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3-L-Dendriform Algebras and Generalized Derivations

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Abstract. The main goal of this paper is to introduce the notion of 3-L-dendriform algebras which are the dendriform version of 3-pre-Lie algebras. In fact they are the algebraic structures behind the $\mathcal{O}$-operator of 3-pre-Lie algebras. They can be also regarded as the ternary analogous of L-dendriform algebras. Moreover, we study the generalized derivations of 3-L-dendriform algebras. Finally, we explore the spaces of quasi-derivations, the centroids and the quasi-centroids and give some properties.

1. Introduction

A dendriform algebra is a module equipped with two binary products whose sum is associative. This concept was introduced by Loday in the late 1990s in the study of periodicity in algebraic K-theory [16]. Several years later, Loday and Ronco introduced the concept of a tridendriform algebra from their study of algebraic topology [17]. It is a module with three binary operations whose sum is associative. Afterward, quite a few similar algebraic structures were introduced, such as the quadi-algebra and ennea algebra [1]. The notion of splitting of associativity was introduced by Loday to describe this phenomena in general for the associative operation (see [8, 12, 18], for more details).

A similar two-part and three-part splittings of the Lie-algebra are found to be the pre-Lie algebra and the post-Lie algebra repectively from operadic study with applications to integrable systems [5, 21]. Further, a two-part and three-part splittings of the associative commutative operation give the Zinbiel algebra and commutative tridendriform algebra respectively [16, 23]. Analogues of the dendriform and tridendriform algebra for Jordan algebra, alternative algebra and Poisson algebra have also been obtained [3, 10, 11].

Some n-ary algebraic structures like n-Lie algebras and n-associative algebras, which have been widely studied in the last few years, were also decomposed into two and three operations (for $n = 3$) giving 3-pre-Lie algebras and partially dendriform 3-algebras [19], using the procedure that splits the operations in algebraic operads.

Rota-Baxter operator (of weight zero) was introduced independently in the 1980s as the operator form of the classical Yang-Baxter equation (see[6]), named after the physicists Yang and Baxter. It gives rise to splittings of various algebraic structures. This is the case, for example, for associative algebras, giving the dendriform and tridendriform algebras, for Lie algebras, giving the pre-Lie and post-Lie algebras and for 3-Lie algebras giving 3-pre-Lie algebras.In [14], Kuperschmidt introduced the notion of Kuperschmidt operators called also $\mathcal{O}$-operators which is generalisation of Rota-Baxter operators.

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It is well known that derivations and generalized derivations on many kinds of algebras play a key role in the research algebraic structures. They have been studied by researchers for many years. The most important paper on the generalized derivation of a Lie algebra was due to Leger and Luks [15]. In [22], authors studied generalized derivations on Lie triple systems.

The following is an outline of the paper. In section 1, we summarize some definitions and known results about 3-Lie algebras and 3-pre-Lie algebras which will be useful in the sequel. In section 2, we introduce the notion of 3-L-dendriform algebra. We establish that it has tow associated 3-pre-Lie algebras (horizontal and vertical). They have the same sub-adjacent 3-Lie algebra. In addition, the left multiplication operator of the first operation (called north-west: $\nwarrow$) and the right multiplication operator of the second operation (called north-east: $\nearrow$) consist a bimodule of the associated horizontal 3-pre-Lie algebra. Section 3 is devoted to the study of generalized derivations on 3-L-dendriform algebras, 3-pre-Lie algebras and 3-Lie algebras.

Throughout this paper $\mathbb{K}$ is a field of characteristic 0 and all vector spaces are over $\mathbb{K}$.

2. Preliminaries and basics

In this section, we give some general results on 3-Lie algebras and 3-pre-Lie algebras which are useful throughout this paper.

Definition 2.1. *[9] A 3-Lie algebra consists of a vector space A equipped with a skew-symmetric linear map called 3-Lie bracket $[\cdot,\cdot,\cdot]:\otimes^3A\to A$ such that the following Fundamental Identity holds (for $x_i\in A, 1\leq i\leq 5$)*

$$[x_1,x_2,[x_3,x_4,x_5]]=[[x_1,x_2,x_3],x_4,x_5]+[x_3,[x_1,x_2,x_4],x_5]+[x_3,x_4,[x_1,x_2,x_5]] \tag{1}$$

In other words, for $x_1,x_2\in A$, the operator

$$ad_{x_1,x_2}:A\to A,\quad ad_{x_1,x_2}x:=[x_1,x_2,x],\quad \forall x\in A, \tag{2}$$

is a derivation in the sense that

$$ad_{x_1,x_2}[x_3,x_4,x_5]=[ad_{x_1,x_2}x_3,x_4,x_5]+[x_3,ad_{x_1,x_2}x_4,x_5]+[x_3,x_4,ad_{x_1,x_2}x_5],\forall x_3,x_4,x_5\in A.$$

A morphism between 3-Lie algebras is a linear map that preserves the 3-Lie brackets.

Example 2.2. *[9] Consider 4-dimensional 3-Lie algebra A generated by (e_1,e_2,e_3,e_4) with the following multiplication*

$$[e_1,e_2,e_3]=e_4,\ [e_1,e_2,e_4]=e_3,\ [e_1,e_3,e_4]=e_2,\ [e_2,e_3,e_4]=e_1.$$

The notion of a representation of an n-Lie algebra was introduced in [13]. See also [7].

Definition 2.3. *A representation of a 3-Lie algebra A on a vector space V is a skew-symmetric linear map $\rho:\otimes^2A\to gl(V)$ satisfying*

- (i) $\rho(x_1,x_2)\rho(x_3,x_4)-\rho(x_3,x_4)\rho(x_1,x_2)=\rho([x_1,x_2,x_3],x_4)-\rho([x_1,x_2,x_4],x_3),$
- (ii) $\rho([x_1,x_2,x_3],x_4)=\rho(x_1,x_2)\rho(x_3,x_4)+\rho(x_2,x_3)\rho(x_1,x_4)+\rho(x_3,x_1)\rho(x_2,x_4),$

for $x_i\in A, 1\leq i\leq 4$.

In fact, (V,ρ) is a representation of a 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$ if and only if there is a 3-Lie algebra structure on the direct sum $A\oplus V$ of the underlying vector spaces A and V given by

$$[x_1+v_1,x_2+v_2,x_3+v_3]_{A\oplus V}=[x_1,x_2,x_3]+\rho(x_1,x_2)v_3+\rho(x_3,x_1)v_2+\rho(x_2,x_3)v_1, \tag{3}$$

for $x_i\in A, v_i\in V, 1\leq i\leq 3$. We denote it by $A\ltimes_\rho V$.

Now we recall the definition of 3-pre-Lie algebra and exhibit construction results in terms of O-operators on 3-Lie algebras (for more details see [4], [19])

Definition 2.4. *Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and (V, ρ) a representation. A linear operator $T : V \longrightarrow A$ is called an O-operator associated to (V, ρ) if T satisfies*

$$[Tu, Tv, Tw] = T\left(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v\right), \quad \forall u, v, w \in V. \tag{4}$$

Definition 2.5. *A 3-pre-Lie algebra is a pair $(A, \{\cdot,\cdot,\cdot\})$ consisting of a a vector space A and a linear map $\{\cdot,\cdot,\cdot\} : A \otimes A \otimes A \longrightarrow A$ such that the following identities hold:*

$$\begin{aligned}
\{x, y, z\} &= -\{y, x, z\}, && (5)\\
\{x_1, x_2, \{x_3, x_4, x_5\}\} &= \{[x_1, x_2, x_3]^C, x_4, x_5\} + \{x_3, [x_1, x_2, x_4]^C, x_5\} \\
&\quad + \{x_3, x_4, \{x_1, x_2, x_5\}\}, && (6)\\
\{[x_1, x_2, x_3]^C, x_4, x_5\} &= \{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_2, x_3, \{x_1, x_4, x_5\}\} \\
&\quad + \{x_3, x_1, \{x_2, x_4, x_5\}\}, && (7)
\end{aligned}$$

where $x, y, z, x_i \in A, 1 \le i \le 5$ and $[\cdot,\cdot,\cdot]^C$ is defined by

$$[x, y, z]^C = \{x, y, z\} + \{y, z, x\} + \{z, x, y\}, \quad \forall x, y, z \in A. \tag{8}$$

Example 2.6. *Let A be a 4-dimensional vector space generated by (e_1, e_2, e_3, e_4) and consider the bracket $\{\cdot,\cdot,\cdot\} : A \otimes A \otimes A \longrightarrow A$ given by*

$$\{e_1, e_2, e_3\} = -\{e_1, e_2, e_4\} = e_1 + e_2$$

and all the other brackets are zero. Then $(A, \{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra.

Example 2.7. *Let A be a 4-dimensional vector space generated by (e_1, e_2, e_3, e_4) and let the bracket $\{\cdot,\cdot,\cdot\} : A \otimes A \otimes A \longrightarrow A$ given by*

$$\begin{cases}
\{e_1, e_2, e_3\} = e_4, \\
\{e_1, e_2, e_4\} = e_3, \\
\{e_1, e_3, e_3\} = -\{e_1, e_4, e_4\} = e_2, \\
\{e_2, e_3, e_3\} = -\{e_2, e_4, e_4\} = e_1,
\end{cases}$$

and all the other brackets are zero. Then $(A, \{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra.

Proposition 2.8. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Then the induced 3-commutator given by Eq. (8) defines a 3-Lie algebra.*

Definition 2.9. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. The 3-Lie algebra $(A, [\cdot,\cdot,\cdot]^C)$ is called the sub-adjacent 3-Lie algebra of $(A, \{\cdot,\cdot,\cdot\})$ and $(A, \{\cdot,\cdot,\cdot\})$ is called a compatible 3-pre-Lie algebra of the 3-Lie algebra $(A, [\cdot,\cdot,\cdot]^C)$.*

Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Define a skew-symmetric linear map $L : \otimes^2 A \longrightarrow gl(A)$ by

$$L(x, y)z = \{x, y, z\}, \quad \forall x, y, z \in A. \tag{9}$$

By the definitions of a 3-pre-Lie algebra and a representation of a 3-Lie algebra, we immediately obtain

Proposition 2.10. *With the above notations, (A, L) is a representation of the 3-Lie algebra $(A, [\cdot,\cdot,\cdot]^C)$. On the other hand, let A be a vector space with a linear map $\{\cdot,\cdot,\cdot\} : A \otimes A \otimes A \longrightarrow A$ satisfying Eq. (5). Then $(A, \{\cdot,\cdot,\cdot\})$ is a 3-pre-Lie algebra if $[\cdot,\cdot,\cdot]^C$ defined by Eq (8) is a 3-Lie algebra and the left multiplication L defined by Eq. (9) gives a representation of this 3-Lie algebra.*

Proposition 2.11. *Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and (V, ρ) a representation. Suppose that the linear map $T : V \longrightarrow A$ is an O-operator associated to (V, ρ). Then there exists a 3-pre-Lie algebra structure on V given by*

$$\{u, v, w\} = \rho(Tu, Tv)w, \quad \forall\ u, v, w \in V. \tag{10}$$

Corollary 2.12. *With the above conditions, $(V, [\cdot,\cdot,\cdot]^C)$ is a 3-Lie algebra as the sub-adjacent 3-Lie algebra of the 3-pre-Lie algebra given in Proposition 2.11, and T is a 3-Lie algebra morphism from $(V, [\cdot,\cdot,\cdot]_C)$ to $(A, [\cdot,\cdot,\cdot])$. Furthermore, $T(V) = \{Tv | v \in V\} \subset A$ is a 3-Lie subalgebra of A and there is an induced 3-pre-Lie algebra structure $\{\cdot,\cdot,\cdot\}_{T(V)}$ on $T(V)$ given by*

$$\{Tu, Tv, Tw\}_{T(V)} := T\{u, v, w\}, \quad \forall u, v, w \in V. \tag{11}$$

Proposition 2.13. *Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra. Then there exists a compatible 3-pre-Lie algebra if and only if there exists an invertible O-operator on A.*

Definition 2.14. *[20] A representation of a 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$ on a vector space V consists of a pair (l, r), where $l : \wedge^2 A \to gl(V)$ is a representation of the 3-Lie algebra A^c on V and $r : \otimes^2 A \to gl(V)$ is a linear map such that for all $x_1, x_2, x_3, x_4 \in A$, the following equalities hold:*

$$\begin{aligned} l(x_1,x_2)r(x_3,x_4) &= r(x_3,x_4)\mu(x_1,x_2) + r([x_1,x_2,x_3]_C,x_4) + r(x_3,\{x_1,x_2,x_4\}), & (12)\\ r([x_1,x_2,x_3]_C,x_4) &= l(x_1,x_2)r(x_3,x_4) + l(x_2,x_3)r(x_1,x_4) + l(x_3,x_1)r(x_2,x_4), & (13)\\ r(x_1,\{x_2,x_3,x_4\}) &= r(x_3,x_4)\mu(x_1,x_2) - r(x_2,x_4)\mu(x_1,x_3) + l(x_2,x_3)r(x_1,x_4), & (14)\\ r(x_3,x_4)\mu(x_1,x_2) &= l(x_1,x_2)r(x_3,x_4) - r(x_2,\{x_1,x_3,x_4\}) + r(x_1,\{x_2,x_3,x_4\}), & (15) \end{aligned}$$

where $\mu(x, y) = l(x, y) + r(x, y) - r(y, x)$, for any $x, y \in A$.

Define the left multiplication $L : \wedge^2 A \longrightarrow gl(A)$ by $L(x, y)z = \{x, y, z\}$ for all $x, y, z \in A$. Then (A, L) is a representation of the 3-Lie algebra A^c. Moreover, we define the right multiplication $R : \otimes^2 A \to gl(A)$ by $R(x, y)z = \{z, x, y\}$. It is obvious that (A, L, R) is a representation of a 3-pre-Lie algebra on itself, which is called the adjoint representation.

Proposition 2.15. *[20] Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra, V a vector space and $l, r : \otimes^2 A \to gl(V)$ two linear maps. Then (V, l, r) is a representation of A if and only if there is a 3-pre-Lie algebra structure (called the semi-direct product) on the direct sum $A \oplus V$ of vector spaces, defined by*

$$[x_1 + u_1, x_2 + u_2, x_3 + u_3]_{A\oplus V} = \{x_1, x_2, x_3\} + l(x_1, x_2)u_3 - r(x_1, x_3)u_2 + r(x_2, x_3)u_1, \tag{16}$$

for $x_i \in A, u_i \in V, 1 \leq i \leq 3$. We denote this semi-direct product 3-Lie algebra by $A \ltimes_{l,r} V$.

Let V be a vector space. Define the switching operator $\tau : \otimes^2 V \longrightarrow \otimes^2 V$ by

$$\tau(T) = x_2 \otimes x_1, \quad \forall T = x_1 \otimes x_2 \in \otimes^2 V.$$

Proposition 2.16. *[20] Let (V, l, r) be a representation of a 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$. Then $l - r\tau + r$ is a representation of the sub-adjacent 3-Lie algebra $(A^c, [\cdot,\cdot,\cdot]_C)$ on the vector space V.*

Proposition 2.17. *[20] Let (l, r) be a representation of a 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$ on a vector space V. Then $(l^* - r^*\tau + r^*, -r^*)$ is a representation of the 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$ on the vector space V^*, which is called the dual representation of the representation (V, l, r).*

If $(l, r) = (L, R)$ is the adjoint representation of a 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$, then we obtain $(l^* - r^*\tau + r^*, -r^*) = (ad^*, -R^*)$.

Definition 2.18. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra and (V, l, r) be a representation. A linear operator $T : V \to A$ is called an O-operator associated to (V, l, r) if T satisfies*

$$\{Tu, Tv, Tw\} = T\left(l(Tu, Tv)w - r(Tu, Tw)v + r(Tv, Tw)u\right), \quad \forall u, v, w \in V. \tag{17}$$

If $V = A$, then T is called a Rota-Baxter operator on A of weight zero. That is

$$\{R(x), R(y), R(z)\} = R\Big(\{R(x), R(y), z\} + \{R(x), y, R(z)\} + \{x, R(y), R(z)\}\Big),$$

for all $x, y, z \in A$.

Example 2.19. *Let the 4-dimensional 3-pre-Lie algebra given in Example 2.6. Define $R : A \to A$ by*

$$R(e_1) = e_1 + e_2, \; R(e_2) = e_3 + e_4, \; R(e_3) = R(e_4) = 0.$$

By a direct computation, we can verify that R is a Rota-Baxter operator.

3. 3-L-dendriform algebras

In this section, we introduce the notion of a 3-L-dendriform algebra which is exactly the ternary version of a L-dendriform algebra. We provide some construction results in terms of $\mathcal{O}$-operator and symplectic structure.

Definition 3.1. *Let A be a vector space with two linear maps $\nwarrow, \nearrow : \otimes^3 A \to A$. The tuple $(A, \nwarrow, \nearrow)$ is called a 3-L-dendriform algebra if the following identities hold*

$$\nwarrow(x_1,x_2,x_3) + \nwarrow(x_2,x_1,x_3) = 0, \tag{18}$$

$$\begin{aligned}&\nwarrow(x_1,x_2,\nwarrow(x_3,x_4,x_5)) - \nwarrow(x_3,x_4,\nwarrow(x_1,x_2,x_5))\\ &\qquad = \nwarrow([x_1,x_2,x_3]^C,x_4,x_5) - \nwarrow([x_1,x_2,x_4]^C,x_3,x_5),\end{aligned} \tag{19}$$

$$\begin{aligned}&\nwarrow(x_1,x_2,\nearrow(x_5,x_3,x_4)) - \nearrow(x_5,x_3,\{x_1,x_2,x_4\}^h)\\ &\qquad = \nearrow(x_5,[x_1,x_2,x_3]^C,x_4) + \nearrow(\{x_1,x_2,x_5\}^v,x_3,x_4),\end{aligned} \tag{20}$$

$$\begin{aligned}&\nearrow(x_5,x_1,\{x_2,x_3,x_4\}^h) - \nwarrow(x_2,x_3,\nearrow(x_5,x_1,x_4))\\ &\qquad = \nearrow(\{x_1,x_2,x_5\}^v,x_3,x_4) - \nearrow(\{x_1,x_3,x_5\}^v,x_2,x_4),\end{aligned} \tag{21}$$

$$\nwarrow([x_1,x_2,x_3]^C,x_4,x_5) = \circlearrowleft_{1,2,3} \nwarrow(x_1,x_2,\nwarrow(x_3,x_4,x_5)), \tag{22}$$

$$\nearrow(x_5,[x_1,x_2,x_3]^C,x_4) = \circlearrowleft_{1,2,3} \nwarrow(x_1,x_2,\nearrow(x_5,x_3,x_4)), \tag{23}$$

$$\begin{aligned}&\nwarrow(x_1,x_2,\nearrow(x_5,x_3,x_4)) + \nearrow(x_5,x_1,\{x_2,x_3,x_4\}^h)\\ &\qquad = \nearrow(x_5,x_2,\{x_1,x_3,x_4\}^h) + \nearrow(\{x_1,x_2,x_5\}^v,x_3,x_4),\end{aligned} \tag{24}$$

for all $x_i \in A$, $1 \leq i \leq 5$, where

$$\{x,y,z\}^h = \nwarrow(x,y,z) + \nearrow(x,y,z) - \nearrow(y,x,z), \tag{25}$$

$$\{x,y,z\}^v = \nwarrow(x,y,z) + \nearrow(z,x,y) - \nearrow(z,y,x), \tag{26}$$

$$[x,y,z]^C = \circlearrowleft_{x,y,z} \{x,y,z\}^h = \circlearrowleft_{x,y,z} \{x,y,z\}^v, \tag{27}$$

for any $x, y, z \in A$.

Remark 3.2. *Let $(A, \nwarrow, \nearrow)$ be a 3-L-dendriform algebra. if $\nearrow = 0$, then $(A, \nwarrow)$ is a 3-pre-Lie algebra.*

Proposition 3.3. *Let $(A, \nwarrow, \nearrow)$ be a 3-L-dendriform algebra.*

1. *The bracket given in (25) defines a 3-pre-Lie algebra structure on A which is called the associated horizontal 3-pre-Lie algebra of $(A, \nwarrow, \nearrow)$ and $(A, \nwarrow, \nearrow)$ is also called a compatible 3-L-dendriform algebra structure on the 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\}^h)$.*
2. *The bracket given in (26) defines a 3-pre-Lie algebra structure on A which is called the associated vertical 3-pre-Lie algebra of $(A, \nwarrow, \nearrow)$ and $(A, \nwarrow, \nearrow)$ is also called a compatible 3-L-dendriform algebra structure on the 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\}^v)$.*

Proof. We will just prove item 1. Note, first that $\{x,y,z\}^h = -\{y,x,z\}^h$ and $\{x,y,z\}^v = -\{y,x,z\}^v$, for any $x, y, z \in A$.
Let $x_i \in A$, $1 \leq i \leq 5$. Then

$$\begin{aligned}&\{x_1,x_2,\{x_3,x_4,x_5\}^h\}^h - \{x_3,x_4,\{x_1,x_2,x_5\}^h\}^h - \{[x_1,x_2,x_3]^C,x_4,x_5\}^h + \{[x_1,x_2,x_4]^C,x_3,x_5\}^h\\ &= r_1 + r_2 + r_3 + r_4 + r_5,\end{aligned}$$

where

$$\begin{aligned}
r_1 =& \nwarrow (x_1,x_2,\nwarrow (x_3,x_4,x_5))- \nwarrow (x_3,x_4,\nwarrow (x_1,x_2,x_5))- \nwarrow ([x_1,x_2,x_3]^C,x_4,x_5)\\
&+ \nwarrow ([x_1,x_2,x_4]^C,x_3,x_5),\\
r_2 =& \nearrow (x_3,[x_1,x_2,x_4]^C,x_5)+ \nearrow (x_3,x_4,\{x_1,x_2,x_5\}^h)- \nwarrow (x_1,x_2,\nearrow (x_3,x_4,x_5))\\
&+ \nearrow (\{x_1,x_2,x_3\}^v,x_4,x_5),\\
r_3 =& \nearrow (x_4,[x_1,x_2,x_3]^C,x_5)+ \nearrow (x_4,x_3,\{x_1,x_2,x_5\}^h)- \nwarrow (x_1,x_2,\nearrow (x_4,x_3,x_5))\\
&+ \nearrow (\{x_1,x_2,x_4\}^v,x_3,x_5),\\
r_4 =& \nearrow (x_1,x_2,\{x_3,x_4,x_5\}^h)- \nwarrow (x_3,x_4,\nearrow (x_1,x_2,x_5))- \nearrow (\{x_2,x_3,x_1\}^v,x_4,x_5)\\
&+ \nearrow (\{x_2,x_4,x_1\}^v,x_3,x_5),\\
r_5 =& \nearrow (x_2,x_1,\{x_3,x_4,x_5\}^h)- \nwarrow (x_3,x_4,\nearrow (x_2,x_1,x_5))- \nearrow (\{x_1,x_3,x_2\}^v,x_4,x_5)\\
&+ \nearrow (\{x_1,x_4,x_2\}^v,x_3,x_5).
\end{aligned}$$

From identities (19)-(21), we obtain immediately $r_i = 0,\ \forall 1 \leq i \leq 5$. This imply that (6) hold.

On the other hand, we have

$$\begin{aligned}
&\{[x_1,x_2,x_3]^C,x_4,x_5\}^h - \{x_1,x_2,\{x_3,x_4,x_5\}^h\}^h - \{x_2,x_3,\{x_1,x_4,x_5\}^h\}^h - \{x_3,x_1,\{x_2,x_4,x_5\}^h\}^h\\
=&s_1 + s_2 + s_3 + s_4 + s_5,
\end{aligned}$$

where

$$\begin{aligned}
s_1 =& \nwarrow ([x_1,x_2,x_3]^C,x_4,x_5)- \circlearrowleft_{1,2,3} \nwarrow (x_1,x_2,\nwarrow (x_3,x_4,x_5)),\\
s_2 =& \nearrow (x_4,[x_1,x_2,x_3]^C,x_5)- \circlearrowleft_{1,2,3} \nwarrow (x_1,x_2,\nearrow (x_4,x_3,x_5)),\\
s_3 =& \nwarrow (x_1,x_2,\nearrow (x_3,x_4,x_5))+ \nearrow (x_3,x_1,\{x_2,x_4,x_5\}^h)\\
&\qquad - \nearrow (x_3,x_2,\{x_1,x_4,x_5\}^h)- \nearrow (\{x_1,x_2,x_3\}^v,x_4,x_5),\\
s_4 =& \nwarrow (x_2,x_3,\nearrow (x_1,x_4,x_5))+ \nearrow (x_1,x_2,\{x_3,x_4,x_5\}^h)\\
&\qquad - \nearrow (x_1,x_3,\{x_2,x_4,x_5\}^h)- \nearrow (\{x_2,x_3,x_1\}^v,x_4,x_5),\\
s_5 =& \nwarrow (x_3,x_1,\nearrow (x_2,x_4,x_5))+ \nearrow (x_2,x_3,\{x_1,x_4,x_5\}^h)\\
&\qquad - \nearrow (x_2,x_1,\{x_3,x_4,x_5\}^h)- \nearrow (\{x_3,x_1,x_2\}^v,x_4,x_5).
\end{aligned}$$

From identities (22)-(24), we obtain immediately $s_i = 0,\ \forall 1 \leq i \leq 5$. This imply that (7) hold.

□

Corollary 3.4. *Let $(A,\nwarrow,\nearrow)$ be a 3-L-dendriform algebra. Then the bracket defined in (27) defines a 3-Lie algebra structure on A which is called the associated 3-Lie algebra of $(A,\nwarrow,\nearrow)$.*

The following Proposition is obvious.

Proposition 3.5. *Let $(A,\nwarrow,\nearrow)$ be a 3-L-dendriform algebra. Define $L_\nwarrow, R_\nearrow : \otimes^2 A \to gl(A)$ by*

$$L_\nwarrow(x,y)z = \nwarrow (x,y,z), \quad R_\nearrow(x,y)z = \nearrow (z,x,y),\ \rho(x,y)z = \nwarrow (x,y,z)+ \nearrow (z,x,y)- \nearrow (z,y,x)$$

for all $x,y,z \in A$. Then

(1) *$(A,L_\nwarrow,R_\nearrow)$ is a representation of its horizontal associated 3-pre-Lie algebra $(A,\{\cdot,\cdot,\cdot\}^h)$ and $(A,L_\nwarrow,-L_\nearrow)$ is a representation of its vertical associated 3-pre-Lie algebra $(A,\{\cdot,\cdot,\cdot\}^v)$.*

(2) *$(A,L_\nwarrow)$ is a representation of its associated 3-Lie algebra $(A,[\cdot,\cdot,\cdot]^C)$.*

(3) (A,ρ) *is a representation of its associated* 3*-Lie algebra* $(A,[\cdot,\cdot,\cdot]^C)$.

Remark 3.6. *In the sense of the above conclusion (1), a* 3*-L-dendriform algebra is understood as a ternary algebra structure whose left and right multiplications give a bimodule structure on the underlying vector space of the* 3*-pre-Lie algebra defined by certain commutators. It can be regarded as the* ***rule*** *of introducing the notion of* 3*-L-dendriform algebra, which more generally, is the* ***rule*** *of introducing the notions of* 3*-pre-Lie algebras, the Loday algebras and their Lie,Jordan and alternative algebraic analogues.*

Theorem 3.7. *Let* $(A,\{\cdot,\cdot,\cdot\})$ *be a* 3*-pre-Lie algebra and* (V,l,r) *be a representation. Suppose that* $T:V\to A$ *is an* $\mathcal{O}$*-operator associated to* (V,l,r)*. Then there exists a* 3*-L-dendriform algebra structure on* V *given by*

$$\nwarrow(u,v,w)=l(Tu,Tv)w,\quad \nearrow(u,v,w)=r(Tv,Tw)u,\forall u,v,w\in V. \tag{28}$$

Therefore, there exists two associated 3*-pre-Lie algebra structures on* V *and* T *is a homomorphism of* 3*-pre-Lie algebras. Moreover,* $T(V)=\{T(v)|v\in V\}$ *is a* 3*-pre-Lie subalgebra of* $(A,\{\cdot,\cdot,\cdot\})$ *and there is an induced* 3*-L-dendriform algebra structure on* $T(V)$ *given by*

$$\nwarrow(Tu,Tv,Tw)=T(\nwarrow(u,v,w)),\quad \nearrow(Tu,Tv,Tw)=T(\nearrow(u,v,w)),\forall u,v,w\in V. \tag{29}$$

Proof. Let $u,v,w\in V$. Define $\{\cdot,\cdot,\cdot\}_V^h,\{\cdot,\cdot,\cdot\}_V^v,[\cdot,\cdot,\cdot]_V^C:\otimes^3V\to V$ by

$$\begin{aligned}
\{u,v,w\}_V^h &= \nwarrow(u,v,w)+\nearrow(u,v,w)-\nearrow(v,u,w),\\
\{u,v,w\}_V^v &= \nwarrow(u,v,w)+\nearrow(w,u,v)-\nearrow(w,v,u),\\
[u,v,w]_V^C &= \circlearrowleft_{u,v,w}\{u,v,w\}_V^h.
\end{aligned}$$

Using identity (17), we have

$$\begin{aligned}
T\{u,v,w\}_V^h &= T(\nwarrow(u,v,w)+\nearrow(u,v,w)-\nearrow(v,u,w))\\
&= T(l(Tu,Tv)w-r(Tu,Tw)v+r(Tv,Tw)u)=\{Tu,Tv,Tw\}^h
\end{aligned}$$

and

$$T[u,v,w]_V^C=\circlearrowleft_{u,v,w}T\{u,v,w\}_V=\circlearrowleft_{u,v,w}\{Tu,Tv,Tw\}^h=[Tu,Tv,Tw]^C.$$

It is straightforward that

$$\nwarrow(u,v,w)+\nwarrow(v,u,w)=(l(Tu,Tv)+l(Tv,Tu))w=0.$$

Furthermore, for any $u_i\in V,\ 1\leq i\leq 5$, we have

$$\begin{aligned}
&\nwarrow(u_1,u_2,\nwarrow(u_3,u_4,u_5))-\nwarrow(u_3,u_4,\nwarrow(u_1,u_2,u_5))\\
&-\nwarrow([u_1,u_2,u_3]_V^C,u_4,u_5)+\nwarrow([u_1,u_2,u_4]_V^C,u_3,u_5)\\
&=l(T(u_1),T(u_2))l(T(u_3),T(u_4))u_5-l(T(u_3),T(u_4))l(T(u_1),T(u_2))u_5\\
&-l(T[u_1,u_2,u_3]_V^C,T(u_4))u_5+l(T[u_1,u_2,u_4]_V^C,T(u_3))u_5\\
&=\Big([l(T(u_1),T(u_2)),l(T(u_3),T(u_4))]-l([T(u_1),T(u_2),T(u_3)]^C,T(u_4))\\
&\qquad+l([T(u_1),T(u_2),T(u_4)]^C,T(u_3))\Big)u_5=0.
\end{aligned}$$

This implies that (19) holds. Moreover, (20) holds. Indeed,

$$\begin{aligned}
&\nearrow(u_5,[u_1,u_2,u_3]_V^C,u_4)+\nearrow(u_5,u_3,\{u_1,u_2,u_4\}_V^h)\\
&-\nwarrow(u_1,u_2,\nearrow(u_5,u_3,u_4))+\nearrow(\{u_1,u_2,u_5\}_V^v,u_3,u_4)\\
&=r(T[u_1,u_2,u_3]_V^C,T(u_4))u_5-r(T(u_3),T\{u_1,u_2,u_4\}_V^h)u_5\\
&-l(T(u_1),T(u_2))r(T(u_3),T(u_4))u_5+r(T(u_3),T(u_4))\{u_1,u_2,u_5\}_V^v\\
&=r([Tu_1,Tu_2,Tu_3]^C,T(u_4))u_5-r(T(u_3),\{T(u_1),T(u_2),T(u_4)\}^h)u_5\\
&-l(T(u_1),T(u_2))r(T(u_3),T(u_4))u_5+r(T(u_3),T(u_4))\mu(T(u_1),T(u_2))u_5\\
&=0
\end{aligned}$$

To prove identity (24), we compute as follow

$$\begin{aligned}&\nwarrow(u_1,u_2,\nearrow(u_5,u_3,u_4))+\nearrow(u_5,u_1,\{u_2,u_3,u_4\}_V^h)\\&\quad-\nearrow(u_5,u_2,\{u_1,u_3,u_4\}_V^h)-\nearrow(\{u_1,u_2,u_5\}_V^v,u_3,u_4)\\&=l(Tu_1,Tu_2)r(Tu_3,Tu_4)u_5+r(Tu_1,\{Tu_2,Tu_3,Tu_4\}^h)u_5\\&\quad-r(Tu_2,\{Tu_1,Tu_3,Tu_4\}^h)u_5-r(Tu_3,Tu_4)\mu(Tu_1,Tu_2)u_5\\&=0.\end{aligned}$$

The other conclusions follow immediately. □

Corollary 3.8. *Let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra and $R:A\to A$ is a Rota-Baxter operator of weight 0. Then there exists a 3-L-dendriform algebra structure on A given by*

$$\nwarrow(x,y,z)=\{R(x),R(y),z\},\quad\nearrow(x,y,z)=\{x,R(y),R(z)\},\tag{30}$$

for all $x,y,z\in A$.

Example 3.9. *Let the 4-dimensional 3-pre-Lie algebra given in Example 2.7. Define $R:A\to A$ by*

$$R(e_1)=e_2,\ R(e_2)=-e_1,\ R(e_3)=e_4,\ R(e_4)=e_3.$$

Then R is a Rota-Baxter operator. Using Corollary 3.8, we can construct a 3-L-dendriform algebra given by the structures $\nwarrow,\nearrow$: $A\otimes A\otimes A\to A$ defined in the basis (e_1,e_2,e_3,e_4), by

$$\begin{cases}\nwarrow(e_1,e_2,e_3)=e_4,\\\nwarrow(e_1,e_2,e_4)=e_3,\\\nearrow(e_1,e_1,e_3)=\nearrow(e_2,e_2,e_3)=e_3,\\\nearrow(e_1,e_1,e_4)=\nearrow(e_2,e_2,e_4)=e_4,\\\nearrow(e_1,e_3,e_3)=-\nearrow(e_1,e_4,e_4)=-e_2,\\\nearrow(e_2,e_3,e_3)=-\nearrow(e_2,e_4,e_4)=-e_1\end{cases}$$

and all the other products are zero.

Proposition 3.10. *Let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra. Then there exists a compatible 3-L-dendriform algebra if and only if there exists an invertible $\mathcal{O}$-operator on A.*

Proof. Let T be an invertible $\mathcal{O}$-operator of A associated to a representation (V,l,r). Then there exists a 3-L-dendriform algebra structure on V defined by

$$\nwarrow(u,v,w)=l(Tu,Tv)w,\quad\nearrow(u,v,w)=r(Tv,Tw)u,\forall u,v,w\in V.\tag{31}$$

In addition there exists a 3-L-dendriform algebra structure on $T(V)=A$ given by

$$\nwarrow(Tu,Tv,Tw)=T(l(Tu,Tv)w),\quad\nearrow(Tu,Tv,Tw)=T(r(Tv,Tw)u),\forall u,v,w\in V.\tag{32}$$

If we put $x=Tu,\ y=Tv$ and $z=Tw$, we get

$$\nwarrow(x,y,z)=T(l(x,y)T^{-1}(z)),\quad\nearrow(x,y,z)=T(r(y,z)T^{-1}(x)),\forall x,y,z\in A.\tag{33}$$

It is a compatible 3-L-dendriform algebra structure on A. Indeed,

$$\begin{aligned}&\nwarrow(x,y,z)+\nearrow(x,y,z)-\nearrow(y,x,z)\\&=T(l(x,y)T^{-1}(z))+T(r(y,z)T^{-1}(x))-T(r(x,z)T^{-1}(y))\\&=\{TT^{-1}(x),TT^{-1}(y),TT^{-1}(z)\}=\{x,y,z\}.\end{aligned}$$

Conversely, let $(A,\{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra and $(A,\nwarrow,\nearrow)$ its compatible 3-L-dendriform algebra. Then the identity map $id:A\to A$ is an $\mathcal{O}$-operator of $(A,\{\cdot,\cdot,\cdot\})$ associated to (A,L,R). □

Definition 3.11. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra and B be a symmetric bilinear form on A. We say that B is closed if it satisfies*

$$B(\{x,y,z\},w) + B(z,[x,y,w]^C) + B(y,\{w,x,z\}) - B(x,\{w,y,z\}) = 0, \tag{34}$$

for any $x,y,z,w \in A$. If in addition B is nondegenerate , then B is called a pseudo-Hessian structure on A.

A 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$ is called pseudo-Hessian if it is equipped with a symmetric, closed and nondegenerate form B. It is denoted by $(A, \{\cdot,\cdot,\cdot\}, B)$.

Proposition 3.12. *Let $(A, \{\cdot,\cdot,\cdot\}, B)$ be a pseudo-Hessian 3-pre-Lie algebra. Then there exists a compatible 3-L-dendriform algebra structure on A given by*

$$B(\nwarrow(x,y,z),w) = -B(z,[x,y,w]^C), \quad B(\nearrow(x,y,z),w) = B(x,\{w,y,z\}), \forall x,y,z,w \in A. \tag{35}$$

Proof. Define the linear map $T : A^* \to A$ by $\langle T^{-1}x, y\rangle = B(x,y)$. Using Eq. (34), we obtain that T is an invertible $\mathcal{O}$-operator on A associated to the dual representation $(A^*, ad^*, -R^*)$. By Proposition 3.10, there exists a compatible 3-L-dendriform algebra structure given by

$$\nwarrow(x,y,z) = T(ad^*(x,y)T^{-1}(z)), \quad \nearrow(x,y,z) = -T(R^*(y,z)T^{-1}(x)), \forall w,y,z \in A. \tag{36}$$

Then we have

$$\begin{aligned} B(\nwarrow(x,y,z),w) =& B(T(ad^*(x,y)T^{-1}(z)),w) = \langle ad^*(x,y)T^{-1}(z),w\rangle \\ =& -\langle T^{-1}(z),[x,y,w]^C\rangle = -B(z,[x,y,w]^C) \end{aligned}$$

and

$$\begin{aligned} B(\nearrow(x,y,z),w) =& -B(T(R^*(y,z)T^{-1}(x))),w) = -\langle R^*(y,z)T^{-1}(x),w\rangle \\ =& \langle T^{-1}(x),\{w,y,z\}\rangle = B(x,\{w,y,z\}). \end{aligned}$$

The proof is finished. □

Corollary 3.13. *Let $(A, \{\cdot,\cdot,\cdot\}, B)$ be a pseudo-Hessian 3-pre-Lie algebra and let $(A, [\cdot,\cdot,\cdot]^C)$ be its associated 3-Lie algebra. Then there exists a 3-pre-Lie algebraic structure $(A, \{\cdot,\cdot,\cdot\}')$ on A given by*

$$B(\{x,y,z\}',w) = -B(z,[x,y,w]^C) + B(z,\{w,x,y\}) - B(z,\{w,y,x\}). \tag{37}$$

Lemma 3.14. *Let $\{R_1, R_2\}$ be a pair of of commuting Rota-Baxter operators (of weight zero) on a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$. Then R_2 is a Rota-Baxter operator (of weight zero) on the associated 3-pre-Lie algebra defined by $\{x,y,z\} = [R_1(x), R_1(y), z]$.*

Proof. For any $x,y,z \in A$, we have

$$\begin{aligned} &\{R_2(x),R_2(y),R_2(z)\} = [R_1R_2(x),R_1R_2(y),R_2(z)] = [R_2R_1(x),R_2R_1(y),R_2(z)] \\ &= R_2([R_2R_1(x),R_2R_1(y),z] + [R_1(x),R_2R_1(y),R_2(z)] + [R_2R_1(x),R_1(y),R_2(z)]) \\ &= R_2(\{R_2(x),R_2(y),z\} + \{x,R_2(y),R_2(z)\} + \{R_2(x),y,R_2(z)\}). \end{aligned}$$

Hence R_2 is a Rota-Baxter operator (of weight zero) on the 3-pre-Lie algebra $(A, \{\cdot,\cdot,\cdot\})$. □

Proposition 3.15. *Let $\{R_1, R_2\}$ be a pair of of commuting Rota-Baxter operators (of weight zero) on a 3-Lie algebra $(A, [\cdot,\cdot,\cdot])$. Then there exists a 3-L-dendriform algebra structure on A defined by*

$$\nwarrow(x,y,z) = [R_1R_2(x), R_1R_2(y), z], \quad \nearrow(x,y,z) = [R_1(x), R_1R_2(y), R_2(z)], \forall x,y,z \in A.$$

Proof. It follows immediately from Lemma 3.14 and Corollary 3.8. □

Remark 3.16. *Let* $(A, [\cdot, \cdot])$ *be a Lie-algebra. Recall that a trace function* $\tau : A \to \mathbb{K}$ *is a linear map such that* $\tau([x, y]) = 0,\ \forall x, y \in A$. *When* τ *is a trace function, it is well known [2] that* $(A, [\cdot, \cdot, \cdot]_\tau)$ *is a* 3*-Lie algebra, where*

$$[x, y, z]_\tau := \circlearrowleft_{x,y,z \in A} \tau(x)[y, z],\ \forall x, y, z \in A.$$

Now let $\{R_1, R_2\}$ *be a pair of of commuting Rota-Baxter operators (of weight zero) on the* 3*-Lie algebra* $(A, [\cdot, \cdot, \cdot]_\tau)$. *Then we can construct a* 3*-L-dendriform structure on A, given by*

$$\begin{aligned}
\nwarrow (x, y, z) =& \tau(R_1R_2(x))[R_1R_2(y), z] + \tau(R_1R_2(y))[z, R_1R_2(x)] + +\tau(z)[R_1R_2(x), R_1R_2(y)],\\
\nearrow (x, y, z) =& \tau(R_1(x))[R_1R_2(y), R_2(z)] + \tau(R_1R_2(y))[R_2(z), R_1(x)] + \tau(R_2(z))[R_1(x), R_1R_2(y)],
\end{aligned}$$

for any $x, y, z \in A$.

4. Generalized derivations of 3-L-dendriform algebras

This section is devoted to investigate generalized derivations of 3-Lie algebras, 3-pre-Lie algebras and 3-L-dendriform algebras. Throughout the sequel $(A, < \cdot, \cdot, \cdot >)$ denotes a 3-(pre)-Lie algebra.

Definition 4.1. *A linear map* $D : A \to A$ *is said to be a derivation on A, if it satisfies the following condition (for* $x, y, z \in A$*)*

$$D(< x, y, z >) =< D(x), y, z > + < x, D(y), z > + < x, y, D(z) > . \tag{38}$$

Definition 4.2. *Let* $(A, \nwarrow, \nearrow)$ *be a* 3*-L-dendriform algebra. A linear map* $D : A \to A$ *is said to be a derivation on A, if it satisfies the following condition (for* $x, y, z \in A$*)*

$$D(\nwarrow (x, y, z)) = \nwarrow (D(x), y, z) + \nwarrow (x, D(y), z) + \nwarrow (x, y, D(z)), \tag{39}$$

$$D(\nearrow (x, y, z)) = \nearrow (D(x), y, z) + \nearrow (x, D(y), z) + \nearrow (x, y, D(z)). \tag{40}$$

We denote the set of all derivations on A by $Der(A)$. We can easily show that $Der(A)$ is equipped with a Lie algebra structure. In fact, for $D \in Der(A)$ and $D' \in Der(A)$, we have $[D, D'] \in Der(A)$, where $[D, D']$ is the standard commutator defined by $[D, D'] = DD' - D'D$.

Definition 4.3. *A linear mapping* $D \in End(A)$ *is said to be a quasi-derivation of A if there exist linear mapping* $D' \in End(A)$ *such that*

$$D'(< x, y, z >) =< D(x), y, z > + < x, D(y), z > + < x, y, D(z) >, \tag{41}$$

for all $x, y, z \in A$. *We say that D associates with* D'.

Definition 4.4. *Let* $(A, \nwarrow, \nearrow)$ *be a* 3*-L-dendriform algebra. A linear map* $D : A \to A$ *is said to be a quasi-derivation on A, if it satisfies the following condition*

$$D'(\nwarrow (x, y, z)) = \nwarrow (D(x), y, z) + \nwarrow (x, D(y), z) + \nwarrow (x, y, D(z)), \tag{42}$$

$$D'(\nearrow (x, y, z)) = \nearrow (D(x), y, z) + \nearrow (x, D(y), z) + \nearrow (x, y, D(z)), \tag{43}$$

for all $x, y, z \in A$. *We say that D associates with* D'.

Definition 4.5. *A linear mapping* $D \in End(A)$ *is said to be a generalized derivation of A if there exist linear mappings* $D', D'', D''' \in End(A)$ *such that*

$$D'''(< x, y, z >) =< D(x), y, z > + < x, D'(y), z > + < x, y, D''(z) >, \tag{44}$$

for all $x, y, z \in A$. *We also say that D associates with* (D', D'', D''').

Definition 4.6. *A linear map $D : A \to A$ is said to be a generalized derivation of a 3-L-dendriform algebra $(A, \nwarrow, \nearrow)$, if it satisfies the following condition*

$$D'''(\nwarrow(x,y,z)) = \nwarrow(D(x),y,z) + \nwarrow(x,D'(y),z) + \nwarrow(x,y,D''(z)), \tag{45}$$

$$D'''(\nearrow(x,y,z)) = \nearrow(D(x),y,z) + \nearrow(x,D'(y),z) + \nearrow(x,y,D''(z)), \tag{46}$$

for all $x,y,z \in A$. We say that D associates with (D', D'', D''').

The sets of quasi-derivations and generalized derivations will be denoted by $QDer(A)$ and $GDer(A)$, respectively. It is easy to see that

$$Der(A) \subset QDer(A) \subset GDer(A).$$

Definition 4.7. *A linear map $\Theta \in End(A)$ is said to be a centroid of A if*

$$\Theta(< x,y,z >) =< \Theta(x),y,z >=< x,\Theta(y),z >=< x,y,\Theta(z) >, \ \forall \ x,y,z \in A. \tag{47}$$

Definition 4.8. *A linear map $\Theta : A \to A$ is said to be a Centroid of a 3-L-dendriform algebra $(A, \nwarrow, \nearrow)$, if it satisfies the following conditions:*

$$\Theta(\nwarrow(x,y,z)) = \nwarrow(\Theta(x),y,z) = \nwarrow(x,\Theta(y),z) = \nwarrow(x,y,\Theta(z)), \tag{48}$$

$$\Theta(\nearrow(x,y,z)) = \nearrow(\Theta(x),y,z) = \nearrow(x,\Theta(y),z) = \nearrow(x,y,\Theta(z)), \ \forall \, x,y,z \in A. \tag{49}$$

The set of centroids of A is denoted by $C(A)$.

Definition 4.9. *A linear map $\Theta \in End(A)$ is said to be a quasi-centroid of A if*

$$< \Theta(x),y,z >=< x,\Theta(y),z >=< x,y,\Theta(z) >, \ \forall \ x,y,z \in A. \tag{50}$$

Definition 4.10. *A linear map $\Theta : A \to A$ is said to be a quasi-Centroid of a 3-L-dendriform algebra $(A, \nwarrow, \nearrow)$, if it satisfies the following conditions:*

$$\nwarrow(\Theta(x),y,z) = \nwarrow(x,\Theta(y),z) = \nwarrow(x,y,\Theta(z)), \tag{51}$$

$$\nearrow(\Theta(x),y,z) = \nearrow(x,\Theta(y),z) = \nearrow(x,y,\Theta(z)), \ \forall \, x,y,z \in A. \tag{52}$$

The set of quasi-centroids of A is denoted by $QC(A)$. It is obvious that $C(A) \subset QC(A)$.

Proposition 4.11. *$GDer(A)$, $QDer(A)$ and $C(A)$ are Lie subalgebras of $gl(A)$.*

Proof. Assume that $D_1, D_2 \in GDer(A)$. For all $x,y,z \in A$, we have

$$\begin{aligned}
\nwarrow(D_1D_2(x),y,z) &= D_1^{'''}D_2^{'''}(\nwarrow(x,y,z)) - D_1^{'''}\nwarrow(x,D_2^{'}(y),z) - D_1^{'''}\nwarrow(x,y,D_2^{''}(z))\\
&- \nwarrow(D_2(x),D_1^{'}(y),z) - \nwarrow(D_2(x),y,D_1^{''}(z))\\
&= D_1^{'''}D_2^{'''}(\nwarrow(x,y,z)) - \nwarrow(D_1(x),D_2^{'}(y),z) - \nwarrow(x,D_1^{'}D_2^{'}(y),z)\\
&- \nwarrow(x,D_2^{'}(y),D_1^{''}(z)) - \nwarrow(D_1(x),y,D_2^{''}(z)) - \nwarrow(x,D_1^{'}(y),D_2^{''}(z))\\
&- \nwarrow(x,y,D_1^{''}D_2^{''}(z)) - \nwarrow(D_2(x),D_1^{'}(y),z) - \nwarrow(D_2(x),y,D_1^{''}(z)),
\end{aligned}$$

and

$$\begin{aligned}
\nwarrow(D_2D_1(x),y,z) &= D_2^{'''}D_1^{'''}(\nwarrow(x,y,z)) - D_2^{'''}\nwarrow(x,D_1^{'}(y),z) - D_2^{'''}\nwarrow(x,y,D_1^{''}(z))\\
&- \nwarrow(D_1(x),D_2^{'}(y),z) - \nwarrow(D_1(x),y,D_2^{''}(z))\\
&= D_2^{'''}D_1^{'''}(\nwarrow(x,y,z)) - \nwarrow(D_2(x),D_1^{'}(y),z) - \nwarrow(x,D_2^{'}D_1^{'}(y),z)\\
&- \nwarrow(x,D_1^{'}(y),D_2^{''}(z)) - \nwarrow(D_2(x),y,D_1^{''}(z)) - \nwarrow(x,D_2^{'}(y),D_1^{''}(z))\\
&- \nwarrow(x,y,D_2^{''}D_1^{''}(z)) - \nwarrow(D_1(x),D_2^{'}(y),z) - \nwarrow(D_1(x),y,D_2^{''}(z))
\end{aligned}$$

Thus for all $x,y,z\in A$, we have

$$\nwarrow([D_1,D_2](x),y,z)=[D_1''',D_2'''](\nwarrow(x,y,z))-\nwarrow(x,y,[D_1'',D_2''](z))-\nwarrow(x,[D_1',D_2'](y),z).$$

Similarly, we can get

$$\nearrow([D_1,D_2](x),y,z)=[D_1''',D_2'''](\nearrow(x,y,z))-\nearrow(x,y,[D_1'',D_2''](z))-\nearrow(x,[D_1',D_2'](y),z).$$

From the definition of generalized derivation, one gets $[D_1,D_2]\in GDer(A)$, so $GDer(A)$ is a Lie-subalgebra of $gl(A)$.

Using a similar computation, we can prove that $QDer(A)$ and $C(A)$ are Lie subalgebras of $gl(A)$. □

Proposition 4.12. *Let $(A,\nwarrow,\nearrow)$ be a 3-L-dendriform algebra, $D\in Der(A)$ and $\Theta\in C(A)$. Then*

$$[D,\Theta]\in C(A).$$

Proof. Assume that $D\in Der(A)$, $\Theta\in C(A)$. For arbitrary $x,y\in A$, we have

$$\begin{aligned}D\Theta(\nwarrow(x,y,z))&=D(\nwarrow(\Theta(x),y,z))\\&=\nwarrow(D\Theta(x),y,z)+\nwarrow(\Theta(x),D(y),z)+\nwarrow(\Theta(x),y,D(z))\end{aligned}\tag{53}$$

and

$$\begin{aligned}\Theta D(\nwarrow(x,y,z))&=\Theta(\nwarrow(D(x),y,z)+\nwarrow(\Theta(x),D(y),z)+\nwarrow(\Theta(x),y,D(z)))\\&=\nwarrow(\Theta D(x),y,z)+\nwarrow(\Theta(x),D(y),z)+\nwarrow(\Theta(x),y,D(z)).\end{aligned}\tag{54}$$

By making the difference of equations (53) and (54), we get

$$[D,\Theta](\nwarrow(x,y,z))=\nwarrow([D,\Theta](x),y,z).$$

Similarly we can proof that, for all $x,y,z\in A$,

$$[D,\Theta](\nwarrow(x,y,z))=\nwarrow(x,[D,\Theta](y),z),\ \ [D,\Theta](\nwarrow(x,y,z>))=\nwarrow(x,y,[D,\Theta](z))$$

and

$$[D,\Theta](\nearrow(x,y,z))=\nearrow([D,\Theta](x),y,z)=\nearrow(x,[D,\Theta](y),z)=\nearrow(x,y,[D,\Theta](z)).$$

□

Proposition 4.13. $C(A)\subseteq QDer(A)$.

Proof. straightforward. □

Proposition 4.14. *Let $(A,\nwarrow,\nearrow)$ be a 3-L-Dendriform algebra and $D\in GDer(A)$ associates with (D',D'',D'''). Then D is a generalized derivation of associated horizontal 3-pre-Lie algebras $(A,\{\cdot,\cdot,\cdot\}^h)$ and vertical 3-pre-Lie algebras $(A,\{\cdot,\cdot,\cdot\}^v)$ defined in Proposition 3.3 associates with (D',D'',D''').*

Proof. Let $x,y,z\in A$ and using definition of bracket $\{\cdot,\cdot,\cdot\}^h$ in (25), we have

$$\begin{aligned}D'''(\{x,y,z\}^h)=&D'''(\nwarrow(x,y,z)+\nearrow(x,y,z)-\nearrow(y,x,z))\\=&\nwarrow(D(x),y,z)+\nwarrow(x,D'(y),z)+\nwarrow(x,y,D''(z))\\&+\nearrow(D(x),y,z)+\nearrow(x,D'(y),z)+\nearrow(x,y,D''(z))\\&-\nearrow(D(y),x,z)-\nearrow(y,D'(x),z)-\nearrow(y,x,D''(z))\\=&\{D(y),x,z\}^h+\{y,D'(x),z\}^h+\{y,x,D''(z)\}^h.\end{aligned}$$

Similarly, we can proof that

$$D'''(\{x,y,z\}^v)=\{D(y),x,z\}^v+\{y,D'(x),z\}^v+\{y,x,D''(z)\}^v.$$

□

Proposition 4.15. *Let $(A, \{\cdot,\cdot,\cdot\})$ be a 3-pre-Lie algebra and $D : A \to A$ be a generalized derivation on A associates with (D', D'', D''') and let $R : A \to A$ be a Rota-Baxter operator of weight 0 commuting with D, D', D'' and D''' . Then D is a generalized derivation of the compatible 3-L-dendriform algebra defined in Corollary 3.8 associates with (D', D'', D''').*

Proof. Let $x, y, z \in A$. Using the definition of structures $\nwarrow, \nearrow$ given in (30), we have

$$\begin{aligned} D'''(\nwarrow(x,y,z)) &= D'''(\{Rx, Ry, z\}) \\ &= \{D(Rx), Ry, z\} + \{Rx, D'R(y), z\} + \{Rx, Ry, D''(z)\} \\ &= \nwarrow(D(x), y, z) + \nwarrow(x, D'(y), z) + \nwarrow(x, y, D''(z)) \end{aligned}$$

and

$$\begin{aligned} D'''(\nearrow(x,y,z)) &= D'''(\{x, Ry, Rz\}) \\ &= \{D(x), Ry, Rz\} + \{x, D'R(y), Rz\} + \{x, Ry, D''(Rz)\} \\ &= \nearrow(D(x), y, z) + \nearrow(x, D'(y), z) + \nearrow(x, y, D''(z)). \end{aligned}$$

□

Proposition 4.16. *Let $(A, \nwarrow, \nearrow)$ be a 3-L-dendriform algebra. Then*

a) $D(A) \oplus C(A) \subset QD(A)$.

b) $QD(A) + QC(A) \subset GD(A)$.

Proof. Straightforward. □

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