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On the Existence and Ulam–Hyers Stability of a New Class of Partial $(\phi, \chi)-$Fractional Differential Equations With Impulses

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Abstract. In this paper, we consider the newly defined partial $(\phi, \chi)-$fractional integral and derivative to study a new class of partial fractional differential equations with impulses. The existence and Ulam–Hyers stability of solutions for the proposed equation are investigated via the means of measure of noncompactness and fixed point theorems. The presented results are quite general in their nature and further complement the existing ones.

1. Introduction

One of the most important tasks in mathematics is to prove results which are more general in their nature. The differential equations of integer order are well known in literature and thus a huge amounts of work have been achieved with respect to their theory and applications. One of the generalizations of ordinary differential equations is so called as fractional differential equations that is based on the fractional calculus which explores various possibilities of determining the differentiation and integration for an arbitrary order. It is worth mentioning that many physical phenomena having memory and genetic characteristics can be adequately described by using systems governed by fractional differential equations. The physical phenomena often demonstrate fractional dynamical behavior due to the property of nonlocal behaviour [18, 19, 28, 30, 31].

Qualitative theory of fractional differential equations have been investigated by many researchers [4, 5, 27, 34]. The theory of impulsive fractional differential equations has also gained considerable attention and as a result many researchers have reported very useful results in these settings [20–22, 29, 35, 36, 38]. It is realized that impulsive fractional differential equations can provide ideal models to explain the actual

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processes that at some moments suddenly deviate from their states and can not be represented using the classical fractional differential equations. Over the past three decades, the problem of impulsive partial fractional differential equations have received a great deal of interest amongst many mathematicians and scientists due to their widespread applications in different real world phenomena [1–3]. A few number of research papers, however, have been comparably reported in this regard.

On the other hand, the study of stability for fractional differential equations appears to be very crucial topic. There are several types of stability. One of its well known types is the so known as Ulam stability which has been introduced by Ulam for functional equation. Ulam originally addressed the consistency of functional equations in a speech given at the University of Wisconsin in 1940. His question was: under what circumstances does an additive mapping occur similar to an approximately additive mapping?. In 1941, Hyers gave the first response to Ulam's question in the case of Banach spaces in [26]. This type of stability is then called the Ulam–Hyers stability. In 1978, Rassias [32] introduced a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been paid to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of integer order fractional differential equations [12–15, 24, 33, 37].

The problem of existence and stability of solutions of fractional order Darboux problem was studied by researchers in the papers [6–11, 16] and many others. In [8], Abbas and Benchohra proved the existence results by applying the nonlinear alternative of Leray–Schauder type fixed point theorem for the following problem

$$\begin{cases} \left(D^{\nu}_{\mathbf{O}}u\right)(t,r) = \mathrm{f}\left(t,r,u(t,r),D^{\nu}_{\mathbf{O}}u(t,r)\right),\ (t,r)\in\mathfrak{J},\ t\neq t_k,\ k=1,2,\cdots,m,\\ u(t_k^+,r) = u(t_k^-,r)+I_k(u(t_k^-,r)),\ r\in[0,b],\ k=1,2,\cdots,m,\\ u(t,0)=\varphi(t),\ t\in[0,a],\ u(0,r)=\psi(r),\ r\in[0,b],\ \varphi(0)=\psi(0), \end{cases}$$

where $a,b>0$, $\mathbf{O}=(0,0)$, $D^{\nu}_{\mathbf{O}}$ is the mixed regularized derivative of order $\boldsymbol{\nu}=(\nu_1,\nu_2)\in(0,1]\times(0,1]$, $0=t_0<t_1\cdots<t_m<t_{m+1}=a$, $\mathrm{f}:\mathfrak{J}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=1,2,\cdots,m$, $\varphi:[0,a]\to B$ and $\psi:[0,b]\to B$ are given absolutely continuous functions. In another work [6], Abbas *et al.* considered a class of partial fractional differential equations with non instantaneous impulses on a Banach space B. They investigated the stability and uniqueness of the solution.

For the sake of a deep understanding of the behavior of complex systems, generalization of fractional differential operators was subject to an intensive debate in the last few years. The objective of this paper is to resume this trend and establish existence and stability criteria for the following class of impulsive partial fractional differential equations

$$\begin{cases} \left({}^cD^{\boldsymbol{\nu},\phi,\chi}_{\mathbf{O}}u\right)(t,r) = \mathrm{f}\left(t,r,u(t,r),{}^cD^{\boldsymbol{\nu},\phi,\chi}_{\mathbf{O}}u(t,r)\right),\ \ (t,r)\in\mathfrak{J}:=[0,a]\times[0,b],\ t\neq t_k,\ k=1,2,\cdots,m,\\ u(t_k^+,r) = u(t_k^-,r)+I_k(u(t_k^-,r)),\ r\in[0,b],\ k=1,2,\cdots,m,\\ u(t,0)=\varphi(t),\ t\in[0,a],\ u(0,r)=\psi(r),\ r\in[0,b],\ \varphi(0)=\psi(0), \end{cases} \tag{1}$$

within the newly defined partial (ϕ,χ)−fractional integral and derivative. Here $a,b>0$, $\mathbf{O}=(0,0)$, ${}^cD^{\boldsymbol{\nu},\phi,\chi}_{\mathbf{O}}$ is the fractional (ϕ,χ)−partial Caputo derivative of order $\boldsymbol{\nu}=(\nu_1,\nu_2)\in(0,1]\times(0,1]$, $0=t_0<t_1\cdots<t_m<t_{m+1}=a$, $\mathrm{f}:\mathfrak{J}\times B\times B\to B$, $I_k:B\to B$, $k=1,2,\cdots,m$, $\varphi:[0,a]\to B$ and $\psi:[0,b]\to B$ are given absolutely continuous functions and B is a Banach space. Furthermore $\phi,\chi\in C^2([0,a],[0,b])$ are second order continuously differentiable strictly increasing and positive functions on $[0,a],[0,b]$, such that $\phi'(t),\chi'(r)\neq 0$ for all $(t,r)\in\mathfrak{J}$. We consider equation (1) in frame of partial fractional derivative with respect to two functions (ϕ,χ) which generalizes the above cited problems to a larger class of functions. Unlike previous results, the main theorems are proved by using different approach that is based on the measure of noncompactness and Mönch's fixed point theorem. Furthermore, we establish sufficient conditions for the generalized Ulam–Hyers–Rassias stability of the solutions of equation (1).

2. Auxiliary preliminaries

By $C(\mathfrak{J})$, we denote the Banach space of all continuous functions from $\mathfrak{J}$ in to B equipped with the norm $\|u\|_\infty = \sup_{(t,r)\in\mathfrak{J}} \|u(t,r)\|$. The $L^1(\mathfrak{J})$ is the space of Lebegue integrable functions $u : \mathfrak{J} \to B$ with the norm

$$\|u\|_{L_1} = \int_0^a \int_0^b \|u(t,r)\| dr dt.$$

Definition 2.1. *Let* $\mathbf{O} = (0,0)$, $\boldsymbol{\nu} = (\nu_1, \nu_2) > \mathbf{O}$, $u \in L^1(\mathfrak{J})$ *and* $\phi, \chi \in C^2([0,a],[0,b])$ *be positive strictly increasing functions such that* $\phi'(t), \chi'(r) \neq 0$ *for all* $(t,r) \in \mathfrak{J}$. *The partial Riemann–Liouville* (ϕ,χ)*–fractional integral of order* $\boldsymbol{\nu}$ *of* $u(t,r)$ *with respect to* ϕ, χ *is defined as*

$$I_{\mathbf{O}}^{\boldsymbol{\nu},\phi,\chi} u(t,r) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^t \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)u(s,\tau)d\tau ds.$$

In particular, we write

$$(I_{\mathbf{O}}^{\mathbf{O},\phi,\chi} u)(t,r) = u(t,r), \quad \text{and} \quad (I_{\mathbf{O}}^{\upsilon,\phi,\chi} u)(t,r) = \int_0^t \int_0^r u(s,\tau)\phi'(s)\chi'(\tau)d\tau ds,$$

for almost all $(t,r) \in \mathfrak{J}$, *where* $\upsilon = (1,1)$. *Moreover*

$$(I_{\mathbf{O}}^{\boldsymbol{\nu},\phi,\chi} u)(t,0) = (I_{\mathbf{O}}^{\boldsymbol{\nu}} u)(0,r) = 0, \ t \in [0,a], \ r \in [0,b].$$

If $\phi = \chi$, *then we write* $I_{\mathbf{O}}^{\boldsymbol{\nu},\phi,\chi} = I_{\mathbf{O}}^{\boldsymbol{\nu},\phi}$.

Example 2.2. *Let* $\boldsymbol{\nu} \in (0,1] \times (0,1]$, *and* $\beta, \gamma > -1$, $\phi(0), \chi(0) = 0$. *Then*

$$I_{\mathbf{O}}^{\boldsymbol{\nu}} (\phi(t))^\beta (\chi(r))^\gamma = \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\nu_1+\beta+1)\Gamma(\nu_2+\gamma+1)} (\phi(t))^{\nu_1+\beta}(\chi(r))^{\nu_2+\gamma},$$

for almost all $\phi(t)$, $\chi(r) \in C^2([0,a],[0,b])$.

Definition 2.3. *Let* $u \in L^1(\mathfrak{J})$ *and* $\boldsymbol{\nu} \in (0,1] \times (0,1]$. *Then the partial Caputo* (ϕ,χ)*–fractional derivative of order* $\boldsymbol{\nu}$ *of* $u(t,r)$ *is defined by*

$$\begin{aligned} {}^cD_{\mathbf{O}}^{\boldsymbol{\nu},\phi,\chi} u(t,r) &= (I_{\mathbf{O}}^{1-\boldsymbol{\nu}} D_{tr}^{\phi,\chi} u)(t,r) \\ &= \frac{1}{\Gamma(1-\nu_1)\Gamma(1-\nu_2)} \int_0^t \int_0^r \frac{(D_{s\tau}^{\phi,\chi} u)(s,\tau)}{(\phi(t)-\phi(s))^{\nu_1}(\chi(r)-\chi(\tau))^{\nu_2}} \chi'(\tau)\phi'(s) d\tau ds. \end{aligned}$$

For $\upsilon = (1,1)$, *we have*

$${}^cD_{\mathbf{O}}^{\upsilon,\phi,\chi} u(t,r) = (D_{tr}^{\phi,\chi} u)(t,r),$$

for almost all $(t,r) \in \mathfrak{J}$. *By* $1-\boldsymbol{\nu}$ *we mean* $(1-\nu_1, \ 1-\nu_2) \in [0,1) \times [0,1)$.

Example 2.4. *Let* $\beta, \gamma > -1$, $\phi(0), \chi(0) = 0$, $\boldsymbol{\nu} \in (0,1] \times (0,1]$. *Then*

$${}^cD_{\mathbf{O}}^{\boldsymbol{\nu},\phi,\chi} (\phi(t))^\beta (\chi(r))^\gamma = \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta-\nu_1+1)\Gamma(\gamma-\nu_2+1)} (\phi(t))^{\beta-\nu_1}(\chi(r))^{\gamma-\nu_2},$$

for almost all $\phi(t)$, $\chi(r) \in C^2([0,a],[0,b])$.

Denote by $D^{\phi,\chi}_{tr} = (\frac{1}{\phi'(t)\chi'(r)}\frac{\partial^2}{\partial t\partial r})$, the mixed order partial $(\phi,\chi)-$fractional derivative. In what follows, we present some properties of partial Riemann–Liouville $(\phi,\chi)-$fractional integral and the mixed order partial $(\phi,\chi)-$fractional derivative.

Lemma 2.5. *The following properties are satisfied:*

(P_1) $I^{\nu,\phi,\chi}_{\mathbf{O}} I^{\bar{\nu},\phi,\chi}_{\mathbf{O}} u = I^{\nu+\bar{\nu},\phi,\chi}_{\mathbf{O}} u$. *(Semi group property).*

(P_2) $I^{1,\phi,\chi}_{\mathbf{O}} D^{\phi,\chi}_{tr} u)(t,r) = u(t,r) - u(0,r) - u(t,0) + u(0,0)$.

Proof. The proof of (P_1) follows from the definition. We proceed to prove (P_2) as follows:

$$\begin{aligned}
(I^{1,\phi,\chi}_{\mathbf{O}} D^{\phi,\chi}_{tr} u)(t,r) &= I^{1,\phi,\chi}_{\mathbf{O}}\left[\left(\frac{1}{\phi'(t)\chi'(r)}\frac{\partial^2}{\partial t\partial r}\right)u(t,r)\right] \\
&= \int_0^t\int_0^r \phi'(s)\chi'(\tau)\left[\left(\frac{1}{\phi'(s)\chi'(\tau)}\frac{\partial^2}{\partial s\partial \tau}\right)u(s,\tau)\right]d\tau ds \\
&= \int_0^t\left[\int_0^r \frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial s}u(s,\tau)\right)d\tau\right]ds \\
&= \int_0^t\left[\frac{\partial}{\partial s}u(s,r) - \frac{\partial}{\partial s}u(s,0)\right]ds \\
&= u(t,r) - u(0,r) - u(t,0) + u(0,0),
\end{aligned}$$

which completes the proof. □

Lemma 2.6. *Let $f : \mathfrak{J}\times B\times B \to B$ be a continuous function. Then $u \in C$ is a solution of the problem:*

$$\begin{cases}
({}^cD^{\nu,\phi,\chi}_{\mathbf{O}} u)(t,r) = f(t,r,u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)),\ (t,r)\in\mathfrak{J} \\
u(t,0) = \varphi(t),\ t\in[0,a], \\
u(0,r) = \psi(r),\ r\in[0,b], \\
\varphi(0) = \psi(0),
\end{cases} \tag{2}$$

if and only if $u(t,r)$ satisfies

$$u(t,r) = \omega(t,r) + I^{\nu,\phi,\chi}_{\mathbf{O}}\mathrm{f}(t,r,u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)),\ (t,r)\in\mathfrak{J}, \tag{3}$$

where $\omega(t,r) = \varphi(t) + \psi(r) - \varphi(0)$.

Proof. Let $u(t,r)$ be a solution of problem (2). Then by the definition of the derivative $({}^cD^{\nu,\phi,\chi}_{\mathbf{O}} u)(t,r)$, we have

$$I^{1-\nu,\phi,\chi}_{\mathbf{O}}(D^{\phi,\chi}_{tr} u)(t,r) = \mathrm{f}(t,r,u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)).$$

Thus, applying $I^{\nu,\phi,\chi}_{\mathbf{O}}$ on both sides of the above equation, we get

$$I^{\nu,\phi,\chi}_{\mathbf{O}}(I^{1-\nu,\phi,\chi}_{\mathbf{O}}(D^{\phi,\chi}_{tr} u))(t,r) = (I^{\nu,\phi,\chi}_{\mathbf{O}}\mathrm{f})((t,r),u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)).$$

Consequently by the semi group property (P_1), we have

$$I^{1,\phi,\chi}_{\mathbf{O}} D^{\phi,\chi}_{tr} u(t,r) = I^{\nu,\phi,\chi}_{\mathbf{O}}\mathrm{f}(t,r,u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)).$$

Since from property (P_2), we have $I^{1,\phi,\chi}_{\mathbf{O}} D^{\phi,\chi}_{tr} u(t,r) = u(t,r) - \omega(t,r)$, thus, we obtain

$$u(t,r) = \omega(t,r) + I^{\nu,\phi,\chi}_{\mathbf{O}}\mathrm{f}(t,r,u(t,r),D^{\nu,\phi,\chi}_{\mathbf{O}} u(t,r)).$$

If we let $u(t,r)$ satisfy (3), then, clearly $u(t,r)$ satisfies (2). □

Lemma 2.7. *[25] Let function* $\mathrm{f} : \mathfrak{J} \times B \times B \to B$ *be continuous. Then problem* (2) *is equivalent to the equation*

$$g(t,r) = \mathrm{f}(t,r,\omega(t,r) + I_{\mathbf{O}}^{\nu,\phi,\chi} g(t,r), g(t,r)),$$

and if $g \in C(\mathfrak{J})$ *is the solution of this equation, then* $u(t,r) = \omega(t,r) + I_{\mathbf{O}}^{\nu,\phi,\chi} g(t,r)$.

To proceed further, we set

$$\mathfrak{J}_k := (t_k, t_{k+1}] \times [0,b].$$

To define the solutions of problem (1), we shall consider the space $PC(\mathfrak{J}, B) = \Big\{u : \mathfrak{J} \to B : u \in C(\mathfrak{J}_k, B);\ k = 0,1.\cdots, m$, and there exist $u(t_k^-, r)$ and $u(t_k^+, r)$; $k = 1,2\cdots, m$, with $u(t_k^-, r) = u(t_k, r)$ for each $r \in [0,b]\Big\}$. The set $PC(\mathfrak{J}, B)$ is a Banach space equipped with the norm

$$\|u\|_{PC} = \sup_{(t,r)\in\mathfrak{J}} \|u(t,r)\|.$$

Set

$$\mathfrak{J}' := \mathfrak{J}/\{(t_1,r),\cdots,(t_m,r),\ r \in [0,b]\}.$$

Definition 2.8. *Let* $u \in PC(\mathfrak{J}, B)$ *and* $\boldsymbol{\nu}$*th−derivative of u exist on* $\mathfrak{J}'$. *Then u is said to be a solution of problem* (1) *if u satisfies* $({}^cD_{\mathbf{O}}^{\nu,\phi,\chi} u)(t,r) = f(t,r,u(t,r), D_{\mathbf{O}}^{\nu,\phi,\chi} u(t,r))$ *on* $\mathfrak{J}'$ *and the conditions of problem* (1) *are satisfied.*

For the next lemma, we let $h \in C([t_k, t_{k+1}] \times [0,b], B)$, $z_k = (t_k, 0)$, *and*

$$\omega_k(t,r) = u(t,0) + u(t_k^+, r) - u(t_k^+, 0),\ \ k = 0,1,\cdots,m. \tag{4}$$

Lemma 2.9. *A function* $u \in AC([t_k, t_{k+1}] \times [0,b], B)$; $k = 0,1\cdots,m$ *is the solution of the differential equation*

$$({}^cD_{z_k^+}^{\nu,\phi,\chi} u)(t,r) = h(t,r);\ (t,r) \in [t_k, t_{k+1}] \times [0,b],$$

with condition (4) *if and only if* $u(t,r)$ *satisfies*

$$u(t,r) = \omega_k(t,r) + (I_{z_k^+}^{\nu,\phi,\chi} h)(t,r);\ (t,r) \in [t_k, t_{k+1}] \times [0,b]. \tag{5}$$

Lemma 2.10 can be proved by using the same algorithms as in the proof of Lemma 3.3 in [10]. Lemma 3.3 in [10] is the special case of Lemma 2.10, if we take $\phi(t) = \chi(t) = t$.

Lemma 2.10. *Let* $\nu_1, \nu_2 \in (0,1]$ *and* $h : \mathfrak{J} \to B$ *be continuous. A function u is a solution of the fractional integral equation*

$$u(t,r) = \begin{cases} \omega(t,r) + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^t \int_0^r (\phi(t) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1} \\ \phi'(s)\chi'(\tau) h(s,\tau) d\tau ds;\ \text{if } (t,r) \in [0,t_1] \times [0,b], \\ \omega(t,r) + \sum_{i=1}^k (I_i(u(t_i^-, r)) - I_i(u(t_i^-, 0))) \\ + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_0^r (\phi(t_i) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1} \\ \phi'(s)\chi'(\tau) h(s,\tau) d\tau ds \\ + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^t \int_0^r (\phi(t) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1} \\ \phi'(s)\chi'(\tau) h(s,\tau) d\tau ds; \\ \text{if } (t,r) \in (t_k, t_{k+1}] \times [0,b],\ k = 1,2\cdots,m, \end{cases} \tag{6}$$

if and only if u is a solution of the fractional initial value problem

$$\begin{aligned} {}^cD_{z_k^+}^{\nu,\phi,\chi} u(t,r) &= h(t,r),\ (t,r) \in \mathfrak{J}',\ k = 1,2,\cdots,m, \\ u(t_k^+, r) &= u(t_k^-, r) + I_k(u(t_k^-, r)),\ r \in [0,b],\ k = 1,2,\cdots,m. \end{aligned} \tag{7}$$

In view of Lemma 2.7 and Lemma 2.10, we state following.

Lemma 2.11. *Let the function* $\mathrm{f} : \mathfrak{J} \times B \times B \to B$ *be continuous. Then Problem* (1) *is equivalent to the problem*

$$g(t,r) = \mathrm{f}(t,r,\xi(t,r), g(t,r)), \tag{8}$$

where

$$\xi(t,r) = \begin{cases} \omega(t,r) + \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^t \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\ \phi'(s)\chi'(\tau) g(s,\tau) d\tau ds; \ \textit{if } (t,r) \in [0,t_1]\times[0,b], \\ \omega(t,r) + \sum_{i=1}^k (I_i(u(t_i^-,r)) - I_i(u(t_i^-,0))) \\ +\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \int_0^r (\phi(t_i)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\ \phi'(s)\chi'(\tau) g(s,\tau) d\tau ds \\ +\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^t \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\ \phi'(s)\chi'(\tau) g(s,\tau) d\tau ds; \\ \textit{if } (t,r) \in (t_k, t_{k+1}]\times[0,b], \ k = 1,2\cdots,m, \end{cases}$$

$$\omega(t,r) = \varphi(t) + \psi(r) - \varphi(0).$$

Furthermore, if $g \in C(\mathfrak{J})$ *is the solution of* (8), *then* $u(t,r) = \xi(t,r)$.

Now we establish the Ulam–Hyers stability for problem (1). Let $\epsilon > 0$ and $\theta : \mathfrak{J} \to [0,\infty)$ be a continuous function.

Definition 2.12. *Problem* (1) *is Ulam–Hyers–Rassias stable with respect to* θ *if there exists a constant* $c_{\Phi,\theta} > 0$ *such that for each solution* $u \in PC$ *of the inequality* $\|u(t,r) - (\Phi u)(t,r)\|_B \le \theta(t,r)$; $(t,r) \in \mathfrak{J}$, *there exists a solution* $\overline{u} \in PC$ *of problem* (1) *with*

$$\|u(t,r) - \overline{u}(t,r)\|_B \le c_{\Phi,\theta}\theta(t,r); \ (t,r) \in \mathfrak{J}.$$

Lemma 2.13. *[6] Let* $\beta > 0$, b *is a nonnegative locally integrable function on* $t \in [0,T)$ *and* η *be a nonnegative, nondecreasing continuous function defined on* $t \in [0,T)$, $\eta(t) \le M$ *and let* $u(t)$ *is nonnegative and locally integrable on* $[0,T)$ *with*

$$u(t) \le b(t) + \eta(t)\int_0^t (t-s)^{\beta-1} u(s) ds.$$

Then for $t \in [0,T)$, *we have*

$$u(t) \le b(t) + \int_0^t \left[\sum_{n=1}^\infty \frac{(\eta(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} b(s) \right] ds.$$

The above lemma can easily be extended to functions of two variables.

Lemma 2.14. *Let* $\nu_1,\nu_2 > 0$, $\phi,\chi \in C^2([0,a],[0,b])$ *are strictly increasing functions such that* $\phi'(t), \chi'(r) \neq 0$ *for all* $(t,r) \in \mathfrak{J}$, $b(t,r)$ *is nonnegative function locally integrable on* $\mathfrak{J}$, $\eta(t,r) \le M$, *and suppose* $u(t,r)$ *is nonnegative and locally integrable on* $\mathfrak{J}$ *with*

$$u(t,r) \le b(t,r) + \eta(t,r)\int_0^t \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} u(s,\tau) ds d\tau.$$

Then

$$u(t,r) \le b(t,r) + \int_0^t \int_0^r \Big[\sum_{n=1}^\infty \frac{(\eta(t,r)\Gamma(\nu_1)\Gamma(\nu_2))^n}{\Gamma(n\nu_1)\Gamma(n\nu_2)} (\phi(t)-\phi(s))^{n\nu_1-1}(\chi(r)-\chi(\tau))^{n\nu_2-1} b(s,\tau) \Big] ds d\tau, \ \textit{on } \mathfrak{J}.$$

Definition 2.15. *[23] Let B be a Banach space and Ω_B be the bounded subset of B. The Kuratowski measure of noncompactness is the map $\gamma : \Omega_B \to [0,+\infty)$ defined by*

$$\gamma(E) = \inf\left\{ d > 0 : E \subset \bigcup_{i=1}^{n} E_i,\ diam(E_i) < d \right\},\ E_i \in \Omega_B.$$

Proposition 2.16. *[23] The Kuratowski measure of noncompactness satisfies the following properties:*

(i) $\gamma(E) = 0 \Longleftrightarrow \overline{E}$ *is compact;*

(ii) $\gamma(E) = \gamma(\overline{E})$;

(iii) $E \subset F \Rightarrow \gamma(E) \leq \gamma(F)$;

(iv) $\gamma(E + F) \leq \gamma(E) + \gamma(F)$;

(v) $\gamma(cE) = |c|\gamma(E);\ c \in \mathbb{R}$;

(vi) $\gamma(convE) = \gamma(E)$.

Theorem 2.17. *[23] Let S be a closed, bounded and convex subset of a Banach space such that $0 \in S$, and let $\Phi : S \to S$ be a continuous mapping. If the implication $W = \overline{conv}\Phi(W)$ or $W = \Phi(W) \cup \{0\} \Longrightarrow \gamma(W) = 0$, hold for every $W \subset S$, then Φ has a fixed point.*

3. Main results

In this section, we investigate the main results for the existence and Ulam–Hyers stability of solution for problem (1). Before we proceed, we set forth these hypotheses:

(S_1) The function $\mathrm{f} : \mathfrak{J} \times B \times B \to B$ be continuous,

(S_2) There exist constants $l_f > 0,\ 0 < l'_f < 1$ such that

$$\|\mathrm{f}(t, r, u, v) - \mathrm{f}(t, r, \overline{u}, \overline{v})\| \leq l_f\|u - \overline{u}\| + l'_f\|v - \overline{v}\|,$$

for each $u, v, \overline{u}, \overline{v} \in B$ and $(t, r) \in \mathfrak{J}$.

(S_3) There exist a constant $l^* > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \leq l^*\|u - \overline{u}\|,$$

for any $u, \overline{u} \in B,\ k = 1, 2, \cdots, m$.

Theorem 3.1. *Suppose that $(S_1) - (S_3)$ are satisfied. If*

$$2ml^* + \frac{2l_f(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1 - l'_f)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)} < 1, \tag{9}$$

then problem (1) has unique solution on $\mathfrak{J}$.

Proof. Let us define an operator $\Phi : PC(\mathfrak{J}) \to PC(\mathfrak{J})$, by

$$\begin{aligned}
\Phi(u)(t, r) &= \omega(t, r) + \sum_{0<t_k<t} (I_k(u(t_k^-, r)) - I_k(u(t_k^-, 0)))\\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k) - \phi(s))^{\nu_1-1}(\chi(r) - \chi(\tau))^{\nu_2-1}\\
&\phi'(s)\chi'(\tau)g(s, \tau)d\tau ds\\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t) - \phi(s))^{\nu_1-1}(\chi(r) - \chi(\tau))^{\nu_2-1}\\
&\phi'(s)\chi'(\tau)g(s, \tau)d\tau ds,
\end{aligned}$$

where $g \in C(\mathfrak{J})$ such that

$$g(t,r) = \mathrm{f}(t,r,u(t,r),g(t,r)).$$

Certainly, the fixed points of the operator Φ are solutions of the problem (1). Let $u,\overline{u} \in PC(\mathfrak{J})$, then we have

$$\begin{aligned}
&\|\Phi(u)(t,r) - \Phi(\overline{u})(t,r)\| \le \sum_{k=1}^{m} \Big(\|I_k(u(t_k^-,r)) - I_k(\overline{u}(t_k^-,r))\| \\
&+ \|I_k(u(t_k^-,0)) - I_k(\overline{u}(t_k^-,0))\| \Big) \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k) - \phi(s))^{\nu_1 - 1} \\
&(\chi(r) - \chi(\tau))^{\nu_2 - 1} \phi'(s)\chi'(\tau)\|g(s,\tau) - h(s,\tau)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|g(s,\tau) - h(s,\tau)\| d\tau ds,
\end{aligned} \tag{10}$$

where $g,h \in C(\mathfrak{J})$ such that $g(t,r) = \mathrm{f}(t,r,u(t,r),g(t,r))$ and $h(t,r) = \mathrm{f}(t,r,\overline{u}(t,r),h(t,r))$.

By (S_2), we get the estimate

$$\|g(t,r) - h(t,r)\| \le l_f \|u(t,r) - \overline{u}(t,r)\| + l'_f \|g(t,r) - h(t,r)\|,$$

which implies

$$\|g(t,r) - h(t,r)\| \le \frac{l_f}{1 - l'_f} \|u(t,r) - \overline{u}(t,r)\| \le \frac{l_f}{1 - l'_f} \|u - \overline{u}\|_{PC}.$$

Thus (S_3) and (10) imply

$$\begin{aligned}
\|\Phi(u)(t,r) - \Phi(\overline{u})(t,r)\|_{PC} \le& \sum_{k=1}^{m} l^* \Big(\|u(t_k^-,r) - \overline{u}(t_k^-,r)\| + \|u(t_k^-,0) - \overline{u}(t_k^-,0)\| \Big) \\
&+ \frac{l_f}{(1 - l'_f)\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k) - \phi(s))^{\nu_1 - 1} \\
&(\chi(r) - \chi(\tau))^{\nu_2 - 1} \phi'(s)\chi'(\tau) \|u - \overline{u}\|_{PC} d\tau ds \\
&+ \frac{l_f}{(1 - l'_f)\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t) - \phi(s))^{\nu_1 - 1} \\
&(\chi(r) - \chi(\tau))^{\nu_2 - 1} \phi'(s)\chi'(\tau) \|u - \overline{u}\|_{PC} d\tau ds, \\
\le& \left(2ml^* + \frac{2 l_f (\phi(a))^{\nu_1} (\chi(b))^{\nu_2}}{(1 - l'_f)\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)} \right) \|u - \overline{u}\|.
\end{aligned}$$

From the above inequalities, we conclude that Φ is a contraction. Hence by contraction mapping principle Φ has a unique fixed point. □

We introduce the notation

$$\varsigma_n := \frac{(l_f)^n}{(1 - l'_f)^n (1 - 2ml^*)^n \Gamma(n\nu_1)\Gamma(n\nu_2)}.$$

Theorem 3.2. *Let the hypothesis* $(S_1)-(S_3)$ *hold and the following assumption hold*
(iv) $\theta \in L^1(\Im,[0,\infty))$ *and there exists* $\lambda_\theta = (\lambda_{\theta_1}+\lambda_{\theta_2})>0$ *such that, for each* $(t,r)\in\Im$*, we have*

$$\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^r\left[\sum_{n=1}^{\infty}\varsigma_n(\phi(t_k)-\phi(s))^{n\nu_1-1}(\chi(r)-\chi(\tau))^{n\nu_2-1}\phi'(s)\chi'(\tau)\theta(s,\tau)\right]d\tau ds \le \lambda_{\theta_1}\theta(t,r),$$

and

$$\int_0^{t_k}\int_0^r\left[\sum_{n=1}^{\infty}\varsigma_n(\phi(t)-\phi(s))^{n\nu_1-1}(\chi(r)-\chi(\tau))^{n\nu_2-1}\phi'(s)\chi'(\tau)\theta(s,\tau)\right]d\tau ds \le \lambda_{\theta_2}\theta(t,r).$$

If condition (9) *holds, then the problem* (1) *is a generalized Ulam–Hyers–Rassias stable.*

Proof. Let $u\in PC$ be a solution of the inequality $\|u-\Phi(u)\|_{PC}\le\theta(t,r)$; $(t,r)\in\Im$. It follows from Theorem 3.1, $\overline{u}$ is a unique fixed point of Φ. Thus for each $(t,r)\in\Im$, we have

$$\begin{aligned}\overline{u}(t,r)=\Phi(\overline{u})(t,r) &= \omega(t,r)+\sum_{0<t_k<t}(I_k(\overline{u}(t_k^-,r))-I_k(\overline{u}(t_k^-,0)))\\ &+\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^r(\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)h(s,\tau)d\tau ds\\ &+\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{t_k}^{t}\int_0^r(\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)h(s,\tau)d\tau ds.\end{aligned}$$

For each $(t,r)\in\Im$, it follows that

$$\begin{aligned}\|u(t,r)-\overline{u}(t,r)\|_{PC}=\|u(t,r)-\Phi(\overline{u})(t,r)\|_{PC} &\le \|u(t,r)-\Phi(u)(t,r)\|_{PC}+\|\Phi(u)(t,r)-\Phi(\overline{u})(t,r)\|_{PC}\\ &\le \theta(t,r)+2ml^*\|u(t,r)-\overline{u}(t,r)\|_{PC}\\ &+\frac{l_f}{(1-l'_f)\Gamma(\nu_1)\Gamma(\nu_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^r(\phi(t_k)-\phi(s))^{\nu_1-1}\\ &(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\|u-\overline{u}\|_{PC}d\tau ds\\ &+\frac{l_f}{(1-l'_f)\Gamma(\nu_1)\Gamma(\nu_2)}\int_{t_k}^{t}\int_0^r(\phi(t)-\phi(s))^{\nu_1-1}\\ &(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\|u-\overline{u}\|_{PC}d\tau ds\end{aligned}$$

or

$$\begin{aligned}\|u(t,r)-\overline{u}(t,r)\|_{PC} &\le \frac{1}{(1-2ml^*)}\theta(t,r)\\ &+\frac{l_f}{(1-l'_f)(1-2ml^*)\Gamma(\nu_1)\Gamma(\nu_2)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^r(\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\|u-\overline{u}\|_{PC}d\tau ds\\ &+\frac{l_f}{(1-l'_f)(1-2ml^*)\Gamma(\nu_1)\Gamma(\nu_2)}\int_{t_k}^{t}\int_0^r(\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\|u-\overline{u}\|_{PC}d\tau ds.\end{aligned}$$

By Lemma 2.14, we obtain

$$\begin{aligned}\|u(t,r)-\overline{u}(t,r)\|_{PC} \leq & \frac{1}{(1-2ml^*)}\theta(t,r)+\frac{1}{(1-2ml^*)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_0^r\Big[\sum_{n=1}^{\infty}\varsigma_n(\phi(t_k)-\phi(s))^{n\nu_1-1}(\chi(r)-\chi(\tau))^{n\nu_2-1} \\ & \phi'(s)\chi'(\tau)\theta(s,\tau)\Big]d\tau ds \\ & +\frac{1}{(1-2ml^*)}\int_{t_k}^{t}\int_0^r\Big[\sum_{n=1}^{\infty}\varsigma_n(\phi(t)-\phi(s))^{n\nu_1-1}(\chi(r)-\chi(\tau))^{n\nu_2-1} \\ & \phi'(s)\chi'(\tau)\theta(s,\tau)\Big]d\tau ds \\ & \leq \frac{1}{(1-2ml^*)}\left(1+\lambda_\theta\right)\theta(t,r)=c_{\Phi,\theta}\theta(t,r),\end{aligned}$$

from which we deduce that problem (1) is a generalized Ulam–Hyers–Rassias stable. □

For our next existence result, we make use of the following hypotheses:

(A_1) $\mathrm{f}:\mathfrak{J}\times B\times B\to B$ satisfies the Caratheodory conditions:

- $\mathrm{f}(t,r,u,v)$ is continuous in u,v for each fixed t and r.
- $\mathrm{f}(t,r,u,v)$ is measurable in t,r for each fixed u and v.

(A_2) There exist $p,q\in L^1(\mathfrak{J},\mathbb{R}^+)\cap C(\mathfrak{J},\mathbb{R}^+)$, such that
$\|\mathrm{f}(t,r,u,v)\|\leq p(t,r)\|u\|+q(t,r)\|v\|$, for $(t,r)\in\mathfrak{J}$ and each $u,v\in B$.

(A_3) There exists a real number $c>0$ such that $\|I_k(u)\|\leq c\|u\|$, for each $u\in B$.

(A_4) $I_k\in C(B,B)$ and for each bounded set $E\subset B$, we assume $\gamma(I_k(E))\leq c\gamma(E),\ k=1,2,\cdots,m$.

(A_5) For each $(t,r)\in\mathfrak{J}$ and each bounded set $E\subset B$, we assume
$\lim_{h\to0+}\gamma(\mathrm{f}(\mathfrak{J}_{(t,r),h}\times E\times E))\leq\frac{p(t,r)}{1-q(t,r)}\gamma(E)$; here $\mathfrak{J}_{(t,r),h}=\Big\{[t-h,t]\times[r-h,r]\Big\}\cap\mathfrak{J}$.

Theorem 3.3. *Suppose the hypotheses (S_2) and $(A_1)-(A_5)$ hold. Let $P^*=\sup_{(t,r)\in\mathfrak{J}}p(t,r)$ and $Q^*=\sup_{(t,r)\in\mathfrak{J}}q(t,r)<1$. If*

$$2mc+\frac{2P^*(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1-Q^*)\Gamma(\nu_1+1)\Gamma(\nu_2+1)}<1, \tag{11}$$

then problem (1) has at least one solution.

Proof. Let us define an operator $\Phi:PC(\mathfrak{J})\to PC(\mathfrak{J})$, by

$$\begin{aligned}\Phi(u)(t,r)=&\omega(t,r)+\sum_{0<t_k<t}(I_k(u(t_k^-,r))-I_k(u(t_k^-,0))) \\ & +\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_0^r(\phi(t_k)-\phi(s))^{\nu_1-1} \\ & (\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\mathrm{f}(s,\tau,u,v)d\tau ds \\ & +\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{t_k}^{t}\int_0^r(\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\ & \phi'(s)\chi'(\tau)\mathrm{f}(s,\tau,u,v)d\tau ds,\end{aligned}$$

where $v(t,r) = {}^cD_\theta^{\nu,\phi,\chi}u(t,r)$. Clearly the fixed points of the operator Φ are solutions of problem (1).
Define a closed bounded and convex set $B_{r_0} = \{u \in PC(\Im, B) : \|u\|_{PC} \leq r_0\}$, where

$$\frac{\|\omega\|_\infty}{1 - 2mc - \frac{2P^*(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1-Q^*)\Gamma(\nu_1+1)\Gamma(\nu_2+1)}} \leq r_0. \tag{12}$$

First we show that Φ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$, in $C([0,t_1] \times [0,b], B)$. In view of assumption (S_2), and for each $(t,r) \in [0,t_1] \times [0,b]$, we have

$$\begin{aligned}\|(\Phi u_n)(t,r) - (\Phi u)(t,r)\| &\leq \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^t \int_0^r (\phi(t_k) - \phi(s))^{\nu_1-1}(\chi(r) - \chi(\tau))^{\nu_2-1} \\ &\phi'(s)\chi'(\tau)\|f(s,t,u_n(s,\tau),v_n(s,\tau)) - f(s,t,u(s,\tau),v(s,\tau))\|d\tau ds \\ &\leq \frac{l_f}{(1-l'_f)\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^t \int_0^r (\phi(t_k) - \phi(s))^{\nu_1-1}(\chi(r) - \chi(\tau))^{\nu_2-1} \\ &\phi'(s)\chi'(\tau)\|u_n(s,\tau) - u(s,\tau)\|d\tau ds.\end{aligned}$$

Since f is Caratheodory type function, then by the Lebsegue dominated convergence theorem [17], we obtain

$$\|(\Phi u_n)(t,r) - (\Phi u)(t,r)\|_{C([0,t_1]\times[0,b],B)} \to 0, \;\; as \;\; n \to \infty.$$

For each $(t,r) \in (t_k, t_{k+1}] \times [0,b]$, $k = 1,2,\cdots,m$, we have

$$\begin{aligned}&\|(\Phi u_n)(t,r) - (\Phi u)(t,r)\| \leq \sum_{k=1}^m \|I_k(u_n(t_k^-,r)) - I_k(u(t_k^-,r))\| \\ &+ \sum_{k=1}^m \|I_k(u_n(t_k^-,0)) - I_k(u(t_k^-,0))\| \\ &+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k) - \phi(s))^{\nu_1-1} \\ &(\chi(r) - \chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau)\|f(s,\tau,u_n,v_n) - f(s,\tau,u,v)\|d\tau ds \\ &+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^t \int_0^r (\phi(t) - \phi(s))^{\nu_1-1}(\chi(r) - \chi(\tau))^{\nu_2-1} \\ &\phi'(s)\chi'(\tau)\|f(s,\tau,u_n,v_n) - f(s,\tau,u,v)\|d\tau ds.\end{aligned}$$

Since I_k is continuous, then again by Lebsegue dominated convergence theorem, we obtain

$$\|(\Phi u_n)(t,r) - (\Phi u)(t,r)\|_{C((t_k,t_{k+1}]\times[0,b],B)} \to 0, \; as \; n \to \infty.$$

Next we show that Φ maps B_{r_0} into itself.

For each $u \in B_{r_0}$, by (A_2), (A_3) and by condition (11), (12), we have for each $(t,r) \in \Im$, we have

$$\begin{aligned}
&\|(\Phi u)(t,r)\| \le \|\omega(t,r)\| + \sum_{k=1}^{m} \|I_k(u(t_k^-,r))\| + \sum_{k=1}^{m} \|I_k(u(t_k^-,0))\| \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s,\tau,u,v)\|d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1}\phi'(s)\chi'(\tau) \\
&\|\mathrm{f}(s,\tau,u,v)\|d\tau ds \\
&\le \|\omega\|_{PC} + 2mc\|u\|_{PC} \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\frac{p(s,\tau)}{1-q(s,\tau)}\|u\|_{PC} d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t)-\phi(s))^{\nu_1-1}(\chi(r)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\frac{p(s,\tau)}{1-q(s,\tau)}\|u\|_{PC} d\tau ds \\
&\le \|\omega\|_\infty + \left[2mc + \frac{2P^*(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1-Q^*)\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\right] r_0 \le r_0.
\end{aligned}$$

Now we show that $\Phi(B_{r_0})$ is bounded and equicontinuous.

As by the previous step, it is clear that $\Phi(B_{r_0})$ is a bounded set of $PC(\Im, B)$. Take (τ_1, r_1), $(\tau_2, r_2) \in \Im$, $\tau_1 < \tau_2$, $r_1 < r_2$, and let $u \in B_{r_0}$, we have

$$\begin{aligned}
\|(\Phi u)(\tau_2,r_2) - (\Phi u)(\tau_1,r_1)\| \le{}& \|\omega(\tau_2,r_2) - \omega(\tau_1,r_1)\| + \sum_{k=1}^{m} (\|I_k(u(t_k^-,r_2) - I_k(u(t_k^-,r_1))\|) \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{r_2} (\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r_2)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s,\tau,u,v)\|d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_2} \int_0^{r_2} (\phi(\tau_2)-\phi(s))^{\nu_1-1}(\chi(r_2)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s,\tau,u,v)\|d\tau ds \\
&- \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{r_1} (\phi(t_k)-\phi(s))^{\nu_1-1}(\chi(r_1)-\chi(\tau))^{\nu_2-1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s,\tau,u,v)\|d\tau ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|(\Phi u)(\tau_2, r_2) - (\Phi u)(\tau_1, r_1)\| - \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_1} \int_0^{r_1} (\phi(\tau_1) - \phi(s))^{\nu_1 - 1} (\chi(r_1) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&\leq \|\omega(\tau_2, r_2) - \omega(\tau_1, r_1)\| + \sum_{k=1}^{m} (\|I_k(u(t_k^-, r_2) - I_k(u(t_k^-, r_1))\|) \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{r_1} (\phi(t_k) - \phi(s))^{\nu_1 - 1} \Big[(\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&- (\chi(r_1) - \chi(\tau))^{\nu_2 - 1}\Big] \phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{r_1}^{r_2} (\phi(t_k) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_1} \int_0^{r_1} \Big[(\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&- (\phi(\tau_1) - \phi(s))^{\nu_1 - 1} (\chi(r_1) - \chi(\tau))^{\nu_2 - 1}\Big] \phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\tau_1}^{\tau_2} \int_0^{r_1} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\tau_1}^{\tau_2} \int_{r_1}^{r_2} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds \\
&+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_1} \int_{r_1}^{r_2} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau)\|\mathrm{f}(s, \tau, u, v)\| d\tau ds.
\end{aligned}$$

From $\|\mathrm{f}(s, \tau, u, v)\| \leq \frac{P^* r_0}{(1 - Q^*)}$, we obtain

$$\begin{aligned}
&\|(\Phi u)(\tau_2, r_2) - (\Phi u)(\tau_1, r_1)\| \leq \|\omega(\tau_2, r_2) - \omega(\tau_1, r_1)\| + \sum_{k=1}^{m} (\|I_k(u(t_k^-, r_2) - I_k(u(t_k^-, r_1))\|) \\
&+ \frac{P^* r_0}{(1 - Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^{r_1} (\phi(t_k) - \phi(s))^{\nu_1 - 1} \\
&\Big[(\chi(r_2) - \chi(\tau))^{\nu_2 - 1} - (\chi(r_1) - \chi(\tau))^{\nu_2 - 1}\Big] \phi'(s)\chi'(\tau) d\tau ds \\
&+ \frac{P^* r_0}{(1 - Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{r_1}^{r_2} (\phi(t_k) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&\phi'(s)\chi'(\tau) d\tau ds \\
&+ \frac{P^* r_0}{(1 - Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_1} \int_0^{r_1} \Big[(\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1} \\
&- (\phi(\tau_1) - \phi(s))^{\nu_1 - 1} (\chi(r_1) - \chi(\tau))^{\nu_2 - 1}\Big] \phi'(s)\chi'(\tau) d\tau ds
\end{aligned}$$

$$+ \frac{P^* r_0}{(1-Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\tau_1}^{\tau_2} \int_0^{r_1} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1}$$
$$\phi'(s)\chi'(\tau) d\tau ds$$
$$+ \frac{P^* r_0}{(1-Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\tau_1}^{\tau_2} \int_{r_1}^{r_2} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1}$$
$$\phi'(s)\chi'(\tau) d\tau ds$$
$$+ \frac{P^* r_0}{(1-Q^*)\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{\tau_1} \int_{r_1}^{r_2} (\phi(\tau_2) - \phi(s))^{\nu_1 - 1} (\chi(r_2) - \chi(\tau))^{\nu_2 - 1}$$
$$\phi'(s)\chi'(\tau) d\tau ds.$$

As $\tau_1 \to \tau_2$ and $r_1 \to r_2$, implies $\|(\Phi u)(\tau_2, r_2) - (\Phi u)(\tau_1, r_1)\| \to 0$.

Now let $W \subset B_{r_0}$ such that $W \subset \overline{conv}(\Phi(W) \cup 0)$. Since W is a bounded and equicontinuous set. Thus the function $(t, r) \to w(t, r) = \gamma(W(t, r))$ is continuous on $\mathfrak{J}$. In view of (A_4), (A_5) and the properties of the measure γ, for each $(t, r) \in \mathfrak{J}$, we have

$$w(t,r) \le \gamma((\Phi W)(t,r) \cup \{0\}) \le \gamma((\Phi W)(t,r)) \le \sum_{k=1}^{m} \gamma(I_k(W(s,t))) + \sum_{k=1}^{m} \gamma(I_k(W(s,0)))$$
$$+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^r (\phi(t_k) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1}$$
$$\phi'(s)\chi'(\tau) \frac{p(s,\tau)}{1 - q(s,\tau)} \gamma(W(s,t)) d\tau ds$$
$$+ \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{t_k}^{t} \int_0^r (\phi(t) - \phi(s))^{\nu_1 - 1} (\chi(r) - \chi(\tau))^{\nu_2 - 1}$$
$$\phi'(s)\chi'(\tau) \frac{p(s,\tau)}{1 - q(s,\tau)} \gamma(W(s,t)) d\tau ds$$
$$\le \|w\|_{PC} \left[2mc + \frac{2P^*(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1-Q^*)\Gamma(\nu_1+1)\Gamma(\nu_2+1)} \right].$$

This means that

$$\|w\|_{PC} \left(1 - \left[2mc + \frac{2P^*(\phi(a))^{\nu_1}(\chi(b))^{\nu_2}}{(1-Q^*)\Gamma(\nu_1+1)\Gamma(\nu_2+1)} \right] \right) \le 0.$$

As a consequence of (11) $\|w\|_{PC} = 0$, that is, for each $(t, r) \in \mathfrak{J}$, we have $w(t, r) = 0$, so $W(t, r)$ is relatively compact in B. Therefore W is relatively compact in B_{r_0}, by the PC-type Ascoli-Arzela theorem. Now from Theorem 2.17, Φ has a fixed point which is a solution of problem (1). □

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