On Minimal Faithful Representations of a Class of Nilpotent Lie Algebras

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Abstract. In this work we consider 2-step nilradicals of parabolic subalgebras of the simple Lie algebra $A_n$ and describe a new family of faithful nil-representations of the nilradicals $n_{a,c}$, $a, c \in \mathbb{N}$. We obtain a sharp upper bound for the minimal dimension $\mu(n_{a,c})$ and for several pairs $(a, c)$ we obtain $\mu(n_{a,c})$.

1. Introduction

All the vector spaces and linear transformations considered in this paper are assumed to be finite dimensional over the complex numbers $\mathbb{C}$. Let $\mathfrak{n}$ be a complex finite dimensional Lie algebra and let $\mu(\mathfrak{n})$ and $\mu_{nil}(\mathfrak{n})$ denote the minimal dimension of a faithful representation and nil-representation, respectively, of $\mathfrak{n}$. By Ado-Iwasawa’s Theorem, see for instance [12, page 202], these numbers are well defined. Clearly, these are invariants of $\mathfrak{n}$ and $\mu(\mathfrak{n}) \leq \mu_{nil}(\mathfrak{n})$.

In the theory of Lie algebras, it is important to study $\mu(\mathfrak{n})$ and to find good upper bounds for it. On the one hand, this is motivated by a Milnor’s conjecture (see [15]), which asserts that any solvable Lie algebra $\mathfrak{n}$ satisfies $\mu(\mathfrak{n}) \leq \dim \mathfrak{n} + 1$. However, the answer to Milnor’s conjecture is negative (see, for instance, [3] and [6] for counterexamples). On the other hand, for computational mathematics, it is interesting to construct faithful representations of a given Lie algebra $\mathfrak{n}$ of the smallest possible dimension; but, in general, this is not easy. In [5], [8] and [11], the authors obtain several methods for such constructions for a given nilpotent Lie algebra $\mathfrak{n}$; but the upper or lower bounds achieved for $\mu$ and $\mu_{nil}$ are far from being sharp.

In [16], Reed established that $\mu(\mathfrak{n}) \leq 1 + (\dim \mathfrak{n})^k$ for any $k$-step nilpotent Lie algebra $\mathfrak{n}$. After that, Burde established in [4], that $\mu(\mathfrak{n}) < \frac{3}{\sqrt{\dim \mathfrak{n}}}^{\frac{1}{2}\dim \mathfrak{n}}$.

A major goal in this area is to prove or disprove that there exists a polynomial $p \in \mathbb{C}[t]$ such that $\mu(\mathfrak{n}) \leq p(\dim \mathfrak{n})$, for all Lie algebras $\mathfrak{n}$ (or at least for a wide class). In fact, the value of $\mu$ or $\mu_{nil}$ has been obtained only for few families (see [7], [9], [14], [17], [18], [19]).

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For \( a, b, c \in \mathbb{N} \), the 2-step nilradical of a parabolic subalgebra of the simple Lie algebra \( A_n \) of dimension \( ab + bc + ac \), \( n_{a,b,c} \), is the Lie algebra with basis
\[
\mathfrak{B} = \{ X_{i,j}, Y_{j,k}, Z_{i,k} : i = 1, \ldots, a; j = 1, \ldots, b; k = 1, \ldots, c \}
\]
and non-zero brackets
\[
[X_{i,j}, Y_{j,k}] = Z_{i,k}
\]
for \( i = 1, \ldots, a; j = 1, \ldots, b \) and \( k = 1, \ldots, c \). This Lie algebra has an standard faithful nilrepresentation \( (\pi_{S}, \mathbb{C}^{a+b+c}) \) that, in terms of the canonical basis of \( \mathbb{C}^{a+b+c} \), is given by
\[
\pi_{S} \left( \sum_{i=1}^{a} \sum_{j=1}^{b} x_{ij}X_{i,j} + \sum_{j=1}^{b} \sum_{k=1}^{c} y_{jk}Y_{j,k} + \sum_{i=1}^{a} \sum_{k=1}^{c} z_{ik}Z_{i,k} \right) = \begin{bmatrix}
0 & x_{11} & \cdots & x_{1b} & z_{11} & \cdots & z_{1c} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_{a1} & x_{ab} & \cdots & z_{a1} & \cdots & \cdots & z_{ac}
\end{bmatrix}
\]

Thus
\[
\mu(n_{a,b,c}) \leq \dim \pi_{S} = a + b + c \quad \text{for all } a, b, c \in \mathbb{N}.
\]

We point out that the nilradical \( n_{1,1,c} \) is the Heisenberg Lie algebra of dimension \( 2b + 1 \). In [4] it is proved that
\[
\mu(n_{1,1,c}) = b + 2 = \dim \pi_{S}.
\]

In [10], 2-step nilradicals of type \( A \) are considered and, in particular, it is proved that if either \( c = b - a \) or \( a = c \) and \( b \leq 2a \), then
\[
\mu(n_{a,b,c}) = a + b + c = \dim \pi_{S}.
\]

These previous examples show that \( \mu(n_{a,b,c}) \) coincides with the dimension of the standard representation \( \pi_{S} \), for certain classes of nilradicals. A first question is whether this is a general phenomenon.

To answer negatively this question, consider the Lie algebra \( n_{1,1,c} \). This is a well-known nilradical of type \( A \), which arise as an extension of \( \mathbb{C} \) by \( \mathbb{C}^{2c} \). The Betti numbers of this Lie algebra have been obtained in [2] and later, its adjoint cohomology has been computed in [1].

Consider the following example:
Example 1.1. For \( n_{1,1,6} \), one has the following faithful nil-representation of dimension 7:

\[
x_{11}X_{11} + \sum_{k=1}^{6} y_{1k} Y_{1k} + \sum_{k=1}^{6} z_{1k} Z_{1k} \rightarrow \begin{pmatrix}
0 & 0 & 0 & z_{11} & z_{12} & z_{13} \\
0 & 0 & 0 & x_{1} & z_{14} & z_{15} \\
0 & 0 & 0 & y_{11} & y_{12} & y_{13} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

therefore

\[
\mu(n_{1,1,6}) < 8 = \dim \pi_S.
\]

We shall prove in this work that,

\[
\mu(n_{1,1,c}) = \left\lceil 2 \sqrt{2c} \right\rceil < 2 + c \quad \text{for} \quad c \geq 6.
\]

An interesting generalization of the Lie algebra \( n_{1,1,c} \) is the class of Lie algebras given by the family \( n_{a,1,c} \) for \( a \leq c \). In particular, these Lie algebras have maximal dimension center among all nilradicals of type A of fixed dimension.

In this paper we will consider the particular case \( b = 1 \), and in order to simplify the notation we denote \( n_{a,c} := n_{a,1,c} \).

The article is organized as follows. In §2 we review some of the standard facts on the representations of \( n_{a,c} \) for every \( a, c \in \mathbb{N} \). In §3, we present the construction of two types of representations and give the value of \( \mu(n_{a,c}) \) for a large class of pairs \( (a, c) \). Finally, in §4 we generalize the previous faithful nil-representations and obtain a bound for \( \mu(n_{a,c}) \) (see Theorem 4.5).

2. Faithful nil-representations for \( n_{a,c} \)

Let \( V \) be a finite dimensional vector space and let \( \mathfrak{n} \) be a nilpotent Lie algebra. A representation \((\pi, V)\) of \( \mathfrak{n} \) on \( V \) is a Lie algebra homomorphism \( \pi : \mathfrak{n} \rightarrow \mathfrak{gl}(V) \) and a nil-representation is, by definition, a representation whose \( \pi(X) \) is a nilpotent endomorphism for every \( X \in \mathfrak{n} \).

Let \( a, c \in \mathbb{N} \) and let \( n_{a,c} \) be the Lie algebra introduced in §1. In order to simplify the notation, let

\[
\mathcal{B} = \{X_i, Y_j, Z_{ij} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c\}
\]

be the basis of \( n_{a,c} \) (see §1). We will denote by

\[
\mathfrak{z}(n_{a,c}) = \text{span}[Z_{ij} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c]\]

the center of \( n_{a,c} \). From (1), we obtain the standard faithful nil-representation \((\pi_S, \mathbb{C}^{a+1+c})\) that, in terms of the canonical basis of \( \mathbb{C}^{a+1+c} \), is given by
Then
\[ \mu(n_{a,c}) \leq a + 1 + c. \]

From now on, we will assume that \( a \leq c \), since the Lie algebra \( n_{a,c} \) is isomorphic to \( n_{c,a} \). It is clear that
\[ a = \text{span}\{Y_j, Z_{i,j} : i = 1, \ldots, a \text{ and } j = 1, \ldots, c\} \]

is an abelian Lie subalgebra of \( n_{a,c} \). From [13], we obtain
\[ \mu_{nil}(a) = \lceil 2 \sqrt{(a + 1)c} \rceil. \]

Therefore
\[ \lceil 2 \sqrt{(a + 1)c} \rceil \leq \mu_{nil}(n_{a,c}) \quad \text{for every } a, c \in \mathbb{N}. \] (4)

**Remark 2.1.** Let \( n \) be a nilpotent Lie algebra and let \( z(n) \) be the center of \( n \). We know that a representation \((\pi, V)\) is faithful if and only if \( \pi|_{z(n)} \) is injective.

### 3. The representations \( \pi_0 \) and \( \pi_1 \)

We now fix the following notation. Let \( a, c \in \mathbb{N} \) such that \( a \leq c \) and let \( q = \left\lfloor \frac{\sqrt{ac}}{c} \right\rfloor \). Let \( M_{m,n} \) be the set of the complex matrices of size \( m \times n \) and let \( \{E_{i,j}\} \subseteq M_{m,n} \) be the canonical basis of \( M_{m,n} \). Let \( r = 0 \) or \( r = 1 \) and let \( z_r = \left\lfloor \frac{\sqrt{ac}}{q+r} \right\rfloor \).

**Definition 3.1.** Consider the usual basis for \( n_{a,c} \) and let \( \pi_r : n_{a,c} \rightarrow \mathfrak{gl}(q+r)(a+1)+z_r \) be the linear map defined by:

1. For \( i = 1, \ldots, a \):
   \[ \pi_r(X_i) = \sum_{p=1}^{q+r} E_i(\nu \cdot (p-1)+p+q+r) a. \]

2. For \( j = 1, \ldots, c \):
   \[ \pi_r(Y_j) = E_{a(q+r)+j,(a+1)(q+r)+ur} \]
   where \( t = \left\lfloor \frac{t}{z} \right\rfloor \) and \( u = j - \left( \left\lfloor \frac{t}{z} \right\rfloor - 1 \right) z_r. \)
3. For $i = 1, \ldots, a$ and $j = 1, \ldots, c$:
\[
\pi_r(Z_{i,j}) = \pi_r(X_i)\pi_r(Y_j) = [\pi_r(X_i), \pi_r(Y_j)].
\]

Remark 3.2. It is easy to see that, by construction, $\pi_r$ is a representation of $\mathfrak{n}_{a,c}$ for $r = 0, 1$.

Example 3.3. Let $a = 3$ and $c = 9$ then $q = 1$.

(i) If $r = 0$ we obtain $z_0 = 9$. Then the representation $(\pi_0, C^{13})$ is
\[
\pi_0 \left( \sum_{i=1}^{3} x_iX_i + \sum_{j=1}^{9} y_jY_j + \sum_{i=1}^{3} \sum_{j=1}^{9} z_{i,j}Z_{i,j} \right) = \begin{bmatrix}
0 & 0 & 0 & 0 & x_1 & z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} & z_{1,7} & z_{1,8} & z_{1,9} \\
0 & 0 & 0 & 0 & x_2 & z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} & z_{2,6} & z_{2,7} & z_{2,8} & z_{2,9} \\
0 & 0 & 0 & 0 & x_3 & z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} & z_{3,7} & z_{3,8} & z_{3,9} \\
0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
Notice that in this case, $\pi_0$ coincides with the standard representation (3) of $\mathfrak{n}_{3,9}$.

(ii) If $r = 1$ we obtain $z_1 = 5$. Then the representation $(\pi_1, C^{13})$ is
\[
\pi_1 \left( \sum_{i=1}^{3} x_iX_i + \sum_{j=1}^{9} y_jY_j + \sum_{i=1}^{3} \sum_{j=1}^{9} z_{i,j}Z_{i,j} \right) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & x_1 & z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} & z_{1,7} & z_{1,8} & z_{1,9} \\
0 & 0 & 0 & 0 & 0 & x_2 & z_{2,1} & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} & z_{2,6} & z_{2,7} & z_{2,8} & z_{2,9} \\
0 & 0 & 0 & 0 & 0 & x_3 & z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} & z_{3,7} & z_{3,8} & z_{3,9} \\
0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Theorem 3.4. Let $a, c \in \mathbb{N}$ and let $(\pi_r, C^{(a+1)(q+r)+u})$ be the representations defined above. Then $\pi_r$ is faithful for $r = 0, 1$.

Proof. From Remark 2.1 we obtain that $\pi_r$ is faithful if and only if $\pi_r|_{\mathfrak{n}_{a,c}}$ is injective. Then
\[
0 = \pi_r \left( \sum_{i=1}^{a} \sum_{j=1}^{c} z_{i,j}Z_{i,j} \right) = \sum_{i=1}^{a} \sum_{j=1}^{c} z_{i,j}\pi_r(Z_{i,j}) = \sum_{i=1}^{a} \sum_{j=1}^{c} z_{i,j}\pi_r(X_i)\pi_r(Y_j)
\]
\[
= \sum_{i=1}^{a} \sum_{j=1}^{c} z_{i,j} \sum_{p=1}^{q+r} \frac{q+r}{i+j} E_{i+j-1, p+r}(x, y) E_{a(q+r)+t, (a+1)(q+r)+u},
\]
where $t = \left\lfloor \frac{i}{a} \right\rfloor$ and $u = j - \left( \left\lfloor \frac{i}{a} \right\rfloor - 1 \right) z_r$. Thus
\[
\sum_{i=1}^{a} \sum_{j=1}^{c} z_{i,j} E_{i+j-1, p+r}(x, y) = 0.
\]
It is easy to see that we must have $z_{i,j} = 0$ for every $i = 1, \ldots, a$, $j = 1, \ldots, c$, and the proof is complete. \(\square\)
Remark 3.5. If $\sqrt{\frac{c}{a+1}} = q + \alpha$ with $0 \leq \alpha < 1$, we obtain

$$2 \sqrt{(a+1)c} = 2(a+1) \sqrt{\frac{c}{a+1}} = 2(a+1)q + 2(a+1)\alpha.$$ 

It follows that

$$\left\lceil 2 \sqrt{(a+1)c} \right\rceil = 2(a+1)q + \lceil 2(a+1)\alpha \rceil,$$

(5)

and

$$2 \sqrt{(a+1)c} \leq 2(a+1)q + \lceil 2(a+1)\alpha \rceil.$$

Hence

$$4(a+1)c \leq 4(a+1)^2 q^2 + 4 \lceil 2(a+1)\alpha \rceil (a+1)q + \lceil 2(a+1)\alpha \rceil^2,$$

thus

$$c \leq (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q + \left\lceil \frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} \right\rceil.$$

(6)

We can now formulate the main result of this section.

Theorem 3.6. Let $a, c \in \mathbb{N}$ and let $\sqrt{\frac{c}{a+1}} = q + \alpha$ with $0 \leq \alpha < 1$. If

$$\lceil 2(a+1)\alpha \rceil < 2 \sqrt{a+1} \quad \text{or} \quad 2(a+1) - 2 \sqrt{a+1} < \lceil 2(a+1)\alpha \rceil$$

then

$$\mu(\pi_{a,c}) = \left\lceil 2 \sqrt{(a+1)c} \right\rceil.$$ 

Proof. Assume that $\lceil 2(a+1)\alpha \rceil < 2 \sqrt{a+1}$, then $\frac{\lceil 2(a+1)\alpha \rceil^2}{4(a+1)} < 1$. Since $c \in \mathbb{N}$, we obtain from (6) that

$$c \leq (a+1)q^2 + \lceil 2(a+1)\alpha \rceil q.$$

It follows that

$$\left\lceil \frac{c}{q} \right\rceil \leq (a+1)q + \lceil 2(a+1)\alpha \rceil,$$

and therefore

$$(a+1)q + \left\lceil \frac{c}{q} \right\rceil \leq 2(a+1)q + \lceil 2(a+1)\alpha \rceil \overset{\text{(5)}}{=} \left\lceil 2 \sqrt{(a+1)c} \right\rceil.$$ 

(7)

Consider the faithful nil-representation $\pi_0$. Then equations (7) and (4) imply that

$$\mu(\pi_{a,c}) = \left\lceil 2 \sqrt{(a+1)c} \right\rceil.$$
Assume now that $2(a + 1) < 2 \sqrt{a + 1}$. Then

$$2(a + 1) - [2(a + 1)\alpha] < 2 \sqrt{a + 1}$$
$$4(a + 1)^2 - 4(a + 1) [2(a + 1)\alpha] + [2(a + 1)\alpha]^2 < 4(a + 1)$$
$$(a + 1) - [2(a + 1)\alpha] + \frac{[2(a + 1)\alpha]^2}{4(a + 1)} < 1$$
$$\frac{[2(a + 1)\alpha]^2}{4(a + 1)} < 1 + [2(a + 1)\alpha] - (a + 1).$$

Therefore

$$\left\lfloor \frac{[2(a + 1)\alpha]^2}{4(a + 1)} \right\rfloor \leq [2(a + 1)\alpha] - (a + 1). \quad (8)$$

It follows that

$$c \leq (a + 1)q^2 + [2(a + 1)\alpha]q + \left\lfloor \frac{[2(a + 1)\alpha]^2}{4(a + 1)} \right\rfloor r \quad (6)$$
$$\leq (a + 1)q^2 + [2(a + 1)\alpha]q + [2(a + 1)\alpha] - (a + 1)$$
$$= (a + 1)q(q + 1) + [2(a + 1)\alpha](q + 1) - (a + 1)(q + 1).$$

Thus

$$\left\lfloor \frac{c}{q + 1} \right\rfloor \leq (a + 1)q + [2(a + 1)\alpha] - (a + 1).$$

and therefore

$$(a + 1)(q + 1) + \left\lfloor \frac{c}{q + 1} \right\rfloor \leq 2(a + 1)q + [2(a + 1)\alpha]$$
$$\quad \leq 2 \sqrt{(a + 1)c}. \quad (9)$$

Consider the faithful nil-representation $\pi_1$. By equations (9) and (4), we obtain that

$$\mu(\pi_{\alpha,c}) = \left\lceil 2 \sqrt{(a + 1)c} \right\rceil,$$

which completes the proof. □

**Corollary 3.7.** If $a = 1$ or $a = 2$ then

$$\mu(\pi_{\alpha,c}) = \left\lceil 2 \sqrt{(a + 1)c} \right\rceil \quad \text{for every } c \geq a.$$

**Proof.** If $a = 1$ or $a = 2$ we obtain $\sqrt{a + 1} < 2$. Thus

$$2(a + 1) - 2 \sqrt{a + 1} < 2 \sqrt{a + 1},$$

by Theorem 3.6 we conclude that $\mu(\pi_{\alpha,c}) = \left\lceil 2 \sqrt{(a + 1)c} \right\rceil$. □

**Theorem 3.8.** Let $a, c \in \mathbb{N}$ and let $\sqrt{\frac{c}{a + 1}} = q + \alpha$ with $0 \leq \alpha < 1$. If
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(i) \(2 \sqrt{a+1} \leq [2(a+1)\alpha] \leq 2(a+1) - 2 \sqrt{a+1}\) and

(ii) \(c \geq (a - \sqrt{a+1})^2(a+1)\)

then

\[
\mu(n_{cc}) = \left\lceil 2 \sqrt{(a+1)c} \right\rceil \quad \text{or} \quad \mu(n_{cc}) = \left\lceil 2 \sqrt{(a+1)c} \right\rceil + 1.
\]

Proof. From (i), we obtain

\[
\frac{1}{2} \leq \frac{2(a+1)\alpha}{4(a+1)} = \frac{1}{2} - \frac{1}{\sqrt{a+1}}.
\]

(10)

Therefore

\[
c \leq (a+1)q + \left\lceil 2(a+1)\alpha \right\rceil q + \frac{\left\lceil 2(a+1)\alpha \right\rceil^2}{4(a+1)}
\]

\[
\leq (a+1)q + \left\lceil 2(a+1)\alpha \right\rceil q + \frac{2(a+1)\alpha}{2} - \frac{2(a+1)\alpha}{2 \sqrt{a+1}}
\]

\[
= ((a+1)q + \left\lceil 2(a+1)\alpha \right\rceil - (a+1))(q+1) + \frac{2(a+1)\alpha}{2} + (a+1) - \frac{2(a+1)\alpha}{2 \sqrt{a+1}}.
\]

(11)

Also, since \(2 \sqrt{(a+1)} \leq [2(a+1)\alpha]\), we obtain

\[
-a - \sqrt{a+1} \leq \frac{2(a+1)\alpha}{2 \sqrt{a+1}}
\]

(12)

Combining (11) and (12), we have

\[
\frac{c}{q+1} \leq (a+1)q + \left\lceil 2(a+1)\alpha \right\rceil - (a+1) + \frac{a - \sqrt{a+1}}{q+1}.
\]

Thus

\[
\left\lceil \frac{c}{q+1} \right\rceil \leq (a+1)q + \left\lceil 2(a+1)\alpha \right\rceil - (a+1) + \frac{a - \sqrt{a+1}}{q+1}.
\]

(13)

Consider \(\pi_1\). From (13) we obtain

\[
\mu(n_{cc}) \leq 2(a+1)q + \left\lceil 2(a+1)\alpha \right\rceil + \left\lceil \frac{a - \sqrt{a+1}}{q+1} \right\rceil.
\]

Under the hypothesis \(c \geq (a - \sqrt{a+1})^2(a+1)\), we obtain

\[
q+1 > \sqrt{\frac{c}{a+1}} \geq a - \sqrt{a+1}.
\]

Then it follows that

\[
\mu(n_{cc}) \leq 2(a+1)q + \left\lceil 2(a+1)\alpha \right\rceil + 1
\]

\[
\mu(n_{cc}) = \left\lceil 2 \sqrt{(a+1)c} \right\rceil + 1.
\]

The proof is completed. \(\Box\)
Corollary 3.9. Let \( a = 3 \).

1. If \( c \neq (2k + 1)^2 \) for \( k \in \mathbb{N} \) then
   \[
   \mu(n_{3,c}) = \left\lfloor 4 \sqrt{c} \right\rfloor.
   \]

2. If \( c = (2k + 1)^2 \) for some \( k \in \mathbb{N} \) then
   \[
   \mu(n_{3,c}) \leq \left\lfloor 4 \sqrt{c} \right\rfloor + 1.
   \]

Proof. If \( a = 3 \), then
   \[
   2 \sqrt{a + 1} = 2(a + 1) - 2 \sqrt{a + 1} = 4.
   \]

   Therefore:

   • If \( \lceil 8\alpha \rceil = 4 \), we obtain by Theorem 3.6 that \( \mu(n_{3,c}) = \left\lfloor 4 \sqrt{c} \right\rfloor \).

   • If \( \lceil 8\alpha \rceil = 4 \), Theorem 3.8 implies that
     \[
     \left\lfloor 4 \sqrt{c} \right\rfloor \leq \mu(n_{3,c}) \leq \left\lfloor 4 \sqrt{c} \right\rfloor + 1 \quad \text{for} \quad c \geq 4.
     \]

   Consider the case \( \lceil 8\alpha \rceil = 4 \), which implies \( 3 \leq \alpha \leq 1/2 \). Assume that \( c \neq (2k + 1)^2 \) for every \( k \in \mathbb{N} \). On the one hand we obtain, by equation (5), that
   \[
   \left\lfloor 2 \sqrt{4c} \right\rfloor = 8q + \lceil 8\alpha \rceil = 8q + 4.
   \]

   On the other hand,
   \[
   c = 4(q + \alpha)^2 \leq 4 \left( q + \frac{1}{2} \right)^2 = 4q^2 + 4q + 1.
   \]

   Since \( c \neq (2q + 1)^2 \) we obtain that \( c \leq 4q(q + 1) \) and then \( \frac{c}{q + 1} \leq 4q \).

   By considering the representation \( n_1 \), we obtain
   \[
   \left\lfloor 2 \sqrt{4c} \right\rfloor \leq \mu(n_{3,c}) \leq \dim n_1 \leq 4(q + 1) + 4q = \left\lfloor 2 \sqrt{4c} \right\rfloor
   \]

   and the result follows. \( \square \)

Remark 3.10. The previous Theorems have shown that there is a large class of pairs \((a, c)\) for which an excellent bound for \( \mu(n_{3,c}) \) is achieved. However, there is still a large class for which these representations are not optimal. Assume that \( 2 \sqrt{a + 1} \leq [2(a + 1)\alpha] \leq 2(a + 1) - 2 \sqrt{a + 1} \) and \( c < (a - \sqrt{a + 1})^2(a + 1) \). Set \( q = \left\lfloor \sqrt{a + 1} \right\rfloor \) and \( \alpha = \sqrt{a + 1} - q \) as before. Recall that \( \left\lfloor 2 \sqrt{c(a + 1)} \right\rfloor = 2(a + 1)q + [2(a + 1)\alpha] \) and \( c = (a + 1)(q + \alpha)^2 \). Then
   \[
   \frac{1}{\sqrt{a + 1}} - \frac{1}{2(a + 1)} < \alpha \leq 1 - \frac{1}{\sqrt{a + 1}}. \tag{14}
   \]

   On the one hand we obtain, by equation (14):
   \[
   c \leq (a + 1)(q + 1)^2 - 2 \sqrt{a + 1}(q + 1) + 1.
   \]

Therefore
   \[
   \frac{c}{q + 1} \leq (a + 1)(q + 1) - 2 \sqrt{a + 1} + \frac{1}{q + 1} \quad \text{and} \quad \left\lfloor \frac{c}{q + 1} \right\rfloor < (a + 1)(q + 1) - 2 \sqrt{a + 1} + \frac{3}{2}.
   \]
By considering the representation $\pi_1$ of dimension $(a + 1)(q + 1) + \left\lceil \frac{\pi}{q+1} \right\rceil$, we obtain
\[
\dim \pi_1 \leq 2(a + 1)(q + 1) + \frac{3}{2} - 2\sqrt{a + 1}
\]
\[
= 2(a + 1)q + 2(a + 1) + \frac{3}{2} - 2\sqrt{a + 1} - 2[2(a + 1)\alpha] + 2(a + 1)\alpha
\]
\[
= 2(a + 1)q + [2(a + 1)\alpha] + \frac{3}{2} + (2(a + 1) - 2\sqrt{a + 1} - [2(a + 1)\alpha])
\]
\[
\leq 2(a + 1)q + [2(a + 1)\alpha] + \frac{3}{2} + 2(a + 1) - 4\sqrt{a + 1}.
\]
Therefore
\[
\dim \pi_1 \leq 2\sqrt{c(a + 1)} + \frac{3}{2} + \frac{1}{2}
\] for $a \geq 3$.

Notice that for $a \geq 12$ we already have $\left( \frac{\sqrt{a + 1}}{2} \right)^2 > a$, so we must consider new representations in order to improve the bound for $\mu(n_{a,c})$.

### 4. New faithful representations for $n_{a,c}$

Extending $\pi_0$ and $\pi_1$.

Let $q$ and $a$ be as before. In the previous section we have obtained minimal representations $\pi_r$ for a large class of pairs $(a, c)$. In particular, if $a = 0$, we have obtained the value of $\mu(n_{a,c})$. We will consider now $0 < a < 1$.

Let $r, s \in \mathbb{N}$ such that $2 \leq s, (s - 1) \left\lceil \frac{a}{s} \right\rceil < a$ and $r < s$. Set
\[
d_1 = qa + r \left\lceil \frac{a}{s} \right\rceil \quad \text{and} \quad d_2 = (s - 1) \left\lceil \frac{c}{qs + 1} \right\rceil + \left\lceil \frac{c - q(s - 1)}{q + 1} \right\rceil.
\]

**Definition 4.1.** Consider the usual basis for $n_{a,c}$ and define the linear map $\pi_{ij} : n_{a,c} \rightarrow gl(d_1 + q + 1 + d_2)$ in the following way:

1. For $1 \leq i \leq a$:
   \[
   \pi_{ij}(X_i) = \sum_{k=1}^{q} E_{(k-1)a+i,d_1+k} + E_{(q-1)a+i+1,d_1+q+1}.
   \]

2. For $1 \leq j \leq (q - 1)d_2 + \left\lceil \frac{c}{q+1} \right\rceil$:
   \[
   \pi_{ij}(Y_j) = E_{d_1+1,\left\lceil \frac{c}{q+1} \right\rceil,d_1+q+1+j} - \left\lceil \frac{c}{q+1} \right\rceil d_1.
   \]

3. For $(q - 1)d_2 + \left\lceil \frac{c}{q+1} \right\rceil + 1 \leq j \leq qd_2$:
   \[
   \pi_{ij}(Y_j) = E_{d_1+1,j-(q-1)d_2+d_1+q+1} + E_{d_1+1,d_1+q+1+j-(q-1)d_2} - \left\lceil \frac{c}{q+1} \right\rceil d_1.
   \]

4. For $qd_2 + 1 \leq j \leq c$:
   \[
   \pi_{ij}(Y_j) = E_{d_1+1,\left\lceil \frac{c}{q+1} \right\rceil+d_1+q+1+j-(q-1)d_2} - \left\lceil \frac{c}{q+1} \right\rceil d_1.
   \]

5. For $1 \leq i \leq a$ and $1 \leq j \leq c$:
   \[
   \pi_{ij}(Z_{ij}) = \pi_{ij}(X_i)\pi_{ij}(Y_j) = [\pi_{ij}(X_i), \pi_{ij}(Y_j)].
   \]
Example 4.2. Consider $a = 5$ and $c = 25$. Then $q = 2$ and we can choose $s = 2$ or $s = 3$.

(i) If $s = 2$, then $r = 1$ and we obtain

$$
\pi_{i_2} \left( \sum_{i=1}^{5} x_i X_i + \sum_{j=1}^{25} y_j Y_j + \sum_{i=1}^{5} \sum_{j=1}^{5} z_{i,j} Z_{i,j} \right) = 
$$

(ii) If $s = 3$, consider $r = 2$. Then we obtain

$$
\pi_{i_3} \left( \sum_{i=1}^{5} x_i X_i + \sum_{j=1}^{25} y_j Y_j + \sum_{i=1}^{5} \sum_{j=1}^{5} z_{i,j} Z_{i,j} \right) = 
$$

Notice that for $\eta_{5,25}$, $\dim \pi_{i_2} = 26$ and $\dim \pi_{i_3} = 28$, while $\dim \pi_5 = 31$.

Theorem 4.3. Let $a, c, r, s \in \mathbb{N}$ such that $a \leq c$, $s \geq 2$, $(s - 1) \left\lceil \frac{r}{s} \right\rceil < a$ and $r < s$. Then $\pi_{i_s}$ is a faithful nil-representation of $\eta_{a,c}$. 

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Proof. From Remark 2.1 we obtain that \( \pi_{ij} \) is faithful if and only if \((\pi_{ij})_{|v(u,v)}\) is injective. Moreover, we obtain the following:

\[
\pi_{ij} \left( \sum_{i=1}^{c} \sum_{j=1}^{c} z_{i,j} Z_{i,j} \right) = \sum_{u=1}^{a} \sum_{v=1}^{d_{i}+q+1+d_{2}} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} + \sum_{u=d_{i}-q+1} \sum_{v=d_{i}+q+2} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} \tag{A}
\]

\[
+ \sum_{u=d_{i}-q+1} \sum_{v=d_{i}+q+2} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} + \sum_{u=d_{i}-q+1} \sum_{v=d_{i}+q+2} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} \tag{B}
\]

\[
+ \sum_{u=1}^{a} \sum_{v=1}^{d_{i}+q+1+d_{2}} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} + \sum_{u=1}^{a} \sum_{v=1}^{d_{i}+q+1+d_{2}} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} \tag{C}
\]

\[
+ \sum_{u=d_{i}-q+1} \sum_{v=d_{i}+q+2} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} + \sum_{u=d_{i}-q+1} \sum_{v=d_{i}+q+2} z_{u-(q-1)\nu,-d_{i}-q-1+(q-1)d_{2}} E_{u,v} \tag{D}
\]

Since the sums (A), (B), (C), (D), (E), (F) and (G) are mutually disjoint, we obtain that if \( \pi_{ij} \left( \sum_{i=1}^{c} \sum_{j=1}^{c} z_{i,j} Z_{i,j} \right) = 0 \) then:

1. From (A): \( z_{i,j} = 0 \) for \( 1 \leq i \leq a \) and \( 1 \leq j \leq (q-1)d_{2} \);
2. From (A): \( z_{i,j} = 0 \) for \( 1 \leq i \leq r \floor*{q-1\nu} \) and \( (q-1)d_{2} + 1 \leq j \leq qd_{2} \);
3. From (B): \( z_{i,j} = 0 \) for \( r \floor*{q-1\nu} + 1 \leq i \leq a \) and \( (q-1)d_{2} + 1 \leq j \leq qd_{2} \);
4. From (C): \( z_{i,j} = 0 \) for \( r \floor*{q-1\nu} + 1 \leq i \leq a \) and \( (q-1)d_{2} + 1 \leq j \leq qd_{2} \);
5. From (D): \( z_{i,j} = 0 \) for \( r \floor*{q-1\nu} + 1 \leq i \leq a \) and \( (q-1)d_{2} + (s-1) \floor*{q-1\nu} + 1 \leq j \leq (s-1) \floor*{q-1\nu} + c - d_{2} \);
6. From (E): \( z_{i,j} = 0 \) for \( r \floor*{q-1\nu} + 1 \leq i \leq a \) and \( (q-1)d_{2} + (s-1) \floor*{q-1\nu} + c - d_{2} + 1 \leq j \leq qd_{2} \);
Corollary 4.4. The dimension of the representation $\pi_{\mu}$ for $n_{a,c}$ is

$$\dim \pi_{\mu} = q(a+1) + 1 + r\left[\frac{a}{s}\right] + \left[\frac{c + (s-1)\left[\frac{c}{q+1}\right]}{q+1}\right].$$

Proof. By construction we have

$$\dim \pi_{\mu} = qa + r\left[\frac{a}{s}\right] + q + 1 + (s-1)\left[\frac{c}{q+1}\right] + \left[\frac{c - (s-1)\left[\frac{c}{q+1}\right]}{q+1}\right].$$
Since
\[ \frac{c - q(s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} \leq \left\lceil \frac{c - q(s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} \right\rceil < \frac{c - q(s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} + 1, \]
then
\[ \frac{c + (s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} \leq (s - 1)\left\lceil \frac{c}{s+1} \right\rceil + \left\lceil \frac{c - q(s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} \right\rceil < \frac{c + (s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} + 1, \]
and
\[ (s - 1)\left\lceil \frac{c}{s+1} \right\rceil + \left\lceil \frac{c - q(s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1} \right\rceil = \frac{c + (s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q + 1}, \]
and the result follows. \(\square\)

Then we obtain:

**Theorem 4.5.** Let \(a, c, s \in \mathbb{N}\) such that \(a \leq c\), \(s \geq 2\) and \((s - 1)\left\lceil \frac{a}{s+1} \right\rceil < a\). Let \(q = \left\lfloor \sqrt{\frac{c}{a+1}} \right\rfloor\). Then
\[
\left\lceil 2 \sqrt{(a+1)c} \right\rceil \leq \mu(n_{a,c}) \leq q(a+1) + 1 + \left\lceil \frac{a}{s} \right\rceil + \left\lceil \frac{c + (s - 1)\left\lceil \frac{c}{s+1} \right\rceil}{q+1} \right\rceil.
\]

**Proof.** It is clear that the dimensions of the representations \(\pi_{ir}\) provide upper bounds for \(\mu(n_{a,c})\). In particular, by taking \(r = 1\) we obtain the minimal dimension among all \(\pi_{ir}\) with fixed \(s\). \(\square\)

**Corollary 4.6.** Under the hypothesis of Theorem 4.5, if \(\sqrt{\frac{c}{a+1}} = q + \frac{1}{s}\), then
\[
\mu(n_{a,c}) = \left\lceil 2 \sqrt{c(a+1)} \right\rceil \quad \text{or} \quad \mu(n_{a,c}) = \left\lceil 2 \sqrt{c(a+1)} \right\rceil + 1.
\]

**Proof.** If \(\sqrt{\frac{c}{a+1}} = q + \frac{1}{s}\) we obtain \(c = \frac{(a+1)(qs + 1)^2}{s^2}\).

Since \(c \in \mathbb{N}\), we must have that \(s^2 \mid (a+1)\). This implies that
\[
\frac{c}{qs + 1} = \frac{(a+1)(qs + 1)^2}{s^2} \in \mathbb{N} \quad \text{and} \quad \left\lceil \frac{a}{s} \right\rceil = \left\lceil \frac{a+1}{s} - \frac{1}{s} \right\rceil = \frac{a+1}{s} \in \mathbb{N}.
\]
Therefore

\[
\dim \pi_{1/4}(\mathfrak{u}_{a,c}) = (a + 1)q + 1 + \left\lfloor \frac{a + 1}{s} \right\rfloor + \left\lfloor \frac{c + (s - 1) \left\lfloor \frac{c}{q+1} \right\rfloor}{q+1} \right\rfloor
\]

\[
= (a + 1)q + 1 + \frac{a + 1}{s} + \left\lfloor \frac{c + (s - 1) \left\lfloor \frac{a + 1}{q+1} \right\rfloor}{q+1} \right\rfloor
\]

\[
= (a + 1) \left( q + \frac{1}{s} \right) + 1 + \left\lfloor \frac{c s^2 + (s - 1)(a + 1)(q + 1)}{s^2(q + 1)} \right\rfloor
\]

\[
= (a + 1) \left( q + \frac{1}{s} \right) + 1 + \left\lfloor \frac{(a + 1)(q + 1)^2 + (s - 1)(a + 1)(q + 1)}{s^2(q + 1)} \right\rfloor
\]

\[
= (a + 1) \left( q + \frac{1}{s} \right) + 1 + \left\lfloor \frac{(a + 1)(q + 1)(q + 1) + (s - 1)(a + 1)(q + 1)}{s} \right\rfloor
\]

\[
= 2(a + 1) \left( q + \frac{1}{s} \right) + 1
\]

\[
= 2 \sqrt{c(a + 1)} + 1
\]

\[
= \left\lfloor 2 \sqrt{c(a + 1)} \right\rfloor + 1.
\]

And the result follows. \(\square\)

Example 4.7. Consider \(\mathfrak{u}_{a,c}\) with \(a = 48\) and \(c = 100\). Notice that this case falls under Remark 3.10. Then we obtain \(q = 1\), \(\alpha = \frac{1}{2}\), and the following results:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>(\pi_0)</th>
<th>(\pi_1)</th>
<th>(\pi_{1/5})</th>
<th>(\pi_{1/4})</th>
<th>(\pi_{1/4})</th>
<th>(\pi_{1/4})</th>
<th>(\left\lfloor 2 \sqrt{49 \cdot 100} \right\rfloor)</th>
</tr>
</thead>
<tbody>
<tr>
<td>149</td>
<td>149</td>
<td>148</td>
<td>146</td>
<td>144</td>
<td>142</td>
<td>141</td>
<td>140</td>
</tr>
</tbody>
</table>

Clearly, the obtained result is very sharp:

\[140 \leq \mu(\mathfrak{u}_{48,100}) \leq 141.\]

References


