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Some Norm Inequalities for Upper Sector Matrices

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Abstract. We generalize some norm inequalities for 2×2 block accretive-dissipative matrices and positive semi-definite matrices that compare the diagonal blocks with the off-diagonal blocks. Moreover, we partially extend a norm inequality of $n\times n$ block accretive-dissipative matrices.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all $n\times n$ complex matrices and I_n be the identity matrix in $\mathbb{M}_n(\mathbb{C})$. For any $T\in\mathbb{M}_n(\mathbb{C})$, T^* stands for the conjugate transpose of T. Every matrix T has the Cartesian (or Toeptliz) decomposition,

$$T = A + iB, \tag{1}$$

in which $A=\frac{1}{2}(T+T^*)$, $B=\frac{1}{2i}(T-T^*)$ are Hermitian. We say that T is called accretive-dissipative if A, B are positive semidefinite. In this paper, we will always represent the decomposition (1) as follows,

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + i\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2}$$

where $T_{jk}\in\mathbb{M}_n(\mathbb{C})$, j,k=1,2.

Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A\in\mathbb{M}_n(\mathbb{C})$ and unitarily matrices $U, V\in\mathbb{M}_n(\mathbb{C})$. For $p\geq 1$ and $A\in\mathbb{M}_n(\mathbb{C})$, let $\|A\|_p = (\sum_{j=1}^n s_j^p(A))^{\frac{1}{p}}$, where $s_1(A)\geq s_2(A)\geq\cdots\geq s_n(A)$ are the singular values of A. This is the Schatten p-norm of A. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A)\geq\lambda_2(A)\geq\cdots\geq\lambda_n(A)$. We denote $s(A) = (s_1(A), s_2(A), ..., s_n(A))$ and $\lambda(A) = (\lambda_1(A), \lambda_2(A), ..., \lambda_n(A))$.

Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$. We rearrange the components of x and y in nonincreasing order: $x_1^\downarrow\geq\cdots\geq x_n^\downarrow$; $y_1^\downarrow\geq\cdots\geq y_n^\downarrow$. If $\sum_{i=1}^k x_i^\downarrow\leq\sum_{i=1}^k y_i^\downarrow$ $\left(\prod_{i=1}^k x_i^\downarrow\leq\prod_{i=1}^k y_i^\downarrow\right)$, $k=1,\ldots,n$. We say that x is

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weakly (log) majorized by y, denoted by $x \prec_\omega y$ $\left(x \prec_{\omega \log} y\right)$. If, in addition, the last inequality is an equality, i.e. $\sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow$ $\left(\prod_{i=1}^n x_i^\downarrow = \prod_{i=1}^n y_i^\downarrow\right)$, we say that x is (log) majorized by y, written as $x \prec y$ $\left(x \prec_{\log} y\right)$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices, the Hadamard product of A, B is the entry-wise product: $A \circ B = (a_{ij}b_{ij})$.

The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} | \Re z \geq 0, |\Im z| \leq (\Re z)\tan(\alpha)\}$$

and let

$$S'_\alpha = \{z \in \mathbb{C} | \Re z \geq 0, \Im z \geq 0, \Im z \leq (\Re z)\tan(\alpha)\}.$$

Clearly, A is positive definite if and only if $W(A) \subseteq S_0$, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A + B) \subseteq S_\alpha$. As $0 \notin S_\alpha$, then A is nonsingular. Some recent studies of sector matrices can be found in [6, 12, 14–17].
Recent research interest in this class of matrices starts with a resolution of a problem from numerical analysis [3].

Lin and Zhou [13, Theorem 3.3, Theorem 3.11] proved the following unitarily invariant norm inequalities:

Theorem 1.1. *[13, Theorem 3.3] Let $T \in \mathcal{B}(\mathscr{H})$ be accretive-dissipative and partitioned as in* (2)*. Then*

$$\|T_{12}\|\,\|T_{21}\| \leq \max\{\|T_{12}\|^2, \|T_{21}\|^2\} \leq 4\|T_{11}\|\,\|T_{22}\| \tag{3}$$

for any unitarily invariant norm $\|\cdot\|$.

Theorem 1.2. *[13, Theorem 3.11] Let $T \in \mathcal{B}(\mathscr{H})$ be accretive-dissipative and partitioned as in* (2)*. Then*

$$\|T\| \leq \sqrt{2}\|T_{11}\| + \|T_{22}\| \tag{4}$$

for any unitarily invariant norm $\|\ \|$. Furthermore, if $T_{12} = T_{21}$, then

$$\|T\| \leq \sqrt{2}\|T_{11} + T_{22}\|. \tag{5}$$

Gumus et al. [7, Theorem 4.2] proved the following Schatten p-norm and quasinorm inequalities.

Theorem 1.3. *[7, Theorem 4.2] Let $T \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative partitioned as in* (2)*. Then*

$$\|T_{12}\|_p^p + \|T_{21}\|_p^p \leq 2^{P-1}\|T_{11}\|_p^{p/2}\|T_{22}\|_p^{p/2}, \qquad for\ p \geq 2$$

and

$$\|T_{12}\|_p^p + \|T_{21}\|_p^p \leq 2^{3-p}\|T_{11}\|_p^{p/2}\|T_{22}\|_p^{p/2}, \qquad for\ 0 < p \leq 2.$$

Basing on the above theorem, Kittaneh and Sakkijha [10, Theorem 2.4] presented the following norm inequalities, which compares the Schatten p-norms and the quasinorms of the off diagonal blocks and those of the diagonal blocks, respectively.

Theorem 1.4. *[10, Theorem 2.4] For i,j = 1,2,$\cdots$,n, let T_{ij} be square matrices of the same size such that the block matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \tag{6}$$

is accretive-dissipative. Then

$$\sum_{i\neq j}\|T_{ij}\|_p^p \le (n-1)2^{|p-2|}\sum_{i=1}^{n}\|T_{ii}\|_p^p, \quad (p\ge 0).$$

Lin and Fu [14, Theorem 2.9] extended the above Theorem 1.4 to the sector matrices.

Theorem 1.5. *[14, Theorem 2.9] Suppose that T is a sector matrix represented as in* (6)*. Then*

$$\sum_{i\neq j}\|T_{ij}\|_p^p \le (n-1)\sec^p(\alpha)\sum_{i=1}^{n}\|T_{ii}\|_p^p \qquad for \quad p>0.$$

Gumus et al. [7, Definition 3.1] introduced the special class C of all nonnegative increasing functions h on $[0,\infty)$ satisfying the following condition: If $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$ are two decreasing sequences of nonnegative real numbers such that $\prod_{j=1}^{k}x_j \le \prod_{j=1}^{k}y_j$ $(k=1,2,\dots,n)$, then $\prod_{j=1}^{k}h\left(x_j\right) \le \prod_{j=1}^{k}h\left(y_j\right)$ $(k=1,2,\dots,n)$.

Afraz et al.[1, Theorem 17] extended Theorem 1.4 to the sector matrices involving the functions of class C.

Theorem 1.6. *Suppose that T is a sector matrix represented as in* (6)*, $h\in C$ is submultiplicative and $\alpha\in\left[0,\frac{\pi}{2}\right)$. If p is positive real number, then*

$$\sum_{i\neq j}\left\|h\left(\left|T_{ij}\right|^2\right)\right\|^p \le (n-1)\sum_{i=1}^{n}\left\|h^2\left(\sec(\alpha)\left|T_{ii}\right|\right)\right\|^p$$

for every unitarily invariant norm $\|\cdot\|$. In particular, we have

$$\sum_{i\neq j}\left\|h\left(\left|T_{ij}\right|^2\right)\right\|_p^p \le (n-1)\sum_{i=1}^{n}\left\|h^2\left(\sec(\alpha)\left|T_{ii}\right|\right)\right\|_p^p.$$

At last, Lee [11, Theorem 2.1] proved the following result which is considered as an extension of the classical Rotfel'd theorem.

Theorem 1.7. *[11, Theorem 2.1] Let f(t) be a non-negative concave function on* $[0,\infty)$ *. Then, given an arbitrary partitioned positive semi-definite matrix,*

$$\|f(\begin{pmatrix} A & X\\ X^* & B\end{pmatrix})\| \le \|f(A)\|+\|f(B)\|. \tag{7}$$

for all unitarily invariant norms.

What are we interested in the above theorem is whether the right-hand side of the inequality (7) can be placed in one norm. And we give a result under some conditions.

Besides, in this paper, we will extend inequalities (3), (4) and (5) to a larger class matrices, i.e. the upper sector matrices. And on the basis of the extension of (3), we partially generalize Theorem 1.4.

2. Main result

We begin this section with some lemmas which are useful to establish our main results.

Lemma 2.1. *[2, p. 54] Let $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$, $\alpha = (\alpha_1, \alpha_2, \cdots)$ be sequence of real numbers with entries arranged in decreasing order. Moreover, we assume the entries of α are nonnegative. If $\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$ for all $k = 1, 2, \cdots$, then*

$$\sum_{j=1}^{k} \alpha_j x_j \leq \sum_{j=1}^{k} \alpha_j y_j$$

for all $k = 1, 2, \cdots$.

Lemma 2.2. *Let $A, B \in \mathbb{M}_n$, $W(A + iB) \subseteq S'_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and $A + iB$ be the Cartesian decomposition of the full matrix like* (1)*. Then*

$$s_j(B) \leq \sin\alpha\, s_j(A + iB). \tag{8}$$

Proof. First, when $\alpha = 0$, inequality (8) is trivial.
Label the eigenvectors of B as $e_1, \cdots, e_n$ in such a way that

$$s_j(B) = |\langle e_j, Be_j \rangle|.$$

For $W(A + iB) \subseteq S_\alpha$, we get

$$B \leq A \tan(\alpha). \tag{9}$$

$$\begin{aligned} \csc\alpha s_j(B) = \csc\alpha |\langle e_j, Be_j\rangle| &= \sqrt{1 + \cot^2\alpha}\,|\langle e_j, Be_j\rangle| \\ &= |\langle e_j, (\cot\alpha B + iB)e_j\rangle| \\ &\leq |\langle e_j, (A + iB)e_j\rangle| \qquad \textit{by } (9) \\ &\leq \|e_j\|\, \|(A + iB)e_j\|. \end{aligned}$$

Since $s_j(A) = \max\limits_{dim(\mathbb{M})=j} \min\limits_{\substack{x\in\mathbb{M} \\ \|x\|=1}} \|Ax\|$ (see, e. g. [2, p.75]), where $\mathbb{M}$ represent a subspace of $\mathbb{C}^n$ for $A \in \mathbb{M}_n$, we deduce the inequality (8). □

Lemma 2.3. *[18, p. 352] Let A, B, C be $n \times n$ complex matrices such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$. Then*

$$s(B) \prec_{\omega \log} \lambda^{\frac{1}{2}}(A) \circ \lambda^{\frac{1}{2}}(C).$$

Lemma 2.4. *[4, 9] Let $A, B \in \mathbb{M}_n^+$ and $W(A + iB) \subseteq S'_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any unitarily invariant norm $\|\cdot\|$,*

$$\|A + iB\| \leq \|A + B\| \leq a\|A + iB\|,$$

where $a = \min\{1 + \tan(\alpha), \sqrt{2}\}$.

The first main result can be stated as follows.

Theorem 2.5. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in* (2) *and assume $W(T) \subseteq S'_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\max\{\|T_{12}\|^2, \|T_{21}\|^2\} \leq (1 + \sin\alpha)^2 \|T_{11}\| \|T_{22}\|, \tag{10}$$

for any unitarily invariant norm $\|\cdot\|$.

Proof. Let $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ be a sequence with nonnegative entries and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. Define $\|X\|_\nu = \sum_{k=1}^{n} \nu_j s_j(X)$ for $X \in \mathbb{M}_n$.

$$\begin{aligned}
\|T_{12}\|_\nu &= \|\sum_{k=1}^{n} \nu_j s_j(T_{12})\| \\
&= \sum_{k=1}^{n} \nu_j s_j(A_{12} + iB_{12}) \\
&\leq \sum_{k=1}^{n} \nu_j [s_j(A_{12}) + s_j(B_{12})] \qquad (by\ Lemma\ 2.1) \\
&\leq \sum_{k=1}^{n} \nu_j [s_j(A_{11})^{1/2} s_j(A_{22})^{1/2} + s_j(B_{11})^{1/2} s_j(B_{22})^{1/2}] \qquad (by\ Lemma\ 2.3) \\
&\leq \sum_{k=1}^{n} \nu_j [s_j(A_{11}) + s_j(B_{11})]^{1/2} [s_j(A_{22}) + s_j(B_{22})]^{1/2} \qquad (by\ Cauchy-Schwarz) \\
&\leq \sum_{k=1}^{n} \nu_j [(1+\sin\alpha) s_j(A_{11} + iB_{11})]^{1/2} [(1+\sin\alpha) s_j(A_{22} + iB_{22})]^{1/2} \qquad (by\ [2, Proposition III.5.1]\ and\ Lemma\ 2.2) \\
&= (1+\sin\alpha) \sum_{k=1}^{n} \nu_j s_j(T_{11})^{1/2} s_j(T_{22})^{1/2} \\
&\leq (1+\sin\alpha) (\sum_{k=1}^{n} \nu_j s_j(T_{11}))^{1/2} (\sum_{k=1}^{n} \nu_j s_j(T_{22}))^{1/2} \qquad (by\ Cauchy-Schwarz) \\
&= (1+\sin\alpha) \|T_{11}\|_\nu^{1/2} \|T_{22}\|_\nu^{1/2}.
\end{aligned}$$

Similarly, we can get

$$\|T_{21}\|_\nu \leq (1+\sin\alpha) \|T_{11}\|_\nu^{1/2} \|T_{22}\|_\nu^{1/2}.$$

As ν is arbitrarily chosen, the alleged inequality follows form [8, Corollary 3.5.9]. □

Seeing this result, we naturally want to make a comparison between the result of the above Theorem 2.5 and that of Lemma 2.6 in [14] (i.e. [17, Theorem 3.2]). Whether $(1+\sin\alpha)^2$ can be less than $\sec^2\alpha$? when $(1+\sin\alpha)^2$ is less than $\sec^2\alpha$? Now we define a funciton

$$f(\alpha) = \cos\alpha(1+\sin\alpha) - 1 \qquad \alpha \in (0, \frac{\pi}{2}),$$

so

$$(1+\sin\alpha)^2 \leq \sec^2\alpha \Leftrightarrow f(\alpha) \leq 0, \tag{11}$$

By the calculation of matlab, we get $f(\alpha) \leq 0$ on $(0.9960, \frac{\pi}{2})$, i.e. $1+\sin\alpha \leq \sec\alpha$, $\alpha \in (0.9960, \frac{\pi}{2})$ and $1+\sin\alpha > \sec\alpha$, $\alpha \in (0, 0.9960)$.

For $p \geq 1$, since the Schatten p-norms are the examples of the unitarily invariant norms, we could get the following two results.

Corollary 2.6. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in (2) and assume $W(T) \subseteq S'_\alpha$. Then*

$$\max\{\|T_{12}\|_p^p, \|T_{21}\|_p^p\} \leq (1+\sin\alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2}, \qquad for\ p \geq 1. \tag{12}$$

Theorem 2.7. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in* (2) *and $W(T) \subseteq S'_\alpha$. Then*

$$\|T_{12}\|_p^p + \|T_{21}\|_p^p \leq 2(1+\sin\alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2}, \qquad for\ p \geq 1. \tag{13}$$

Proof.

$$\begin{aligned}\|T_{12}\|_p^p + \|T_{21}\|_p^p &\leq (1+\sin\alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2} + (1+\sin\alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2} \quad (12)\\ &= 2(1+\sin\alpha)^p \|T_{11}\|_p^{p/2} \|T_{22}\|_p^{p/2}.\end{aligned}$$

□

In view of the above results, we give a generalization of the Theorem 1.4 in the case $p \geq 1$.

Theorem 2.8. *For i,j = 1,2,$\cdots$,n, let T_{ij} be square matrices of the same size such that*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

and assume $W(T) \subseteq S'_\alpha$. Then

$$\sum_{i \neq j} \|T_{ij}\|_p^p \leq (n-1)(1+\sin\alpha)^p \sum_{i=1}^n \|T_{ii}\|_p^p \qquad for\ p \geq 1.$$

Proof. It is easy to obtain that a principal submatrix $\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}$ of T is also accretive-dissipative and its numerical range is contained in S'_α. Now, applying (13) to $\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}$, we get

$$\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leq 2(1+\sin\alpha)^p \|T_{ii}\|_p^{p/2} \|T_{jj}\|_p^{p/2}$$

for $i \neq j$ and $p \geq 1$.

Consequently, using the arithmetic-geometric mean inequality, we have

$$\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leq (1+\sin\alpha)^p (\|T_{ii}\|_p^p + \|T_{jj}\|_p^p)$$

for $i \neq j$ and $p \geq 1$.
Adding up the previous inequalities for $i, j = 1, 2, \cdots, n$, we get

$$\sum_{i \neq j} \|T_{ij}\|_p^p \leq (n-1)(1+\sin\alpha)^p \sum_{i=1}^n \|T_{ii}\|_p^p,$$

which proves the inequality. □

Remark 2.9. *From inequality* (11)*, we know that the results of Theorem* 2.7 *and Theorem* 2.8 *are tigher than that of [14, Theorem 2.8, 2.9], correspondingly, when $\alpha \in (0.9960, \frac{\pi}{2})$, for $p \geq 1$.*

Next, we extend Theorem 1.2 to the upper sector matrices.

Theorem 2.10. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in* (2) *and assume $W(T) \subseteq S'_\alpha$. Then*

$$\|T\| \leq a(\|T_{11}\| + \|T_{22}\|), \tag{14}$$

for any unitarily invariant norm $\|\cdot\|$. Furthermore, if the off diagonal blocks of $\Re T$ and $\Im T$ are Hermitian or skew-Hermitian, then

$$\|T\| \leq a(\|T_{11} + T_{22}\|), \tag{15}$$

where $a = \min\{1 + \tan(\alpha), \sqrt{2}\}$.

Proof. Consider the Cartesian decomposition $T = A + iB$, where A and B are positive semi-definite. Compute

$$\begin{aligned}\|T\| &= \|A + iB\| \\ &\leq \|A + B\| \qquad (\textit{by Lemma } 2.4) \\ &\leq \|A_{11} + B_{11}\| + \|A_{22} + B_{22}\| \qquad (\textit{by } (7)) \\ &\leq a(\|A_{11} + iB_{11}\| + \|A_{22} + iB_{22}\|) \qquad (\textit{by Lemma } 2.4) \\ &= a(\|T_{11}\| + \|T_{22}\|),\end{aligned}$$

which prove the first inequality.

Now we prove the second inequality. we assume that $A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$ is positive with Hermitian off diagonal blocks and using the simple fact that $T^*T \cong TT^*$ (unitarily congruent) we then deduce

$$A + B \cong J(A + B)J^* = \begin{pmatrix} \frac{A_{11}+B_{11}+A_{22}+B_{22}}{2} & \star \\ \star & \frac{A_{11}+B_{11}+A_{22}+B_{22}}{2} \end{pmatrix},$$

where $J = \frac{1}{\sqrt{2}}\begin{pmatrix} iI & -I \\ iI & I \end{pmatrix}$ is a unitary matrix, I is an identity matrix in I_n and $\star$ stands for the unspecified matrices. Then

$$\begin{aligned}\|T\| &= \|A + iB\| \\ &\leq \|A + B\| \qquad (\textit{by Lemma } 2.4) \\ &= \left\|\begin{pmatrix} \frac{A_{11}+B_{11}+A_{22}+B_{22}}{2} & \star \\ \star & \frac{A_{11}+B_{11}+A_{22}+B_{22}}{2} \end{pmatrix}\right\| \\ &\leq \left\|\frac{A_{11} + B_{11} + A_{22} + B_{22}}{2}\right\| + \left\|\frac{A_{11} + B_{11} + A_{22} + B_{22}}{2}\right\| \qquad (\textit{by } (7)) \\ &= \|A_{11} + B_{11} + A_{22} + B_{22}\| \\ &\leq a\|A_{11} + A_{22} + i(B_{11} + B_{22})\| \qquad (\textit{by Lemma } 2.4) \\ &= a\|A_{11} + iB_{11} + A_{22} + iB_{22}\| \\ &= a\|T_{11} + T_{22}\|.\end{aligned}$$

Similarly, if $A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$ is positive with skew Hermitian off diagonal blocks and still using the simple fact that $T^*T \cong TT^*$ (unitarily congruent), $T = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ we then deduce the same result. □

Remark 2.11. *It is clear, when $a \leq \sqrt{2}$, i.e. $0 \leq \alpha \leq \arctan(\sqrt{2} - 1)$, the result in Theorem* 2.10 *is tigher than that of Theorem* 1.2.
For example, we take $\alpha = 15°$, i.e. $W(T) \subseteq S'_{15°}$, we can get

$$\|T\| \leq 1.268(\|T_{11}\| + \|T_{22}\|).$$

Inequality (15) *correspondingly becomes*

$$\|T\| \leq 1.268(\|T_{11} + T_{22}\|).$$

At the end, we generalize the Theorem 1.7, as follows.

Theorem 2.12. *Let* $A,\ B \in \mathbb{M}_n$, *and* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$ *with Hermitian or skew-Hermitian off diagonal blocks. If f(t) is a non-negative concave function on* $[0, \infty)$, *then*

$$\|f(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix})\| \leq 2\|f(\frac{1}{2}A) + f(\frac{1}{2}B)\|. \tag{16}$$

for all unitarily invariant norm.

Proof. If $X = X^*$

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \cong J\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}J^* = \begin{pmatrix} \frac{A+B}{2} & \star \\ \star & \frac{A+B}{2} \end{pmatrix}$$

where $J = \frac{1}{\sqrt{2}}\begin{pmatrix} iI & -I \\ iI & I \end{pmatrix}$ is a unitary matrix, I is an identity matrix in I_n and $\star$ stands for the unspecified matrices. Then

$$\begin{aligned} \|f(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix})\| &= f(J\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}J^*) \\ &\leq 2\|f(\frac{A+B}{2})\| \qquad by\ (7) \\ &\leq 2\|f(\frac{1}{2}A) + f(\frac{1}{2}B)\|. \end{aligned}$$

The last inequality is by [5, theorem 1.1]. Similarly, if $X^* = -X$, let $T = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$, we still deduce the same result. □

Corollary 2.13. *Let* $A,\ B \in \mathbb{M}_n$, *and* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$ *with Hermitian or skew-Hermitian off diagonal blocks. Then for all unitarily invariant norm* $\|\cdot\|$

$$\|\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}^p\| \leq 2^{1-p}\|A^p + B^p\| \qquad (0 < p \leq 1),$$

$$\|\log(I + \begin{pmatrix} A & X \\ X^* & B \end{pmatrix})\| \leq 2\|\log(I + A/2) + \log(I + B/2)\|.$$

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