neighbourhood of C that is disjoint from A and B . If $K \subseteq X$ is a connected set that intersects A and B nontrivially, then \bar{K} is a continuum in X that intersects A and B nontrivially, but is disjoint from C . This is a contradiction. The case where D is a closed neighbourhood of C is similar. \square

The following result appears in [8, Corollary 6-8].

Lemma 3.5. Let X be a compact Hausdorff space. Let A and B disjoint closed subsets of X . If there is no connected set K such that $K \cap A \neq \emptyset$ and $K \cap B \neq \emptyset$, then the empty set is a separator between A and B .

With these in mind we may interpret Lemma 3.5 as saying that if the empty set is a cut between disjoint closed subsets of a compact Hausdorff space, then it is also a separator between them.

4. Main Result

In this final section, we will prove that the proximity inductive dimension and the Brouwer dimension coincide on compact Hausdorff spaces. To do this, we first characterize cuts within compact Hausdorff spaces.

Proposition 4.1. Let X be a compact Hausdorff space and $A, B \subseteq X$ nonempty disjoint closed subsets. Given a closed subset $C \subseteq X$ that is disjoint from $A \cup B$, the following are equivalent:

1. C is a cut in X between A and B .
2. Every closed neighbourhood of C that is disjoint from $A \cup B$ is a separator between A and B .

Proof. Let X, A, B , and C be given as in the statement of the proposition. The case where either A or B is empty is trivial, so assume that A and B are both nonempty.

(2) \implies (1) Assume that every closed neighbourhood D of C that is disjoint from $A \cup B$ is a separator between A and B . If $C = \emptyset$ then C is a closed neighbourhood of itself that is disjoint from A and B . This implies that the empty set is a separator between A and B . As every separator is a cut this implies that C is a cut between A and B . Assume then, that $C \neq \emptyset$ and, assume further towards a contradiction that C is not a cut between A and B . Then, there is a continuum $K \subseteq X$ such that $K \cap A \neq \emptyset$, $K \cap B \neq \emptyset$, but $K \cap C = \emptyset$. As $K \cup A \cup B$ and C are disjoint closed sets, the normality of X implies that there is a closed neighbourhood D of C that is disjoint from $K \cup A \cup B$. Then $K \subseteq X \setminus D = U \cup V$ where U and V are disjoint open sets of X containing A and B respectively. This would imply that $K \cap U$ and $K \cap V$ are open subsets of K that witness K being disconnected, contradicting the connectedness of K . Therefore, C is a cut between A and B .

(1) \implies (2) Assume that C is a cut between A and B . If $C = \emptyset$ then by Lemma 3.5 we have that C is also a separator between A and B . Then so is every set containing C that is disjoint from A and B . This includes every closed neighbourhood of C that is disjoint from A and B . Otherwise assume $C \neq \emptyset$ and let D be a closed neighbourhood of C that is disjoint from A and B . Then $C \subseteq \text{int}(D)$. Because C is a cut between A

and B , we have by Lemma 3.4 that there is no connected set K in the compact Hausdorff space $X \setminus \text{int}(D)$ that intersects both A and B nontrivially. Therefore, by Lemma 3.5 the empty set is a separator in $X \setminus \text{int}(D)$ between A and B . Let U and V be disjoint open subsets of $X \setminus \text{int}(D)$ that contain A and B , respectively. Then $U' = U \cap (X \setminus D)$ and $V' = V \cap (X \setminus D)$ are disjoint open subsets of $X \setminus D$ and consequently disjoint open subsets of X such that $X \setminus D = U' \cup V'$, $A \subseteq U'$, and $B \subseteq V'$. Therefore, D is a separator in X between A and B . \square

Proposition 4.2. *Let X be a compact Hausdorff space, and let $A, B \subseteq X$ disjoint closed sets. Given a closed subset $C \subseteq X$ that is disjoint from A and B , the following are equivalent:*

1. C is a cut between A and B .
2. C frees A and B .

Proof. Let X, A, B , and C be as given in the statement of the proposition. If either of A or B is empty then every closed subset of X disjoint from A and B trivially satisfies both properties in the statement of the proposition. We will then assume that neither A nor B is empty.

(2) \implies (1) Assume that C frees A and B . The proof of the case where $C = \emptyset$ is the same as the proof of the case where $C \neq \emptyset$, so we will not treat them differently as we did in Proposition 4.1. Let D be a closed neighbourhood of C that is disjoint from A and B . Note that by Lemma 2.8 every closed neighbourhood of C is a δ -neighbourhood of C . Then, by the definition of a freeing set, we have that $X \setminus D = U \cup V$, where $A \subseteq U$, $B \subseteq V$, and $U \delta V$. Then $U \delta V$ in the subspace $X \setminus D$. As $U \delta V$, we have that $\{x\} \delta V$ and $\{y\} \delta U$ for all $x \in U$ and $y \in V$. Likewise, U and V must be disjoint. The remarks following Proposition 2.2 then imply that U and V are open in $X \setminus D$, and are therefore open in X . This means that D is a separator between A and B . As D was an arbitrary close neighbourhood of C , we have that C is a cut between A and B by Proposition 4.1.

(1) \implies (2) Assume that C is a cut between A and B . If $C = \emptyset$, then Lemma 3.5 tells us that C is also a separator between A and B in X . Then $X = U \cup V$ where U and V are disjoint open subset of X containing A and B , respectively. Then we have that U and V are closed in X as well, so $U \delta V$. Thus C is a δ -separator between A and B in X . Consequently, so is every δ -neighbourhood of C that is disjoint from A and B . Thus C frees A and B .

Now assume that $C \neq \emptyset$, and let $D \subseteq X$ be a δ -neighbourhood of C that is disjoint from $A \cup B$. We may assume that D is an open neighbourhood of C because if $C \ll D$, then $C \subseteq \text{int}(D)$ by Lemma 2.8, and if a subset of D is a δ -separator, then D is as well. Then let D' be a closed neighbourhood of C such that $C \subseteq D' \subseteq D$ and $D' \cap (X \setminus D) = \emptyset$. Then D' is a closed neighbourhood of C that is disjoint from A and B , so by Proposition 4.1 D' is a separator in X between A and B . Then let U and V be disjoint open subsets of X so that $X \setminus D' = U \cup V$, $A \subseteq U$, and $B \subseteq V$. We then consider

$$U' = U \cap (X \setminus D) \text{ and } V' = V \cap (X \setminus D)$$

and claim that $U' \delta V'$. To see this, we note that because U' and V' are subsets of the closed set $X \setminus D$, we must have that $\overline{U'}$ and $\overline{V'}$ are also subsets of $X \setminus D$. Then $\overline{U'} \cap \overline{V'} \subseteq X \setminus D$. However, as $U' \subseteq U$ and $V' \subseteq V$, we must have that $\overline{U'} \subseteq \overline{U}$ and $\overline{V'} \subseteq \overline{V}$. Therefore, we have

$$\overline{U'} \cap \overline{V'} \subseteq X \setminus D \text{ and } \overline{U'} \cap \overline{V'} \subseteq \overline{U} \cap \overline{V}.$$

However, $\overline{U} \cap \overline{V}$ is a subset of D' and $D' \cap (X \setminus D) = \emptyset$. Therefore, we have that $\overline{U'} \cap \overline{V'} = \emptyset$, which gives us that $U' \delta V'$. Summarizing, $X \setminus D$ is the union of the disjoint sets U' and V' that contain A and B respectively, and are not close. That is, D is a δ -separator between A and B , then implies that C frees A and B . \square

Theorem 4.3. *For every compact Hausdorff space X , $Dg(X) = \delta \text{Ind}(X)$.*

Proof. We will show that $\delta\text{Ind}(X) \geq Dg(X)$ by induction on $\delta\text{Ind}(X)$. The result is obvious when $\delta\text{Ind}(X) = -1$. Assume then that the result holds for $\delta\text{Ind}(X) < n$ for $n \geq 0$, and assume that $\delta\text{Ind}(X) = n$. If A and B are disjoint closed subsets of X , then there must be a closed set $C \subseteq X$ that frees A and B and satisfies $\delta\text{Ind}(C) \leq n - 1$. By Proposition 4.2, C is a cut between A and B , and the inductive hypothesis gives us that $Dg(C) \leq \delta\text{Ind}(C) \leq n - 1$. Therefore $Dg(X) \leq n$. Clearly, if $\delta\text{Ind}(X) = \infty$, then $Dg(X) \leq \delta\text{Ind}(X)$. Therefore $\delta\text{Ind}(X) \geq Dg(X)$. The argument showing that $Dg(X) \geq \delta\text{Ind}(X)$ is a similar induction argument that follows from Proposition 4.2. \square

In [4], a compact Hausdorff space B was constructed with the property that $Dg(X) = 3$ and $\dim(X) = 2$. Then Theorem 4.3 and Theorem 2.11 imply the following corollary.

Corollary 4.4. *There is a compact Hausdorff space B such that $\delta\text{Ind}(B) = 3$ and $\delta d(B) = 2$.*

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