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On Some Operations on Soft Topological Spaces

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Abstract. The aim of the article is to point out a one-to-one correspondence between soft topological spaces over a universe U with respect to a parameter set E and topological ones on the Cartesian product $E \times U$. From this point of view, all soft topological terms, soft operations, soft functions and properties of soft topological spaces are actually topological concepts. Because the set valued mappings and set valued analysis have great application potential, it is necessary to look for their meaningful use with respect to standard topological methods and set valued analysis procedures.

1. Introduction

There are several papers that document certain problems relating to the fundamentals of soft set theory and soft topological spaces. In [17] the authors claim that soft topology is exactly a special subcase of general topology. Also in [14] it is stated that a soft topology is nothing but a topology on the Cartesian product.

Based on a one-to-one correspondence between soft topology and general topology (see below), we show that each soft topological concept has its own topological equivalent. Some soft terms (for example soft compactness, soft paracompactness, soft Lindelöfness, soft nomality, soft connectedness, soft hyperconnectedness, soft topological sum [1, 5, 6, 20]) correspond to known commonly used and studied topological terms. Others (for example soft separation axioms [2, 3, 4, 5, 6, 7, 11, 12, 16, 18, 19, 20]) correspond to topological terms that bring new challenges to research. In principle, any soft concept can be studied by standard topological methods.

In this article, we will deal exclusively with soft concepts that have known topological equivalents. This means there is no need to prove them and they are the consequences of known results of general topology. Other soft notions (for example soft separation axioms), which correspond to the new topological ones will be mentioned only marginally in this article (see the end of Section 3).

In the following sections we present a definition of soft topological space as a topological space on the product of two sets. The whole issue of soft topology and soft topological terms will be simplified and in particular the known topological methods can be used more effectively.

We give only a selection of the most important terms of soft topology, which we will convert to methods of general topology and it is only for further examination how other soft topology concepts, procedures and proofs can be transformed into methods of general topology. We show that the use of topological methods

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fully supersede and simplify the cumbersome approaches and proofs that are often used in the theory of soft topological spaces.

In the first part we present a one-to-one correspondence between binary relations and set valued mappings, in the next part between soft topological spaces and topological spaces on the product of two sets. We also deal with fundamental soft terms such as soft mappings, soft continuity, soft homeomorphism, soft homogeneity and some operations on soft topological spaces that correspond to the topological sum and the product of topologies.

2. Relations and Set Valued Mappings

Any subset S of the Cartesian product $E \times U$ is a binary relation from a set E to a set U. Let $S(e) = \{u \in U : (e,u) \in S\}$ and $U_e = \{e\} \times U$, $e \in E$. By $\mathbf{R}(E,U)$ we denote the set of all binary relations from E to U.

The operations of sum $S \cup T$, $\cup_{t\in T} S_t$, intersection $S \cap T$, $\cap_{t\in T} S_t$, complement S^c and difference $S \setminus T$ of relations are defined in the obvious way as in set theory.

By $F : E \to 2^U$ we denote a set valued mapping (multifunction) from E to power set 2^U of U. The set of all set valued mappings from E to 2^U is denoted by $\mathbf{F}(E,U)$ and the set of all constant set valued mappings $F : E \to 2^U$ for which $F(e) = A \subset U$ for any $e \in E$ is denoted by $\mathbf{C}(E,U)$. A set valued mapping F for which $F(e) = \{u\}$ and it is empty valued otherwise is denoted by F^u_e (P^v_e, Q^w_f).

If F, G are two set valued mappings, then $F \subset G$ ($F = G$) means $F(e) \subset G(e)$ ($F(e) = G(e)$) for any $e \in E$. The intersection (union) of family $\{G_t : t \in T\}$ of set valued mappings is defined as a set valued mapping $H : E \to 2^U$ for which $H(e) = \cap_{t\in T} G_t(e)$ ($H(e) = \cup_{t\in T} G_t(e)$) for any $e \in E$ and it is denoted by $\cap_{t\in T} G_t$ ($\cup_{t\in T} G_t$). The complement F^c of F is defined as a set valued mapping for which $F^c(e) = U \setminus F(e)$ for all $e \in E$.

A graph of $G \in \mathbf{F}(E,U)$ is a set $Gr(G) = \{(e,u) \in E \times U : u \in G(e)\}$ and it is a subset of $E \times U$, hence $Gr(G) \in \mathbf{R}(E,U)$. So, any set valued mapping G determines a relation from $\mathbf{R}(E,U)$ denoted by $\mathbf{R}_G = \{(e,u) \in E \times U : u \in G(e)\} = Gr(G)$.

On the other hand, any relation $S \in \mathbf{R}(E,U)$ determines a set valued mapping $\mathbf{F}_S$ from E to 2^U where $\mathbf{F}_S(e) = S(e)$. So there is a one-to-one correspondence between $\mathbf{R}(E,U)$ and $\mathbf{F}(E,U)$ and the following lemma deals with the obvious facts of mutually inverse operators $\mathbf{R}$ and $\mathbf{F}$.

Lemma 2.1. *If $S, Q \in \mathbf{R}(E,U)$ and $G, H \in \mathbf{F}(E,U)$, then:*

(1) $S \mapsto \mathbf{F}_S \in \mathbf{F}(E,U)$, $\quad \mathbf{F}_S(e) = S(e)$,

(2) $G \mapsto \mathbf{R}_G = Gr(G) \in \mathbf{R}(E,U)$, $\quad \mathbf{R}_G(e) = G(e)$,

(3) $\mathbf{F}_{\mathbf{R}_G} = G$, $\ \mathbf{R}_{\mathbf{F}_S} = S$, $\ \mathbf{R}_G = S \Leftrightarrow G = \mathbf{F}_S$,

(4) $S(e) = Q(e)$ *for any* $e \in E \Leftrightarrow S = Q \Leftrightarrow \mathbf{F}_S = \mathbf{F}_Q$,

(5) $S(e) \subset Q(e)$ *for any* $e \in E \Leftrightarrow S \subset Q \Leftrightarrow \mathbf{F}_S \subset \mathbf{F}_Q$,

(6) $H(e) = G(e)$ *for any* $e \in E \Leftrightarrow H = G \Leftrightarrow \mathbf{R}_H = \mathbf{R}_G$,

(7) $H(e) \subset G(e)$ *for any* $e \in E \Leftrightarrow H \subset G \Leftrightarrow \mathbf{R}_H \subset \mathbf{R}_G$,

(8) $\mathbf{R}_{F^u_e} = \{(e,u)\} \Leftrightarrow F^u_e = \mathbf{F}_{\{(e,u)\}}$,

(9) $S = \cup_{e\in E}\{e\} \times S(e)$, $\quad \mathbf{R}_G = \cup_{e\in E}\{e\} \times G(e)$,

(10) $\mathbf{R}_{H^c} = (\mathbf{R}_H)^c$, $\ d\, \mathbf{F}_{S^c} = (\mathbf{F}_S)^c$.

3. Soft Topology and Topology on $E \times U$

In the following we will show that many terms of soft topology can be derived and investigated by means of general topology. In the literature (see references) a definition of a soft set is introduced by a set valued mapping. The next definition introduces the basic operations on the set of all soft sets with respect to a fixed set of parameters E. In the literature a soft set is usually denoted by (F, E) or F_E. Since E is fixed, we prefer a notation F where F is a set valued mapping.

Definition 3.1. ([1-7], [11-21]) Let E, U be two nonempty sets.

(1) If $F : E \to 2^U$ is a set valued mapping, then F is called a soft set over U with respect to E. A soft set F for which $F(e) = \emptyset$ ($F(e) = U$) for any $e \in E$ is called the null soft set (the full soft set) and F_e^u is called a soft point.

(2) A soft set F is a soft subset of G (F is contained in G or G contains F), if $F(e) \subset G(e)$ for any $e \in E$. The complement of soft set F is defined as a soft set F^c where $F^c(e) = U \setminus F(e)$ for all $e \in E$. The intersection (union) of a family of soft sets $\{G_t : t \in T\}$ is defined as a soft set G where $G(e) = \cap_{t\in T} G_t(e)$ ($G(e) = \cup_{t\in T} G_t(e)$) for all $e \in E$.

(3) The family of all soft sets over U with respect to E is denoted by $SS(E, U)$. It is clear $SS(E, U) = \mathbf{F}(E, U)$. The family of all soft points is denoted by $SP(E, U)$.

As we said above there is no difference between a set valued mapping and a relation. So, a soft set can be defined equivalently by the following way.

Definition 3.2. ([14]) Let E, U be two nonempty sets. A soft set over U with respect to E is any subset A of $E \times U$. The sets $\emptyset$ and $E \times U$ (both subsets of $E \times U$) represent the null soft set $\mathbf{F}_\emptyset$ and the full soft set $\mathbf{F}_{E\times U}$, respectively.

In the next definition we denote a soft topological space as a triplet (E, U, τ). This notation is more natural than that of (U, τ, E), because first the sets are given and then the structure on them.

Definition 3.3. ([1-7], [11-21]) Let E, U be two nonempty sets.

(1) A soft topological space over U with respect to E is a triplet (E, U, τ), where $\tau \subset SS(E, U)$ is closed under finite intersection, arbitrary union of soft sets and contains the null soft set and the absolute soft set. A soft set from τ is called a soft open set and its complement is called a soft closed set.

(2) If H is a soft set, then a soft closure (a soft interior) of H denoted by $scl(H)$ ($sint(H)$) is defined as the intersection (union) of all soft closed (soft open) sets containing H (contained in H).

(3) A soft topological space (E, U, τ) is called enriched (see [3, 4]), if for any $e \in E$ a soft set F for which $F(e) = U$ and $F(f) = \emptyset$ for $f \neq e$ is soft open.

In the following we will denote by (X, τ) a topological space where τ is a topology on X and by (A, τ_A) a topological subspace of (X, τ) where $A \subset X$ and τ_A is a relative topology. If (X, τ_1) and (Y, τ_2) are topological spaces, then $(X \times Y, \tau_1 \times \tau_2)$ is a topological space equipped with the product topology (the Cartesian product) of τ_1 and τ_2. By $(E \times U, \tau)$ we denote a topological space where τ is a topology on $E \times U$. If A is a subset of $E \times U$, by $cl(A)$ ($int(A)$) we denote a closure (an interior) of A in the topological space $(E \times U, \tau)$.

Since a soft topology is represented by a family of set valued mappings which have a binary relation representation, any soft topological space can be transformed to a corresponding topological space by the following theorem.

Theorem 3.4. ([14]) *There is a one-to-one correspondence between the family of all soft topological spaces over U with respect to E and the family of all topological spaces $(E \times U, \tau)$ as follows:*

(1) If (E, U, τ) is a soft topological space, then $(E \times U, \mathbf{R}_\tau)$ is a topological space where $\mathbf{R}_\tau = \{\mathbf{R}_G : G \in \tau\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{R}_G \in \mathbf{R}_\tau$.

(2) If $(E \times U, \tau)$ is a topological space, then $(E, U, \mathbf{F}_\tau)$ is a soft topological space where $\mathbf{F}_\tau = \{\mathbf{F}_G : G \in \tau\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{F}_G \in \mathbf{F}_\tau$.

Definition 3.5. A topological space $(E \times U, \mathbf{R}_\tau)$ from Theorem 3.4 item (1) is called the corresponding topological space to (E, U, τ) or $(E \times U, \mathbf{R}_\tau)$ is given by (E, U, τ). A soft topological space $(E, U, \mathbf{F}_\tau)$ from Theorem 3.4 item (2) is called the corresponding soft topological space to $(E \times U, \tau)$ or $(E, U, \mathbf{F}_\tau)$ is given by $(E \times U, \tau)$. We say (E, U, τ) is given by a topological space $(E \times U, \theta)$ (one is given by the other or they are mutually correspondence) if $\tau = \mathbf{F}_\theta$ and $\theta = \mathbf{R}_\tau$. Similarly we say that a soft set G and a relation A are mutually correspondence if $G = \mathbf{F}_A$ and $\mathbf{R}_G = A$.

Theorem 3.6. *Let (E, U, τ) be a soft topological space. If $H, G, G_t \in S(E, U)$ and $A, B, S_t \in \mathbf{R}(E, U)$, $t \in T$, then (in the items (4), (5) cl, int is the closure, the interior operator in the corresponding topological space, respectively)*

(1) a soft set H is soft open (soft closed) if and only if a set $\mathbf{R}_H$ is open (closed) in $(E \times U, \mathbf{R}_\tau)$,

(2) $\mathbf{F}_{A\cap B} = \mathbf{F}_A \cap \mathbf{F}_B$, $\quad \mathbf{F}_{A\cup B} = \mathbf{F}_A \cup \mathbf{F}_B$, $\quad \mathbf{F}_{\cap_{t\in T} S_t} = \cap_{t\in T}\mathbf{F}_{S_t}$, $\quad \mathbf{F}_{\cup_{t\in T} S_t} = \cup_{t\in T}\mathbf{F}_{S_t}$,

(3) $\mathbf{R}_{H\cap G} = \mathbf{R}_H \cap \mathbf{R}_G$, $\quad \mathbf{R}_{H\cup G} = \mathbf{R}_H \cup \mathbf{R}_G$, $\quad \mathbf{R}_{\cap_{t\in T} G_t} = \cap_{t\in T}\mathbf{R}_{G_t}$, $\quad \mathbf{R}_{\cup_{t\in T} G_t} = \cup_{t\in T}\mathbf{R}_{G_t}$,

(4) $scl(H) = \mathbf{F}_{cl(\mathbf{R}_H)}$, $\quad sint(H) = \mathbf{F}_{int(\mathbf{R}_H)}$, $\quad scl(\mathbf{F}_A) = \mathbf{F}_{cl(A)}$, $\quad sint(\mathbf{F}_A) = \mathbf{F}_{int(A)}$,

(5) $cl(A) = \mathbf{R}_{scl(\mathbf{F}_A)}$, $\quad int(A) = \mathbf{R}_{sint(\mathbf{F}_A)}$, $\quad cl(\mathbf{R}_H) = \mathbf{R}_{scl(H)}$, $\quad int(\mathbf{R}_H) = \mathbf{R}_{sint(H)}$,

(6) $scl(H \cup G) = scl(H) \cup scl(G)$, $\qquad sint(H \cap G) = sint(H) \cap sint(G)$.

Proof. The item (1) is clear, since $H \in \tau$ if and only if $\mathbf{R}_H \in \mathbf{R}_\tau$ ($H^c \in \tau$ if and only if $\mathbf{R}_{H^c} \in \mathbf{R}_\tau$), by Theorem 3.4.

From items (2) and (3) we will prove $\mathbf{R}_{\cap_{t\in T} G_t} = \cap_{t\in T}\mathbf{R}_{G_t}$. The rest equations are similar. For any $e \in E$, by Lemma 2.1 (4), we have

$\mathbf{R}_{\cap_{t\in T} G_t}(e) = (\cap_{t\in T} G_t)(e) = \cap_{t\in T} G_t(e) = \cap_{t\in T}\mathbf{R}_{G_t}(e) = (\cap_{t\in T}\mathbf{R}_{G_t})(e)$, so $\mathbf{R}_{\cap_{t\in T} G_t} = \cap_{t\in T}\mathbf{R}_{G_t}$.

From item (4) we will prove only the second equation. Since $sint(H) = \cup\{H_t \in \tau : H_t \subset H\}$, by item (3), $\mathbf{R}_{sint(H)} = \mathbf{R}_{\cup\{H_t\in\tau : H_t\subset H\}} = \cup\{\mathbf{R}_{H_t} \in \mathbf{R}_\tau : \mathbf{R}_{H_t} \subset \mathbf{R}_H\} = int(\mathbf{R}_H)$. That means $sint(H) = \mathbf{F}_{int(\mathbf{R}_H)}$.

The item (5) follows from (4).

From item (6) we will prove only the second equation.

By (4), (3), (2), $sint(H \cap G) = \mathbf{F}_{int(\mathbf{R}_{H\cap G})} = \mathbf{F}_{int(\mathbf{R}_H\cap\mathbf{R}_G)} = \mathbf{F}_{int(\mathbf{R}_H)\cap int(\mathbf{R}_G)} = \mathbf{F}_{int(\mathbf{R}_H)} \cap \mathbf{F}_{int(\mathbf{R}_G)} = sint(H) \cap sint(G)$. □

Definition 3.7. Let (E, U, τ) be a soft topological space and $(E \times U, \mathbf{R}_\tau)$ be the corresponding topological space.

(1) For any $e \in E$, $u \in U$, a system $\tau_e = \{F(e) : F \in \tau\}$, $\tau_u = \{\{e : u \in F(e)\} : F \in \tau\}$ defines a topological space denoted by (U, τ_e), (E, τ_u), respectively.

(2) By (U_e, τ_{U_e}) we denote a subspace of $(E \times U, \mathbf{R}_\tau)$ with a subspace topology $\tau_{U_e} = \{U_e \cap G : G \in \mathbf{R}_\tau\}$.

The previous facts allow us to provide an alternative definition of soft topological space.

Definition 3.8. A topological space $(E \times U, \tau)$ where τ is a topology on $E \times U$ is called a soft topological one.

In the following, however, we will continue to denote a soft topological space by (E, U, τ) and the corresponding topological space by $(E \times U, \mathbf{R}_\tau)$.

Remark 3.9. Let (E, U, τ) be a soft topological space and $(E \times U, \mathbf{R}_\tau)$ be the corresponding topological space. Since $\tau_e = \{F(e) : F \in \tau\}$ and $\tau_{U_e} = \{U_e \cap F : F \in \mathbf{R}_\tau\} = \{\{e\} \times F(e) : F \in \mathbf{R}_\tau\}$, a subspace (U_e, τ_{U_e}) of $(E \times U, \mathbf{R}_\tau)$ and a topological space (U, τ_e) are homeomorphic with a homeomorphism $f : U_e \to U$ given by $f((e, u)) = u$.

The next theorem follows from the one-to-one correspondence between soft topological spaces and topological spaces and from the definitions of soft normal, soft compact, soft Lindeöf and soft disconnected topological space, respectively (see for example [4, 6, 21]).

Theorem 3.10. *A soft topological space (E, U, τ) is soft normal (soft compact, soft Lindelöf, soft disconnected) if and only if its corresponding topological space $(E \times U, \mathbf{R}_\tau)$ is a normal (compact, Lindelöf, disconnected) topological space.*

Proof. Normality and soft normality: By Theorem 3.6 (1), H, G are disjoint soft closed if and only if $\mathbf{R}_H, \mathbf{R}_G$ are disjoint closed in $(E \times U, \mathbf{R}_\tau)$ and H_1, G_1 are soft open and disjoint containing H, G , respectively, if and only if $\mathbf{R}_{H_1}, \mathbf{R}_{G_1}$ are open and disjoint in $(E \times U, \mathbf{R}_\tau)$ containing $\mathbf{R}_H, \mathbf{R}_G$, respectively, by Lemma 2.1 (7).

Lindelöfness and soft Lindelöfness: There is a one-to-one correspondence between the family of soft open covers $\{G_t \in \tau : t \in T\}$ of (E, U, τ) and the family of open covers $\{\mathbf{R}_{G_t} \in \mathbf{R}_\tau : t \in T\}$ of $(E \times U, \mathbf{R}_\tau)$. So (E, U, τ) is soft compact (soft Lindelöf) if and only if $(E \times U, \mathbf{R}_\tau)$ is compact (Lindelöf).

Disconnectedness and soft disconnectedness: The union of two disjoint soft open sets H, G is equal to the full soft set if and only if the union of disjoint open sets $\mathbf{R}_H, \mathbf{R}_G$ is equal to $E \times U$. □

Many authors mention that a soft topology is not topological notion (not topology). But as we have shown, it can be identified with a topology on $E \times U$. The previous theorem shows that terms of soft theory are actually terms of general topology. For example, a soft topological subspace can be introduced by the following way: Let (E, U, τ) be a soft topological space and $Y \subset U$. Then the collection $\tau_{E \times Y} = \{(E \times Y) \cap \mathbf{R}_H : H \in \tau\}$ is a subspace topology on $E \times Y$ and a soft topological space given by $(E \times Y, \tau_{E \times Y})$ is a soft topological subspace of (E, U, τ), denoted by (E, Y, τ_Y). Similarly, we can reformulate each soft concept into an equivalent concept of general topology (known or new). We will briefly mention the issue of soft separation axioms which is diversified and quite problematic (see exhaustive work [2]). No matter how we introduce some soft separation axiom, we have the opportunity to study it within a general topology. Recall a few definitions well known from general topology. Two subsets A, B of a topological space are

- topologically distinguishable if they have not the same system of neighbourhoods (at least one of them has a neighbourhood that is not a neighbourhood of the other),
- mutually topologically distinguishable if each set has a neighbourhood that is not a neighbourhood of the other,
- topologically disjoint if there exists a neighbourhood of one set that is disjoint from the other set,
- separated if each set has a neighbourhoods that is disjoint from the other set (if each is disjoint from the other's closure),
- separated by neighbourhoods if they have disjoint neighbourhoods.

All conditions for separation of sets may also be applied to points (or to a point and a set) by using singleton sets. For example, two points x and y will be considered separated if and only if their singleton sets $\{x\}$ and $\{y\}$ are separated.

In the context of topology, the soft separation axioms can be defined in a soft topological space as follows. Two soft subsets A, B of soft topological space (E, U, τ) are soft topologically distinguishable, soft mutually topologically distinguishable, soft topologically disjoint, soft separated, soft separated by neighbourhoods if the corresponding subsets $\mathbf{R}_A$, $\mathbf{R}_B$ are topologically distinguishable, mutually topologically distinguishable, topologically disjoint, separated, separated by neighbourhoods in $(E \times U, \mathbf{R}_\tau)$, respectively.

In general topology, the separation axioms separate two different points or a closed set and a point or two disjoint closed sets. On the other hand in soft topological space the whole problem of diversity of soft separation axioms is caused by a wide range of selection and specification of soft sets A, B. Denote $f_u = E \times \{u\}$ where $u \in U$, $Z[e] = \{e\} \times Z$ where $e \in E$ and $\emptyset \neq Z \subset U$.

By selecting an appropriate soft separation axiom above and the following commonly used pairs of soft sets $A = \mathbf{F}_{f_u}$ and $B = \mathbf{F}_{f_v}$, $u \neq v$ (see for example [6], [16]) or $A = \mathbf{F}_{Z_1[e]}$ and $B = \mathbf{F}_{Z_2[f]}$, $Z_1[e] \cap Z_2[f] = \emptyset$ (see for example [12]), we can defined ten soft separation axioms.

For example (see [16]) a soft topological space (E, U, τ) is soft T_0-space (soft T_1-space, soft T_2-space) if and only if for any distinct points $u, v \in U$, f_u and f_v are topologically distinguishable (mutually topologically distinguishable, separated by neighbourhoods) in $(E \times U, \mathbf{R}_\tau)$ equivalently for any distinct points $u, v \in U$, $\mathbf{F}_{f_u}$ and $\mathbf{F}_{f_v}$ are soft topologically distinguishable (soft mutually topologically distinguishable, soft separated by neighbourhoods) in (E, U, τ). Moreover, (see [6]) a soft topological space (E, U, τ) is p-soft T_0-space (p-soft T_1-space, p-soft T_2-space) if and only if for any distinct points $u, v \in U$, f_u and f_v are topologically disjoint (separated, separated by neighbourhoods) in $(E \times U, \mathbf{R}_\tau)$ equivalently for any distinct points $u, v \in U$, $\mathbf{F}_{f_u}$ and $\mathbf{F}_{f_v}$ are soft topologically disjoint (soft separated, soft separated by neighbourhoods) in (E, U, τ).

Similar characterization holds for the pair $A = \mathbf{F}_{Z_1[e]}$ and $B = \mathbf{F}_{Z_2[f]}$, $Z_1[e] \cap Z_2[f] = \emptyset$ (see [12]). Some selected problems related to soft separation axioms were solved in [15]. Moreover we can separate by an appropriate soft separation axiom a soft point F_e^u and a soft set $\mathbf{F}_{f_v}$ for $u \neq v$ (F_e^u and a soft set $\mathbf{F}_{Z[f]}$ for $e \neq f$, F_e^u and a soft set $\mathbf{F}_{Z[e]}$ for $u \notin Z$). There are many other combinations and it is a challenge for further research to introduce a systematic and uniform approach to the soft separation axioms (see Conclusion).

For more details we recommend [2] describing different types of soft separation axioms as well as the causes of diversity including the development of the definitions of soft points, the ways of defining the distinct soft points and the different forms of belong and non-belong relations.

4. Topological Sum and Homogeneous Topological Spaces

In this section we recall some topological notions and facts concerning a topological sum and a homogeneous space which will be used in Section 6, 7 and 8.

Let $\{(X_i, \tau_i),\ i \in I\}$ be collection of topological spaces. A topological sum denoted by $\oplus_{i\in I}(X_i, \tau_i)$ is a topological space $(\oplus_{i\in I}X_i, \oplus_{i\in I}\tau_i)$, where $\oplus_{i\in I}X_i$ is a disjoint union of X_i ($\oplus_{i\in I}X_i = \cup_{i\in I}\{i\} \times X_i$) and $\oplus_{i\in I}\tau_i$ is a topology defined as the finest topology on $\oplus_{i\in I}X_i$ for which all canonical injections φ_i are continuous, where $\varphi_i : X_i \to \oplus_{i\in I}X_i$ is defined by $\varphi_i(x) = (i, x)$ for $x \in X_i$.

Definition 4.1. ([9, 10]) A topological space (X, τ) is homogeneous if for any points x, y there is a homeomorphism $f : X \to X$ for which $f(x) = y$.

The next theorem presents the basic properties of topological sum (see [8]) and it can be useful for further investigation of enriched soft ([4]) and soft homogeneous topological spaces ([1]).

Theorem 4.2. *Let $\{(X_i, \tau_i),\ i \in I\}$ be collection of topological spaces and $A \subset \oplus_{i\in I}X_i$. Then:*

(1) a set A is open (closed) in $\oplus_{i\in I}(X_i, \tau_i)$ if and only if $\varphi_i^{-1}(A)$ is open (closed) in (X_i, τ_i) for any $i \in I$,

(2) If $X_i \neq \emptyset$ for all $i \in I$, then $\oplus_{i\in I}(X_i, \tau_i)$ is compact if and only if (X_i, τ_i) is compact for any $i \in I$ and I is finite,

(3) a canonical injection φ_i is a continuous, open and closed map for any $i \in I$,

(4) a map $f : \oplus_{i\in I}(X_i, \tau_i) \to (Y, \sigma)$ is continuous if and only if $f \circ \varphi_i : (X_i, \tau_i) \to (Y, \sigma)$ is continuous for any $i \in I$.

(5) If a topological space (X, τ) can be represented as the union of family $\{X_s\}_{s\in S}$ of pairwise disjoint open subsets, then $\tau = \oplus_{s\in S}\tau_{X_s}$ and a map $f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if $f_s : (X_s, \tau_{X_s}) \to (Y, \sigma)$ is continuous for any $s \in S$, where f_s is a restriction of f to X_s and τ_{X_s} is a subspace topology.

(6) If each (X_i, τ_i) is homeomorphic to a fixed topological space (Z, θ), then the disjoint union $\oplus_{i\in I}(X_i, \tau_i)$ is homeomorphic to a product space $(I \times Z, \tau_{dis} \times \theta)$, where τ_{dis} is the discrete topology on I.

(7) ([9]) Let (X, τ) be a topological space which contains a nonempty open set G such that (G, τ_G) is indiscrete. Then (X, τ) is homogeneous if and only if (X, τ) is homeomorphic to a topological sum of indiscrete topological subspaces of (X, τ) all of which are homeomorphic to one another.

(8) ([10]) A finite topological space is homogeneous if and only if it is homeomorphic to the Cartesian product of a finite discrete topological space and a finite indiscrete topological one.

5. Soft Mappings

In soft theory (see for example [1, 3, 4, 6, 13]) a soft mapping f_{pu} between two families of soft sets $SS(E_1, U_1)$ and $SS(E_2, U_2)$ is usually defined by two mappings $u : E_1 \to E_2$ and $p : U_1 \to U_2$. If $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$, then the image $f_{pu}(A)$ of A, the inverse image $f_{pu}^{-1}(B)$ of B under f_{pu} is defined as a soft set from $SS(E_2, U_2)$, $SS(E_1, U_1)$ given by

$$f_{pu}(A) : e_2 \mapsto \cup_{e_1 \in u^{-1}(e_2)} p(A(e_1)) \text{ for } e_2 \in E_2, \quad f_{pu}^{-1}(B) : e_1 \mapsto p^{-1}(B(u(e_1))) \text{ for } e_1 \in E_1, \text{ respectively.}$$

A soft mapping $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$ (a restriction of f_{pu} to $SP(E_1, U_1)$) is the corresponding soft mapping to the Cartesian product $[u \times p]$ of u and p (see Theorem 5.3 below).

Definition 5.1. Let $u : E_1 \to E_2$, $p : U_1 \to U_2$. A Cartesian product $[u \times p]$ of u and p is a function from $E_1 \times U_1$ to $E_2 \times U_2$ for which $[u \times p]((e_1, u_1)) = (u(e_1), p(u_1))$. The image of $A \subset E_1 \times U_1$, the inverse image of $B \subset E_2 \times U_2$ is denoted by $[u \times p](A)$, $[u \times p]^{-1}(B)$, respectively.

Lemma 5.2. *A function $[u \times p]$ is injective (surjective, bijective) if and only if u and p are injective (surjective, bijective) and if $A_i \subset E_i$, $B_i \subset U_i$, $i = 1, 2$, then*

$$[u \times p]^{-1}(A_2 \times B_2) = u^{-1}(A_2) \times p^{-1}(B_2), \quad [u \times p](A_1 \times B_1) = u(A_1) \times p(B_1).$$

Theorem 5.3. *Let $u : E_1 \to E_2$, $p : U_1 \to U_2$, $A, F_{e_1}^{u_1} \in SS(E_1, U_1)$, $B, F_{e_2}^{u_2} \in SS(E_2, U_2)$ and $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$ (a restriction of f_{pu} to $SP(E_1, U_1)$). Then*

$$f_{pu}(A) = \mathbf{F}_{[u \times p](\mathbf{R}_A)}, \quad f_{pu}(F_{e_1}^{u_1}) = \mathbf{F}_{\{(u(e_1), p(u_1))\}} = F_{u(e_1)}^{p(u_1)}, \quad f_{pu}^{-1}(B) = \mathbf{F}_{[u \times p]^{-1}(\mathbf{R}_B)}, \quad f_{pu}^{-1}(F_{e_2}^{u_2}) = \mathbf{F}_{u^{-1}(\{e_2\}) \times p^{-1}(\{u_2\})}.$$

Proof. We will prove that the values of $f_{pu}(A)$ and $\mathbf{F}_{[u \times p](\mathbf{R}_A)}$ at any $e_2 \in E_2$ are equal. By Lemma 2.1 (9) and Lemma 5.2, $[u \times p](\mathbf{R}_A) = [u \times p](\cup_{e_1 \in E_1} \{e_1\} \times A(e_1)) = \cup_{e_1 \in E_1} [u \times p](\{e_1\} \times A(e_1)) = \cup_{e_1 \in E_1} \{u(e_1)\} \times p(A(e_1))$. That means, by Lemma 2.1 (1), $\mathbf{F}_{[u \times p](\mathbf{R}_A)}(e_2) = ([u \times p](\mathbf{R}_A))(e_2) = \cup_{e_1 \in u^{-1}(e_2)} p(A(e_1)) = f_{pu}(A)(e_2)$.

The equation $f_{pu}^{-1}(B) = \mathbf{F}_{[u \times p]^{-1}(\mathbf{R}_B)}$ is similar and the equations for the soft points are the consequences of previous proven equations. □

Recall that f_{pu} is a special mapping. Generally, if every soft point $F_{e_1}^{u_1}$ in $SP(E_1, U_1)$ is uniquely associated with a soft point $h(F_{e_1}^{u_1})$ in $SP(E_2, U_2)$, then we say that a soft mapping $h : SP(E_1, U_1) \to SP(E_2, U_2)$ is given (compare with [21]).

In the following definition we introduce a one-to-one correspondence between a mapping $g : E_1 \times U_1 \to E_2 \times U_2$ (a soft mapping $h : SP(E_1, U_1) \to SP(E_2, U_2)$) and a soft mapping $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$ (a mapping $\Psi_h : E_1 \times U_1 \to E_2 \times U_2$).

Definition 5.4. Let $g : E_1 \times U_1 \to E_2 \times U_2$. Then a soft mapping $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$ is defined by

$$\Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow g((e_1, u_1)) = (e_2, u_2), \text{ equivalently, } \Phi_g(F_{e_1}^{u_1}) = \mathbf{F}_{\{g((e_1, u_1))\}}.$$

On the other hand, if $h : SP(E_1, U_1) \to SP(E_2, U_2)$ is a soft mapping, then for $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$, we define the image of A, the inverse image of B under h by

$$h(A) = \cup\{h(F_{e_1}^{u_1}) : F_{e_1}^{u_1} \text{ is a soft subset of } A\}, \quad h^{-1}(B) = \cup\{F_{e_1}^{u_1} : h(F_{e_1}^{u_1}) \text{ is a soft subset of } B\},$$

respectively, and a function $\Psi_h : E_1 \times U_1 \to E_2 \times U_2$ is defined by

$$\Psi_h((e_1, u_1)) = (e_2, u_2) \Leftrightarrow h(F_{e_1}^{u_1}) = F_{e_2}^{u_2}.$$

A soft mapping Φ_g, f_{pu}, h and a mapping g, $[u \times p]$, Ψ_h are said to be mutually correspondence, respectively. A soft mapping f_1 and a mapping f_2 are mutually correspondence if $\Psi_{f_1} = f_2$ and $\Phi_{f_2} = f_1$.

Since $\Psi_{\Phi_g}((e_1, u_1)) = (e_2, u_2) \Leftrightarrow \Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow g((e_1, u_1)) = (e_2, u_2)$, $\Psi_{\Phi_g} = g$ and since $\Phi_{\Psi_h}(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow \Psi_h((e_1, u_1)) = (e_2, u_2) \Leftrightarrow h(F_{e_1}^{u_1}) = F_{e_2}^{u_2}$, $\Phi_{\Psi_h} = h$.

The next theorem says $\Phi_g(A)$ ($\Phi_g^{-1}(B)$) is a soft set for which its corresponding relation is equal to the image (the inverse image) of the corresponding relation $\mathbf{R}_A$ ($\mathbf{R}_B$) of A (B) under $g : E_1 \times U_1 \to E_2 \times U_2$ and $\Psi_h(C)$ ($\Psi_h^{-1}(D)$) is a relation for which its corresponding soft set is equal to the image (the inverse image) of the corresponding soft set $\mathbf{F}_C$ ($\mathbf{F}_D$) of C (D) under $h : SP(E_1, U_1) \to SP(E_2, U_2)$.

Theorem 5.5. *Let $g : E_1 \times U_1 \to E_2 \times U_2$, $h : SP(E_1, U_1) \to SP(E_2, U_2)$. If $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$ and $C \subset E_1 \times U_1$, $D \subset E_2 \times U_2$, then*

$$\Phi_g(A) = \mathbf{F}_{g(\mathbf{R}_A)} \qquad \Phi_g^{-1}(B) = \mathbf{F}_{g^{-1}(\mathbf{R}_B)} \qquad \Psi_h(\mathbf{R}_A) = \mathbf{R}_{h(A)} \qquad \Psi_h^{-1}(\mathbf{R}_B) = \mathbf{R}_{h^{-1}(B)}$$
$$\Phi_g(\mathbf{F}_C) = \mathbf{F}_{g(C)} \qquad \Phi_g^{-1}(\mathbf{F}_D) = \mathbf{F}_{g^{-1}(D)} \qquad \Psi_h(C) = \mathbf{R}_{h(\mathbf{F}_C)} \qquad \Psi_h^{-1}(D) = \mathbf{R}_{h^{-1}(\mathbf{F}_D)}$$

consequently, $\Phi_g^{-1}(F_{e_2}^{u_2}) = \mathbf{F}_{g^{-1}(\{(e_2,u_2)\})}$ *and* $\Psi_h^{-1}(\{(e_2, u_2)\}) = \mathbf{R}_{h^{-1}(F_{e_2}^{u_2})}$.

Proof. Let $\{F_{e_{1,t}}^{u_{1,t}} : t \in T\}$ be an indexed family of soft points which union is equal to A, so $A = \cup_{t\in T} F_{e_{1,t}}^{u_{1,t}}$. Clearly $F_{e_{1,t}}^{u_{1,t}}$ is a soft subset of A if and only if $(e_{1,t}, u_{1,t}) \in \mathbf{R}_A$. Then

$$\Phi_g(A) = \Phi_g(\cup_{t\in T} F_{e_{1,t}}^{u_{1,t}}) = \cup_{t\in T}\Phi_g(F_{e_{1,t}}^{u_{1,t}}) = \cup_{t\in T}\mathbf{F}_{\{g((e_{1,t},u_{1,t}))\}} = \mathbf{F}_{\cup_{t\in T}\{g((e_{1,t},u_{1,t}))\}} = \mathbf{F}_{g(\mathbf{R}_A)}.$$

Similarly we can prove the rest equations. □

By Lemma 2.1 (3), we have the next lemma.

Lemma 5.6. *If $g : E_1 \times U_1 \to E_2 \times U_2$, then:*
(1) $\Phi_g(A) = \mathbf{F}_{g(\mathbf{R}_A)} \Leftrightarrow \mathbf{R}_{\Phi_g(A)} = g(\mathbf{R}_A)$,
(2) $\Phi_g^{-1}(B) = \mathbf{F}_{g^{-1}(\mathbf{R}_B)} \Leftrightarrow \mathbf{R}_{\Phi_g^{-1}(B)} = g^{-1}(\mathbf{R}_B)$,
(3) $\Phi_g(F_{e_1}^{u_1}) = \mathbf{F}_{\{g((e_1,u_1))\}} \Leftrightarrow \mathbf{R}_{\Phi_g(F_{e_1}^{u_1})} = \{g((e_1, u_1))\}$,
(4) $\Phi_g^{-1}(F_{e_2}^{u_2}) = \mathbf{F}_{g^{-1}(\{(e_2,u_2)\})} \Leftrightarrow \mathbf{R}_{\Phi_g^{-1}(F_{e_2}^{u_2})} = g^{-1}(\{(e_2, u_2)\})$.

Lemma 5.7. *If (E_1, U_1, τ), (E_2, U_2, θ) are soft topological spaces and $g : E_1 \times U_1 \to E_2 \times U_2$, then:*
(1) $\Phi_g(A) \in \theta \Leftrightarrow g(\mathbf{R}_A) \in \mathbf{R}_\theta$, $A \in \tau$,
(2) $\Phi_g(\mathbf{F}_A) \in \theta \Leftrightarrow g(A) \in \mathbf{R}_\theta$, $A \in \mathbf{R}_\tau$,
(3) $\Phi_g^{-1}(B) \in \tau \Leftrightarrow g^{-1}(\mathbf{R}_B) \in \mathbf{R}_\tau$, $B \in \theta$,
(4) $\Phi_g^{-1}(\mathbf{F}_B)) \in \tau \Leftrightarrow g^{-1}(B) \in \mathbf{R}_\tau$, $B \in \mathbf{R}_\theta$.

Proof. We will prove item (1), the others are similar. By Theorem 5.5, $\Phi_g(A) = \mathbf{F}_{g(\mathbf{R}_A)}$. By Theorem 3.4 (1) and Lemma 2.1 (3), $\Phi_g(A) \in \theta$ if and only if $\mathbf{R}_{\Phi_g(A)} = g(\mathbf{R}_A) \in \mathbf{R}_\theta$. □

Analogous two lemmas to those above are also true for a soft mapping $h : SP(E_1, U_1) \to SP(E_2, U_2)$ and the corresponding mapping $\Psi_h : E_1 \times U_1 \to E_2 \times U_2$.

6. Soft Continuity and Soft Homogeneity

Definition 6.1. Let (E_1, U_1, τ), (E_2, U_2, θ) be soft topological spaces and $g : E_1 \times U_1 \to E_2 \times U_2$. Then the soft mapping $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$

(1) is soft continuous if $\Phi_g^{-1}(B) \in \tau$ for any $B \in \theta$,

(2) is soft open if $\Phi_g(A) \in \theta$ for any $A \in \tau$,

(3) is a soft homeomorphism if it is soft continuous and soft open bijection.

(4) (E_1, U_1, τ) and (E_2, U_2, θ) are $\times$-soft homeomorphic if there are $u : E_1 \to E_2$ and $p : U_1 \to U_2$ such that $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$ is a soft homeomorphism.

(5) (E_1, U_1, τ) and (E_2, U_2, θ) are soft homeomorphic if there is $g : E_1 \times U_1 \to E_2 \times U_2$ such that $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$ is a soft homeomorphism.

Definition 6.2. A soft topological space (E, U, τ) is said to be soft homogeneous ($\times$-soft homogeneous, see [1]) if for any soft points P_e^v, Q_f^w there is a soft homeomorphism Φ_g (f_{pu}) from $SP(E, U)$ to $SP(E, U)$ such that $\Phi_g(P_e^v) = Q_f^w$ ($f_{pu}(P_e^v) = Q_f^w$.

The following theorem confirms that the soft continuity properties of soft mappings can be characterized by the corresponding topological continuity properties of the corresponding functions and its proof follows from Lemma 5.7.

Theorem 6.3. *Let (E_1, U_1, τ), (E_2, U_2, θ) be soft topological spaces. Then*

(1) $\Phi_g, f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$ is soft continuous (soft open, a soft homeomorphism) if and only if g, $[u \times p] : (E_1 \times U_1, \mathbf{R}_\tau) \to (E_2 \times U_2, \mathbf{R}_\theta)$ is continuous (open, a homeomorphism),

(2) (E_1, U_1, τ) and (E_2, U_2, θ) are soft homeomorphic if and only if $(E_1 \times U_1, \mathbf{R}_\tau)$ and $(E_2 \times U_2, \mathbf{R}_\theta)$ are homeomorphic.

(3) (E_1, U_1, τ) and (E_2, U_2, θ) are $\times$-soft homeomorphic if and only if there are $u : E_1 \to E_2$ and $p : U_1 \to U_2$ such that $[u \times p] : (E_1 \times U_1, \mathbf{R}_\tau) \to (E_2 \times U_2, \mathbf{R}_\theta)$ is a homeomorphism.

(4) A soft topological space (E_1, U_1, τ) is $\times$-soft homogeneous if and only if for any points $(e, v), (f, w) \in E_1 \times U_1$ there are $u : E_1 \to E_1$ and $p : U_1 \to U_1$ such that $[u \times p] : (E_1 \times U_1, \mathbf{R}_\tau) \to (E_1 \times U_1, \mathbf{R}_\tau)$ is a homeomorphism and $[u \times p]((e, v)) = (f, w)$.

(5) A soft topological space is soft homogeneous if and only if its corresponding topological space is homogenous.

Remark 6.4. Note an $\times$-soft homeomorphism ($\times$-soft homogeneity), in the literature it is referred as a soft homeomorphism (a soft homogeneity) with respect to f_{pu}, is a special case of soft homeomorphism (soft homogeneity), see Example 6.5 and Example 7.6.

For a priori given soft function h, we define the terms of Definition 6.1 (items (1), (2), (3), (5)) analogously. Namely, h is soft continuous (soft open) if the inverse image (the image) of any soft open set is soft open. In the context of the previous theorem a soft function is soft continuous (soft open) if and only if its corresponding function is continuous (open).

Example 6.5. Let $E_1 = \{a, b\}$, $U_1 = \{u, v\}$, $E_2 = \{c, d\}$, $U_2 = \{s, t\}$, $A = \{(a, u), (b, v)\}$, $B = \{(d, s), (d, t)\}$ and $\tau = \{\mathbf{F}_\emptyset, \mathbf{F}_{E_1 \times U_1}, \mathbf{F}_A\}$ and $\theta = \{\mathbf{F}_\emptyset, \mathbf{F}_{E_2 \times U_2}, \mathbf{F}_B\}$. Define a function $g : E_1 \times U_1 \to E_2 \times U_2$ by the following way: $g((a, u)) = (d, s)$, $g((b, u)) = (c, s)$, $g((b, v)) = (d, t)$ and $g((a, v)) = (c, t)$. Φ_g is a soft homeomorphism but there is no soft homeomorphism f_{pu} since $[u \times p]^{-1}(\{d\} \times U_2) = u^{-1}(\{d\}) \times p^{-1}(U_2) \neq A$ (because A is not a Cartesian product, by Lemma 5.2).

Theorem 6.6. *Let (E_1, τ_1), (U_1, θ_1), (E_2, τ_2), (U_2, θ_2) be topological spaces. If $u : E_1 \to E_2$, $p : U_1 \to U_2$, then $[u \times p] : (E_1 \times U_1, \tau_1 \times \theta_1) \to (E_2 \times U_2, \tau_2 \times \theta_2)$*

(1) is continuous if and only if $p : (U_1, \theta_1) \to (U_2, \theta_2)$ and $u : (E_1, \tau_1) \to (E_2, \tau_2)$ are continuous,

(2) is open if and only if $p : (U_1, \theta_1) \to (U_2, \theta_2)$ and $u : (E_1, \tau_1) \to (E_2, \tau_2)$ are open.

(3) A topological space $(E_1 \times U_1, \tau_1 \times \theta_1)$ is homogenous if and only if (E_1, τ_1) and (U_1, θ_1) are homogenous if and only if for any $(e_1, u_1), (e_2, u_2) \in E_1 \times U_1$ there are $u : E_1 \to E_1$ and $p : U_1 \to U_1$ such that $[u \times p]$ is a homeomorphism from $(E_1 \times U_1, \tau_1 \times \theta_1)$ to $(E_1 \times U_1, \tau_1 \times \theta_1)$ and $[u \times p]((e_1, u_1)) = (e_2, u_2)$.

Proof. The items (1), (2) are well known topological results (see [8]).

Item (3): Let $(E_1 \times U_1, \tau_1 \times \theta_1)$ be homogenous and $u_1, v_1 \in U_1$, $e_1 \in E_1$. Then there is a homomorphism $f : (E_1 \times U_1, \tau_1 \times \theta_1) \to (E_1 \times U_1, \tau_1 \times \theta_1)$ such that $f((e_1, u_1)) = (e_1, v_1)$. Since $\{e_1\} \times U_1$ is homeomorphic to U_1 where $p : (e_1, x) \mapsto x$ is the required homeomorphism, $p \circ f \circ p^{-1}$ is a homeomorphism from (U_1, θ_1) to (U_1, θ_1) and $p \circ f \circ p^{-1}(u_1) = p(f((e_1, u_1))) = p((e_1, v_1)) = v_1$. That means (U_1, θ_1) is homogenous. Similarly we can prove that (E_1, τ_1) is homogenous.

Let (E_1, τ_1), (U_1, θ_1) be homogenous spaces and $(e_1, u_1), (e_2, u_2) \in E_1 \times U_1$. Then there is a homeomorphism u from (E_1, τ_1) to (E_1, τ_1) and a homeomorphism p from (U_1, θ_1) to (U_1, θ_1) such that $u(e_1) = e_2$ and $p(u_1) = u_2$. Then $[u \times p] : (E_1 \times U_1, \tau_1 \times \theta_1) \to (E_1 \times U_1, \tau_1 \times \theta_1)$ is a homeomorphism (see Lemma 5.2) and $[u \times p]((e_1, u_1)) = (u(e_1), p(u_1)) = (e_2, u_2)$. The last implication is obvious. □

The next corollary is a consequence of Theorem 6.3 and Theorem 6.6 which reflects the fact that the whole approach to soft topology is based on topological methods including the study of soft continuity and operations on soft topological spaces.

Corollary 6.7. *Let (E_1, τ_1), (U_1, θ_1), (E_2, τ_2), (U_2, θ_2) be topological spaces and (E_1, U_1, μ), (E_2, U_2, σ) be soft topological spaces given by $(E_1 \times U_1, \tau_1 \times \theta_1)$, $(E_2 \times U_2, \tau_2 \times \theta_2)$, respectively.*

If $u : E_1 \to E_2$, $p : U_1 \to U_2$, then a soft mapping $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$

(1) is soft continuous if and only if $p : (U_1, \theta_1) \to (U_2, \theta_2)$ and $u : (E_1, \tau_1) \to (E_2, \tau_2)$ are continuous,

(2) is soft open if and only if $p : (U_1, \theta_1) \to (U_2, \theta_2)$ and $u : (E_1, \tau_1) \to (E_2, \tau_2)$ are open.

(3) (E_1, U_1, μ) is soft homogenous if and only if (E_1, τ_1) and (U_1, θ_1) are homogenous if and only if (E_1, U_1, μ) is $\times$-soft homogenous.

7. Soft Stable Topological Spaces

Definition 7.1. ([3]) A soft topological space (E, U, τ) is called stable if $\tau \subset \mathbf{C}(E, U)$.

The next theorem characterizes a stable soft topological space and in its light all assertions numbered from 3.15 to 3.22 in [1] are obvious.

Theorem 7.2. *A soft topological space (E, U, τ) is stable if and only if its corresponding topological space is given by $(E \times U, \tau_{ind} \times \theta)$, where τ_{ind} is the indiscrete topology on E and (U, θ) is a topological space (we say (E, U, τ) is generated by (U, θ)).*

Proof. Let (E, U, τ) be a stable soft topological space. Then $H \in \tau \subset \mathbf{C}(E, U)$ if and only if $\mathbf{R}_H = E \times A$, where $A = H(e)$ for any $e \in E$. Then $\theta = \{A : \mathbf{R}_H = E \times A, H \in \tau\}$ is a topology on U if and only if τ is a soft topology. That means $(E \times U, \tau_{ind} \times \theta)$ is the corresponding topological space. □

Since a stable soft topological space and a function f_{pu} are specific cases, many soft properties of stable soft topological spaces and soft continuity of f_{pu} are strictly connected to the properties of function p. The next theorem covers the following theorems and corollaries of [1] numbered as 5.19–5.24 and 5.31–5.33.

Theorem 7.3. *Suppose (E_1, U_1, τ_1), (E_2, U_2, τ_2) are stable soft topological spaces generated by (U_1, θ_1), (U_2, θ_2), respectively, and $p : (U_1, \theta_1) \to (U_2, \theta_2)$.*
Then $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$ is:

(1) soft continuous if and only if p is continuous,

(2) soft open if and only if p is open, provided u is surjective,

(3) a soft homeomorphism if and only if p is a homeomorphism, provided u is bijective.

Proof. Since any function (surjection, bijection) u from an indiscrete space to an indiscrete space is continuous (open, a homeomorphism) and the corresponding soft mapping to f_{pu} is equal to $[u \times p]$, a proof follows from Corollary 6.7. □

Theorem 7.4. *If (E, U, τ) is a stable soft topological space generated by a topological space (U, θ), then the following conditions are equivalent.*

(1) (E, U, τ) is $\times$-soft homogeneous,

(2) (E, U, τ) is soft homogeneous,

(3) (U, θ) is homogenous.

Proof. Since any indiscrete topological space is homogeneous and the corresponding soft mapping to f_{pu} is equal to $[u \times p]$, a proof follows from Theorem 6.3, Theorem 6.6 (3) and Theorem 7.2. □

Remark 7.5. By Theorem 4.2, the soft topological spaces in Examples 5.3–5.5 of [1] are trivially soft homogeneous and $\times$-soft homogeneity follows from vertical, horizontal and diagonal symmetry.

Example 7.6. Let $E = \{e_1, e_2\}$, $U = \{u_1, u_2, u_3\}$, $A_{ij} = (e_i, u_j)$ $(i = 1, 2, j = 1, 2, 3)$, $A = \{A_{11}, A_{12}, A_{21}\}$, $B = \{A_{22}, A_{23}, A_{13}\}$. Then a topological space $(E \times U, \theta)$ where $\theta = \{\emptyset, E \times U, A, B\}$ is homogeneous (by Theorem 4.2 (7)) and the corresponding soft topological space (E, U, τ) where $\tau = \mathbf{F}_\theta = \{\mathbf{F}_\emptyset, \mathbf{F}_{E\times U}, \mathbf{F}_A, \mathbf{F}_B\}$ is soft homogeneous (by Theorem 6.3 (5)) but it is not $\times$-homogeneous, since both topological spaces (U, τ_{e_1}), (U, τ_{e_2}) are not homogenous, by Theorem 5.14 of [1]. The soft topological space (E, U, τ) is soft homeomorphic to a soft topological space over three point set with respect to two point set which corresponding topological space is homeomorphic to a topological sum, by Theorem 4.2 (7), (to the Cartesian product of a discrete space and an indiscrete space, by Theorem 4.2 (8)).

8. Soft Enriched Topological Spaces

Let $\{(U, \theta_e) : e \in E\}$ be an indexed family of topological spaces, $U_e = \{e\} \times U$. Since $U_{e_1} \cap U_{e_2} = \emptyset$ for $e_1 \neq e_2$ and $\oplus_{e\in E} U = \cup_{e\in E} U_e = E \times U$, $\oplus_{e\in E}\theta_e$ is a topology on $E \times U$ with the basis $\mathcal{B} = \{\{\varphi_e(G) : G \in \theta_e\} : e \in E\} = \{\{\{e\} \times G : G \in \theta_e\} : e \in E\}$ where $\varphi_e : U \to E \times U$ is the canonical injection given by $\varphi_e(u) = (e, u)$. This allows us to define a soft topological space generated by an indexed family of topological spaces ([1]).

Definition 8.1. A soft topological space generated by an indexed family $\{(U, \theta_e) : e \in E\}$ of topological spaces is given by the topological space $(E \times U, \oplus_{e\in E}\theta_e)$ and it is denoted by $(E, U, \mathbf{F}_{\oplus_{e\in E}\theta_e})$, where $\mathbf{F}_{\oplus_{e\in E}\theta_e}$ is the corresponding soft topology to $\oplus_{e\in E}\theta_e$, i.e., $F \in \mathbf{F}_{\oplus_{e\in E}\theta_e}$ if and only if $F(e) \in \theta_e$ for any $e \in E$ (see [1]).

The next theorem follows directly from definition of an extended soft topology (see Proposition 3 and Remark 3 of [4]) and from Theorem 4.2 (5).

Theorem 8.2. *A soft topological space (E, U, τ) is extended if and only if (E, U, τ) is enriched if and only if its corresponding topological space is equal to $(E \times U, \oplus_{e\in E}\tau_e)$.*

Remark 8.3. Note some operations that generate soft topologies known from [1, 20].

(1) For a topological space (U, Σ) (see [20]), two soft topologies $\mathcal{T}(\Sigma$ and $\hat{\mathcal{T}}(\Sigma)$ were introduced. The soft topology $\mathcal{T}(\Sigma)$, $\hat{\mathcal{T}}(\Sigma)$ is given by $\tau_{dis} \times \Sigma$, $\tau_{ind} \times \Sigma$, where τ_{dis}, τ_{ind} is the discrete, the indiscrete topology on E, respectively (see Theorem 4.2 (6) and Theorem 7.2).

(2) For a topological space $(U, \Im)$, an indexed family of topological spaces $\{(U, \Im_e) : e \in E\}$ (see [1]), a soft topology $\tau(\Im)$, $\oplus_{e\in E}\Im_e$ was introduced, respectively. The soft topology $\tau(\Im)$, $\oplus_{e\in E}\Im_e$ is given by $\tau_{dis} \times \Im$, the topological sum of $\{(U, \Im_e) : e \in E\}$, respectively.

In the end we mention a theorem dealing with a soft continuity of soft mapping defined on an extended soft topological space. It generalizes Theorem 4 of [4] that only talks about a soft continuity of f_{pu}.

Theorem 8.4. *Let (E_1, U_1, τ) be an enriched (extended) soft topological space, (E_2, U_2, θ) be a soft topological space and g be a function from $E_1 \times U_1$ to $E_2 \times U_2$. Then the next conditions are equivalent.*

(1) $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$ is soft continuous,

(2) for any $e_1 \in E_1$, $g : \{e_1\} \times U_1 \to (E_2 \times U_2, \mathbf{R}_\theta)$ is continuous where $\{e_1\} \times U_1$ is the subspace of $(E_1 \times U_1, \mathbf{R}_\tau)$,

(3) for any $e_1 \in E_1$, $g \circ \varphi_{e_1} : (U_1, \tau_{e_1}) \to (E_2 \times U_2, \mathbf{R}_\theta)$ is continuous.

Proof. Since (E_1, U_1, τ) is enriched, the corresponding topological space $(E_1 \times U_1, \mathbf{R}_\tau)$ can be represented as a union of family $\{\{e_1\} \times U_1 : e_1 \in E_1\}$ of pairwise disjoint open subsets of $(E_1 \times U_1, \mathbf{R}_\tau)$ and a proof follows from Theorem 4.2 (5) and (4) and Theorem 6.3. □

9. Conclusion

From what has been shown in the article, the whole issue of soft theory and its applications can be developed as part of general topology. We suggest identifying a soft set with a subset of the Cartesian product. For example, in the set valued analysis if $F \in \mathbf{F}(E, U)$, it is commonly used a notation $F \subset E \times U$ that means a set valued mapping $F : E \to 2^U$ is identified with its graph and a set valued mapping F_e^u (a soft point) is identified with $\{(e, u)\}$. Each soft topological term has a corresponding topological term and vice versa, which should be taken into account in further research and applications. From this point of view, it is more convenient to define a soft topology as a topology on the Cartesian product and to use known methods of general topology. The correspondence from the view of categorical theory would be desirable for further research.

As for the separation axioms, the fundamental problem is to define the soft point. For a nonempty family $\mathcal{M} \subset SS(E, U) = \mathbf{F}(E, U)$ (without the null soft set) it would be appropriate to introduce a topology on $\mathbf{R}_{\mathcal{M}} = \{\mathbf{R}_M : M \in \mathcal{M}\}$ by using appropriate hypertopology where the underlying set is $\mathbf{R}_{\mathcal{M}}$. So a soft point is an element of $\mathbf{R}_{\mathcal{M}}$ (a graph of set valued mapping from $\mathcal{M}$). More precisely, over a topological space $(E \times U, \tau)$, a hyperspace is a topological space whose points are the elements from $\mathbf{R}_{\mathcal{M}}$ with some topology. For example $\mathbf{R}_{\mathcal{M}}$ can be considered as the hyperspace equipped with the lower Vietoris, the upper Vietoris, the Vietoris topology, respectively.

Below we summarize the correspondence between topological and soft topological objects.

topological object $\mapsto$ **soft topological object**	**soft topological object** $\mapsto$ **topological object**
$\{(e,u)\} \mapsto F_e^u = \mathbf{F}_{\{(e,u)\}}$	$F_e^u \mapsto \{(e,u)\} = \mathbf{R}_{F_e^u}$
$A \mapsto \mathbf{F}_A$	$A \mapsto \mathbf{R}_A$
$\tau \mapsto \mathbf{F}_\tau$	$\tau \mapsto \mathbf{R}_\tau$
$(E \times U, \tau) \mapsto (E, U, \mathbf{F}_\tau)$	$(E, U, \tau) \mapsto (E \times U, \mathbf{R}_\tau)$
$g \mapsto \Phi_g$	$h \mapsto \Psi_h$
$g(A) \mapsto \Phi_g(\mathbf{F}_A) = \mathbf{F}_{g(A)}$	$h(A) \mapsto \Psi_h(\mathbf{R}_A) = \mathbf{R}_{h(A)}$
$g^{-1}(B) \mapsto \Phi_g^{-1}(\mathbf{F}_B) = \mathbf{F}_{g^{-1}(B)}$	$h^{-1}(B) \mapsto \Psi_h^{-1}(\mathbf{R}_B) = \mathbf{R}_{h^{-1}(B)}$
$[u \times p] \mapsto f_{pu}$	$f_{pu} \mapsto [u \times p]$
$[u \times p](A) \mapsto f_{pu}(\mathbf{F}_A) = \mathbf{F}_{[u\times p](A)}$	$f_{pu}(A) \mapsto [u \times p](\mathbf{R}_A)$
$[u \times p]^{-1}(B) \mapsto f_{pu}^{-1}(\mathbf{F}_B) = \mathbf{F}_{[u\times p]^{-1}(B)}$	$f_{pu}^{-1}(B) \mapsto [u \times p]^{-1}(\mathbf{R}_B)$

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