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Hermite–Hadamard Type Inequalities via New Exponential Type Convexity and Their Applications

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Abstract. In this paper, authors study the concept of (s,m)–exponential type convex functions and their algebraic properties. New generalizations of Hermite–Hadamard type inequality for the (s,m)–exponential type convex function ψ and for the products of two (s,m)–exponential type convex functions ψ and ϕ are proved. Some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power are (s,m)–exponential type convex are obtain. Finally, many new bounds for special means and new error estimates for the trapezoidal and midpoint formula are provided as well.

1. Introduction

Theory of convexity also played significant role in the development of theory of inequalities. Many famously known results in inequalities theory can be obtained using the convexity property of the functions. Hermite–Hadamard's double inequality is one of the most intensively studied result involving convex functions. This result provides us necessary and sufficient condition for a function to be convex. It is also known as classical equation of (H–H) inequality.

The Hermite–Hadamard inequality assert that, if a function $\psi : J \subset \Re \to \Re$ is convex in J for $a_1, a_2 \in J$ and $a_1 < a_2$, then

$$\psi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \psi(\chi) d\chi \leq \frac{\psi(a_1) + \psi(a_2)}{2}. \tag{1}$$

Interested readers can refer to [1]–[8].

Definition 1.1. *[9] A function $\psi : [0, +\infty) \to \Re$ is said to be s–convex in the second sense for a real number $s \in (0,1]$ or ψ belongs to the class of K_s^2, if*

$$\psi(\chi\theta_1 + (1-\chi)\theta_2) \leq \chi^s \psi(\theta_1) + (1-\chi)^s \psi(\theta_2) \tag{2}$$

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holds $\forall\ \theta_1, \theta_2 \in [0, +\infty)$ *and* $\chi \in [0, 1]$.

An s–convex function was introduced in Breckner's article in [9] and a number of properties and connections with s–convexity in the first sense are discussed in [5]. Usually, convexity means for s–convexity when $s = 1$. Dragomir et al. proved a variants of Hadamard's inequality in [3], which holds for s–convex functions in the second sense.

G. Toader introduced the class of m–convex functions in [10].

Definition 1.2. *[10] A function* $\psi : [0, a_2] \to \Re$, $a_2 > 0$, *is said to be m–convex, where* $m \in (0, 1]$, *if*

$$\psi(\chi\theta_1 + m(1-\chi)\theta_2) \le \chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2) \tag{3}$$

holds $\forall\ \theta_1, \theta_2 \in [0, a_2]$ *and* $\chi \in [0, 1]$. *Otherwise* ψ *is m–concave if* $(-\psi)$ *is m–convex.*

In a recent paper, Eftekhari [4] defined the class of (s, m)–convex functions in the second sense as follows:

Definition 1.3. *A function* $\psi : [0, +\infty) \to \Re$ *is said to be* (s, m)*–convex for some fixed real numbers* $s, m \in (0, 1]$, *if*

$$\psi(\chi\theta_1 + m(1-\chi)\theta_2) \le \chi^s\psi(\theta_1) + m(1-\chi)^s\psi(\theta_2) \tag{4}$$

holds $\forall\ \theta_1, \theta_2 \in [0, +\infty)$ *and* $\chi \in [0, 1]$.

Regarding recently published papers in the field of integral inequalities about their refinements and generalizations pertaining convex, harmonically convex, exponentially, convex, co-ordinated convex interval–valued and preinvex class of functions using some useful fractional integral operators or quantum calculus, please see [11]–[36] and references therein.

Motivated by above results and literatures, we will give first in Sect. 2 the concept of (s, m)–exponential type convex function and we will study some of their algebraic properties. In Sect. 3, we will prove new generalizations of Hermite–Hadamard type inequality for the (s, m)–exponential type convex function ψ and for the products of two (s, m)–exponential type convex functions ψ and ϕ. In Sect. 4, we will be obtain some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power are (s, m)–exponential type convex. In Sect. 5, some new bounds for special means and error estimates for the trapezoidal and midpoint formula will be provided. In Sect. 6, a briefly conclusion will be given as well.

2. Some algebraic properties of (s, m)–exponential type convex functions

In this section, we will give a new definition, which is called (s, m)–exponential type convex function and we will study some basic algebraic properties of it.

Definition 2.1. *A nonnegative function* $\psi : J \to \Re$, *is said* (s, m)*–exponential type convex for some fixed* $s, m \in (0, 1]$, *if*

$$\psi(\chi\theta_1 + m(1-\chi)\theta_2) \le (e^{s\chi} - 1)\psi(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)\psi(\theta_2) \tag{5}$$

holds $\forall\ \theta_1, \theta_2 \in J$ *and* $\chi \in [0, 1]$.

Remark 2.2. *For* $m = s = 1$, *we get exponential type convexity given by İşcan in [6].*

Remark 2.3. *The range of the* (s, m)*–exponential type convex functions for some fixed* $m \in (0, 1]$ *and* $s \in [\ln 2.5, 1]$ *is* $[0, +\infty)$.

Proof. Let $\theta \in J$ be arbitrary for some fixed $m \in (0,1]$ and $s \in [\ln 2.5, 1]$. Using the definition 2.1 for $\chi = 1$, we have

$$\psi(\theta) \leq (e^s - 1)\psi(\theta) \Longrightarrow (e^s - 2)\psi(\theta) \geq 0 \Longrightarrow \psi(\theta) \geq 0.$$

□

Lemma 2.4. *For all $\chi \in [0,1]$ and for some fixed $m \in (0,1]$ and $s \in [\ln 2.5, 1]$ the following inequalities $(e^{s\chi} - 1) \geq \chi^s$ and $(e^{(1-\chi)s} - 1) \geq (1-\chi)^s$ hold.*

Proof. The proof is evident. □

Proposition 2.5. *Every nonnegative (s,m)–convex function is (s,m)–exponential type convex function for some fixed $m \in (0,1]$ and $s \in [\ln 2.5, 1]$.*

Proof. By using Lemma 2.4, for some fixed $m \in (0,1]$ and $s \in [\ln 2.5, 1]$, we have

$$\begin{aligned} &\psi(\chi\theta_1 + m(1-\chi)\theta_2) \leq \chi^s \psi(\theta_1) + m(1-\chi)^s \psi(\theta_2) \\ &\leq (e^{s\chi} - 1)\psi(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)\psi(\theta_2). \end{aligned}$$

□

Theorem 2.6. *Let $\psi, \phi : [a_1, a_2] \to \Re$. If ψ and ϕ are (s,m)–exponential type convex functions for some fixed $s, m \in (0,1]$, then*

1. *$\psi + \phi$ is (s,m)–exponential type convex function;*
2. *For nonnegative real number c, $c\psi$ is (s,m)–exponential type convex function.*

Proof. By definition 2.1 for some fixed $s, m \in (0,1]$, the proof is obvious. □

Theorem 2.7. *Let $\psi : [0, a_2] \to J$ be m–convex function for $a_2 > 0$ and some fixed $m \in (0,1]$ and $\phi : J \to \Re$ is non-decreasing and (s,m)–exponential type convex function for some fixed $s \in (0,1]$. Then for the same fixed numbers $s, m \in (0,1]$, the function $\phi \circ \psi : [0, a_2] \to \Re$ is (s,m)–exponential type convex.*

Proof. For all $\theta_1, \theta_2 \in [0, a_2]$ and $\chi \in [0,1]$, and for the some fixed numbers $s, m \in (0,1]$, we have

$$\begin{aligned} &(\phi \circ \psi)(\chi\theta_1 + m(1-\chi)\theta_2) = \phi(\psi(\chi\theta_1 + m(1-\chi)\theta_2)) \leq \phi(\chi\psi(\theta_1) + m(1-\chi)\psi(\theta_2)) \\ &\leq (e^{s\chi} - 1)(\phi \circ \psi)(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)(\phi \circ \psi)(\theta_2). \end{aligned}$$

□

Theorem 2.8. *Let $\psi_i : [a_1, a_2] \to \Re$ be an arbitrary family of (s,m)–exponential type convex functions for the same fixed $s, m \in (0,1]$ and let $\psi(\theta) = \sup_i \psi_i(\theta)$. If $A = \{\theta \in [a_1, a_2] : \psi(\theta) < +\infty\} \neq \emptyset$, then A is an interval and ψ is (s,m)–exponential type convex function on A.*

Proof. For all $\theta_1, \theta_2 \in A$ and $\chi \in [0,1]$, and for the same fixed numbers $s, m \in (0,1]$, we have

$$\begin{aligned} &\psi(\chi\theta_1 + m(1-\psi)\theta_2) = \sup_i \psi_i(\chi\theta_1 + m(1-\chi)\theta_2) \\ &\leq \sup_i \left[(e^{s\chi} - 1)\psi_i(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)\psi_i(\theta_2)\right] \\ &\leq (e^{s\chi} - 1)\sup_i \psi_i(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)\sup_i \psi_i(\theta_2) \\ &= (e^{s\chi} - 1)\psi(\theta_1) + m\left(e^{(1-\chi)s} - 1\right)\psi(\theta_2) < +\infty. \end{aligned}$$

This shows simultaneously that A is an interval, since it contains every point between any two of its points, and that ψ is (s,m)–exponential type convex function on A. □

Theorem 2.9. *If the function $\psi : [a_1, a_2] \rightarrow \Re$ is (s, m)–exponential type convex for some fixed $s, m \in (0, 1]$, then ψ is bounded on $[a_1, ma_2]$.*

Proof. Let $L = \max\left\{\psi(a_1), \psi\left(\frac{a_2}{m}\right)\right\}$ and $x \in [a_1, a_2]$ be an arbitrary point for some fixed $m \in (0, 1]$. Then there exists $\chi \in [0, 1]$ such that $x = \chi a_1 + (1 - \chi)a_2$. Thus, since $e^{s\chi} \leq e^s$ and $e^{(1-\chi)s} \leq e^s$ for some fixed $s \in (0, 1]$, we have

$$\psi(x) = \psi(\chi a_1 + (1 - \chi)a_2) \leq (e^{s\chi} - 1)\,\psi(a_1) + m\left(e^{(1-\chi)s} - 1\right)\psi\left(\frac{a_2}{m}\right)$$

$$\leq (e^s - 1)\,L + m\,(e^s - 1)\,L = (m + 1)L(e^s - 1) = M.$$

We have shown that ψ is bounded above from real number M. Interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 2.4 in [6]. □

3. New generalizations of (H–H) type inequality

Let's find some new generalizations of Hermite–Hadamard type inequality for the (s, m)–exponential type convex function ψ and for the products of two (s, m)–exponential type convex functions ψ and ϕ. Throughout the paper the space $L_1([a_1, a_2])$ denotes the space of integrable functions over $[a_1, a_2]$.

Theorem 3.1. *Let $\psi : [a_1, ma_2] \rightarrow \Re$ be (s, m)–exponential type convex function for some fixed $s, m \in (0, 1]$ and $a_1 < ma_2$. If $\psi \in L_1([a_1, ma_2])$, then*

$$\frac{1}{\left(e^{\frac{s}{2}} - 1\right)}\psi\left(\frac{a_1 + ma_2}{2}\right) \leq \frac{2}{(ma_2 - a_1)}\int_{a_1}^{ma_2}\psi(x)dx \tag{6}$$

$$\leq \left(\frac{e^s - s - 1}{s}\right)\left[\psi(a_1) + \psi(ma_2) + m\left(\psi\left(\frac{a_1}{m}\right) + \psi(a_2)\right)\right].$$

Proof. Let denote, respectively,

$$\theta_1 = \chi a_1 + m(1 - \chi)a_2, \quad \theta_2 = (1 - \chi)\frac{a_1}{m} + \chi a_2, \quad \forall\, \chi \in [0, 1].$$

Using (s, m)–exponential type convexity of ψ, we have

$$\psi\left(\frac{a_1 + ma_2}{2}\right) = \psi\left(\frac{\theta_1 + m\theta_2}{2}\right)$$

$$= \psi\left(\frac{[\chi a_1 + m(1 - \chi)a_2] + [(1 - \chi)a_1 + m\chi a_2]}{2}\right)$$

$$\leq \left(e^{\frac{s}{2}} - 1\right)\left[\psi(\chi a_1 + m(1 - \chi)a_2) + \psi((1 - \chi)a_1 + m\chi a_2)\right].$$

Now, integrating on both sides in the last inequality with respect to χ over $[0, 1]$, we get

$$\psi\left(\frac{a_1 + ma_2}{2}\right) \leq \left(e^{\frac{s}{2}} - 1\right)$$

$$\times\left[\int_0^1 \psi(\chi a_1 + m(1 - \chi)a_2)d\chi + \int_0^1 \psi\left((1 - \chi)a_1 + m\chi a_2\right)d\chi\right]$$

$$= \frac{2\left(e^{\frac{s}{2}} - 1\right)}{(ma_2 - a_1)}\int_{a_1}^{ma_2}\psi(x)dx,$$

which completes the left side inequality. For the right side inequality, using (s,m)–exponential type convextyi of ψ, we obtain

$$\begin{aligned}
&\frac{2}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(x)dx\\
&=\int_0^1\psi(\chi a_1+m(1-\chi)a_2)d\chi+\int_0^1((1-\chi)a_1+m\chi a_2)\,d\chi\\
&\le\int_0^1\left[(e^{s\chi}-1)\,\psi\,(a_1)+m\left(e^{(1-\chi)s}-1\right)\psi\,(a_2)\right]d\chi\\
&+\int_0^1\left[(e^{s\chi}-1)\,\psi\,(ma_2)+m\left(e^{(1-\chi)s}-1\right)\psi\left(\frac{a_1}{m}\right)\right]d\chi\\
&=\left(\frac{e^s-s-1}{s}\right)\left[\psi(a_1)+\psi(ma_2)+m\left(\psi\left(\frac{a_1}{m}\right)+\psi(a_2)\right)\right],
\end{aligned}$$

which give the right side inequality. □

Corollary 3.2. *By choosing $m=s=1$ in Theorem 3.1, we get (Theorem 3.1, [6]).*

Theorem 3.3. *Assume that $\psi,\phi:[a_1,ma_2]\to\Re$ are respectively, (s_1,m) and (s_2,m)–exponential type convex functions for the same fixed $m\in(0,1]$ and for some fixed $s_1,s_2\in(0,1]$, where $s_1<s_2$ and $a_1<ma_2$. If ψ,ϕ are synchronous functions and $\psi,\phi,\psi\phi\in L_1([a_1,ma_2])$, then*

$$\begin{aligned}
&\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)d\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)d\theta\le\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\phi(\theta)d\theta\\
&\le A(s_1,s_2)M_m(a_1,a_2)+B(s_1,s_2)N_m(a_1,a_2),
\end{aligned}\tag{7}$$

where

$$M_m(a_1,a_2):=\psi(a_1)\phi(a_1)+m^2\psi(a_2)\phi(a_2),\quad N_m(a_1,a_2):=m\psi(a_1)\phi(a_2)+\psi(a_2)\phi(a_1),$$

and

$$\begin{aligned}
A(s_1,s_2)&:=\frac{e^{s_1+s_2}+s_1+s_2-1}{s_1+s_2}-\frac{s_1\left(e^{s_2}-1\right)+s_2\left(e^{s_1}-1\right)}{s_1s_2},\\
B(s_1,s_2)&:=\frac{e^{s_2}-e^{s_1}}{s_2-s_1}-\frac{s_1\left(e^{s_2}-1\right)+s_2\left(e^{s_1}-1\right)}{s_1s_2}+1.
\end{aligned}$$

Proof. Let denote, $\theta=\chi a_1+m(1-\chi)a_2$ for all $\chi\in[0,1]$. Using the property of the (s_1,m) and (s_2,m)–exponential type convex functions ψ and ϕ, respectively, we have

$$\psi(\chi a_1+m(1-\chi)a_2)\le(e^{s_1\chi}-1)\,\psi\,(a_1)+m\left(e^{(1-\chi)s_1}-1\right)\psi\,(a_2)$$

and

$$\phi(\chi a_1+m(1-\chi)a_2)\le(e^{s_2\chi}-1)\,\phi\,(a_1)+m\left(e^{(1-\chi)s_2}-1\right)\phi\,(a_2).$$

Multiplying above inequalities on both sides, we get

$$\psi(\chi a_1+m(1-\chi)a_2)\phi(\chi a_1+m(1-\chi)a_2)\le\left[(e^{s_1\chi}-1)\,\psi\,(a_1)+m\left(e^{(1-\chi)s_1}-1\right)\psi\,(a_2)\right]$$

$$\times\Big[(e^{s_2\chi}-1)\,\phi(a_1) + m\left(e^{(1-\chi)s_2}-1\right)\phi(a_2)\Big]$$
$$= (e^{s_1\chi}-1)(e^{s_2\chi}-1)\,\psi(a_1)\phi(a_1)$$
$$+m\Big[(e^{s_1\chi}-1)\left(e^{(1-\chi)s_2}-1\right)\psi(a_1)\phi(a_2) + (e^{s_2\chi}-1)\left(e^{(1-\chi)s_1}-1\right)\psi(a_2)\phi(a_1)\Big]$$
$$+m^2\left(e^{(1-\chi)s_1}-1\right)\left(e^{(1-\chi)s_2}-1\right)\psi(a_2)\phi(a_2). \tag{8}$$

Applying Chebyshev integral inequality (see [37]), we obtain

$$\int_0^1 \psi(\chi a_1 + m(1-\chi)a_2)\phi(\chi a_1 + m(1-\chi)a_2)d\chi$$
$$\geq \int_0^1 \psi(\chi a_1 + m(1-\chi)a_2)d\chi \cdot \int_0^1 \phi(\chi a_1 + m(1-\chi)a_2)d\chi.$$

Changing the variable of integration, we get

$$\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)\phi(\theta)d\theta \geq \frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)d\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)d\theta,$$

which completes the left side inequality. For the right side inequality, integrating on both sides of the inequality (8) with respect to χ over $[0,1]$, we have

$$\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\psi(\theta)d\theta\cdot\frac{1}{(ma_2-a_1)}\int_{a_1}^{ma_2}\phi(\theta)d\theta$$
$$\leq \left[\int_0^1 (e^{s_1\chi}-1)(e^{s_2\chi}-1)\,d\chi\right]\psi(a_1)\phi(a_1)$$
$$+m\left[\left(\int_0^1 (e^{s_1\chi}-1)\left(e^{(1-\chi)s_2}-1\right)d\chi\right)\psi(a_1)\phi(a_2)\right.$$
$$\left.+\left(\int_0^1 (e^{s_2\chi}-1)\left(e^{(1-\chi)s_1}-1\right)d\chi\right)\psi(a_2)\phi(a_1)\right]$$
$$+m^2\left[\int_0^1 \left(e^{(1-\chi)s_1}-1\right)\left(e^{(1-\chi)s_2}-1\right)d\chi\right]\psi(a_2)\phi(a_2).$$
$$= A(s_1,s_2)M_m(a_1,a_2) + B(s_1,s_2)N_m(a_1,a_2),$$

which give the right side inequality. □

4. Refinements of (H–H) type inequality

Let obtain some refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power is (s,m)–exponential type convex.

Lemma 4.1. *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : \left[a_1, \frac{a_2}{k}\right] \to \Re$ *is differentiable on* $\left(a_1, \frac{a_2}{k}\right)$ *with* $0 < a_1 < a_2$. *If* $\psi' \in L_1\left[a_1, \frac{a_2}{k}\right]$, *then*

$$\frac{\psi(a_1)+\psi\left(\frac{a_2}{k}\right)}{2} - \frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta = \left(\frac{a_2-ka_1}{2k}\right) \tag{9}$$
$$\times\int_0^1 (1-2\chi)\,\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)d\chi.$$

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2-ka_1}{2k}\right)\int_0^1 (1-2\chi)\,\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi$$
$$=\left(\frac{a_2-ka_1}{2k}\right)\left\{\frac{(1-2\chi)\,\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)}{a_1-\frac{a_2}{k}}\Bigg|_0^1-\int_0^1\frac{\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)}{a_1-\frac{a_2}{k}}(-2)\,d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{2k}\right)\left\{\frac{-\psi(a_1)-\psi\left(\frac{a_2}{k}\right)}{\frac{ka_1-a_2}{k}}+\frac{2k}{ka_1-a_2}\int_0^1\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{2k}\right)\left\{\frac{k\left(\psi(a_1)+\psi\left(\frac{a_2}{k}\right)\right)}{a_2-ka_1}-\frac{2k}{a_2-ka_1}\int_0^1\psi\left(\chi a_1+(1-t)\frac{a_2}{k}\right)d\chi\right\}$$
$$=\frac{\psi(a_1)+\psi\left(\frac{a_2}{k}\right)}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta,$$

which completes the proof. □

Lemma 4.2. *Suppose* $0<k\le 1$ *and a mapping* $\psi:[ka_1,a_2]\to\mathfrak{R}$ *is differentiable on* (ka_1,a_2) *with* $0<a_1<a_2$. *If* $\psi'\in L_1[ka_1,a_2]$, *then*

$$\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta=\left(\frac{a_2-ka_1}{2}\right) \tag{10}$$
$$\times\int_0^1(2\chi-1)\,\psi'(k(1-\chi)a_1+\chi a_2)\,d\chi.$$

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2-ka_1}{2}\right)\int_0^1(2\chi-1)\,\psi'(k(1-\chi)a_1+\chi a_2)$$
$$=\left(\frac{a_2-ka_1}{2}\right)$$
$$\times\left\{\frac{(2\chi-1)\,\psi(k(1-\chi)a_1+\chi a_2)}{a_2-ka_1}\Bigg|_0^1-\int_0^1\frac{\psi(k(1-\chi)a_1+\chi a_2)}{a_2-ka_1}(2)\;d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{2}\right)\left\{\frac{\psi(a_2)+\psi(ka_1)}{a_2-ka_1}-\frac{2}{a_2-ka_1}\int_0^1\psi(k(1-\chi)a_1+\chi a_2)\,d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{2}\right)\left\{\frac{\psi(a_2)+\psi(ka_1)}{a_2-ka_1}-\frac{2}{a_2-ka_1}\int_0^1\psi(k(1-\chi)a_1+\chi a_2)\,d\chi\right\}$$
$$=\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta,$$

which completes the proof. □

Lemma 4.3. *Suppose* $0<k\le 1$ *and a mapping* $\psi:[ka_1,a_2]\to\mathfrak{R}$ *is differentiable on* (ka_1,a_2) *with* $0<a_1<a_2$. *If* $\psi'\in L_1[ka_1,a_2]$, *then*

$$\frac{\psi(ka_1)+\psi(a_2)}{k+1}-\frac{2}{(k+1)(a_2-ka_1)}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta=\left(\frac{a_2-ka_1}{k+1}\right) \tag{11}$$
$$\times\int_0^1(2\chi-1)\,\psi'(k(1-\chi)a_1+\chi a_2)\,d\chi.$$

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2-ka_1}{k+1}\right)\int_0^1 (2\chi-1)\,\psi'\left(k(1-\chi)a_1+\chi a_2\right)$$
$$=\left(\frac{a_2-ka_1}{k+1}\right)$$
$$\times\left\{\frac{(2\chi-1)\,\psi\left(k(1-\chi)a_1+\chi a_2\right)}{a_2-ka_1}\bigg|_0^1-\int_0^1\frac{\psi\left(k(1-\chi)a_1+\chi a_2\right)}{a_2-ka_1}(2)\,d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{k+1}\right)\left\{\frac{\psi(ka_1)+\psi(a_2)}{a_2-ka_1}-\frac{2}{a_2-ka_1}\int_0^1\psi\left(k(1-\chi)a_1+\chi a_2\right)d\chi\right\}$$
$$=\frac{\psi(ka_1)+\psi(a_2)}{k+1}-\frac{2}{k+1}\int_0^1\psi\left(k(1-\chi)a_1+\chi a_2\right)d\chi$$
$$=\frac{\psi(ka_1)+\psi(a_2)}{k+1}-\frac{2}{(k+1)(a_2-ka_1)}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta,$$

which completes the proof. □

Lemma 4.4. *Suppose* $0<k\le 1$ *and a mapping* $\psi:\left[a_1,\frac{a_2}{k}\right]\to\Re$ *is differentiable on* $\left(a_1,\frac{a_2}{k}\right)$ *with* $0<a_1<a_2$. *If* $\psi'\in L_1\left[a_1,\frac{a_2}{k}\right]$, *then*

$$\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta-\psi\left(\frac{a_1+a_2}{2k}\right)=\left(\frac{a_2-ka_1}{k}\right) \tag{12}$$
$$\times\left\{\int_0^1\chi\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi-\int_{\frac{1}{2}}^1\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi\right\}.$$

Proof. Using the integrating by parts, we have

$$\left(\frac{a_2-ka_1}{k}\right)\left\{\int_0^1\chi\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi-\int_{\frac{1}{2}}^1\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi\right\}$$
$$=\left(\frac{a_2-ka_1}{k}\right)$$
$$\times\left\{\frac{\chi\,\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)}{a_1-\frac{a_2}{k}}\bigg|_0^1-\int_0^1\frac{\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)}{a_1-\frac{a_2}{k}}d\chi-\frac{\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)}{a_1-\frac{a_2}{k}}\bigg|_{\frac{1}{2}}^1\right\}$$
$$=\left(\frac{a_2-ka_1}{k}\right)\left\{\frac{k\psi(a_1)}{ka_1-a_2}-\frac{k}{ka_1-a_2}\int_0^1\psi\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)d\chi\right.$$
$$\left.-\frac{k}{ka_1-a_2}\left(\psi(a_1)-\psi\left(\frac{a_1+a_2}{2k}\right)\right)\right\}$$
$$=\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta-\psi\left(\frac{a_1+a_2}{2k}\right),$$

which completes the proof. □

Lemma 4.5. *Suppose* $0<k\le 1$ *and a mapping* $\psi:[ka_1,a_2]\to\Re$ *is differentiable on* (ka_1,a_2) *with* $0<a_1<a_2$. *If* $\psi'\in L_1[ka_1,a_2]$, *then*

$$\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)=(a_2-ka_1) \tag{13}$$

$$\times\left\{\int_0^1 -\chi\psi'\left(k(1-\chi)a_1+\chi a_2\right)d\chi+\int_{\frac{1}{2}}^1 \psi'\left(k(1-\chi)a_1+\chi a_2\right)d\chi\right\}.$$

Proof. Using the integrating by parts, we have

$$\begin{aligned}
&(a_2-ka_1)\left\{\int_0^1 -\chi\psi'\left(k(1-\chi)a_1+\chi a_2\right)d\chi+\int_{\frac{1}{2}}^1 \psi'\left(k(1-\chi)a_1+\chi a_2\right)d\chi\right\}\\
&=(a_2-ka_1)\left\{\frac{-\chi\,\psi\left(k(1-\chi)a_1+\chi a_2\right)}{a_2-ka_1}\bigg|_0^1-\int_0^1\frac{\psi\left(k(1-\chi)a_1+\chi a_2\right)}{a_2-ka_1}(-1)\,d\chi\right.\\
&\left.+\frac{\psi\left(k(1-\chi)a_1+\chi a_2\right)}{a_2-ka_1}\bigg|_{\frac{1}{2}}^1\right\}\\
&=(a_2-ka_1)\\
&\times\left\{\frac{-\psi(a_2)}{a_2-ka_1}+\frac{1}{a_2-ka_1}\int_0^1\psi\left(k(1-\chi)a_1+\chi a_2\right)d\chi+\frac{\psi(a_2)-\psi\left(\frac{ka_1+a_2}{2}\right)}{a_2-ka_1}\right\}\\
&=\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.6. *Suppose $0<k\le 1$ and a mapping $\psi:\left(0,\frac{a_2}{mk}\right]\to\mathfrak{R}$ is differentiable on $\left(0,\frac{a_2}{mk}\right)$ with $0<a_1<a_2$. If $\left|\psi'\right|^q$ is (s,m)–exponential type convex on $\left(0,\frac{a_2}{mk}\right]$ for $q>1$ and $q^{-1}+p^{-1}=1$, then for some fixed $s,m\in(0,1]$, the following inequality holds:*

$$\left|\frac{\psi(a_1)+\psi\left(\frac{a_2}{k}\right)}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right|\le\left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\tag{14}$$
$$\times\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(a_1)\right|^q+m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 4.1, Hölder's inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi\left(\frac{a_2}{k}\right)}{2}-\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta\right|\\
&\le\left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\chi|^p\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right\}^{\frac{1}{q}}\\
&\le\left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\chi|^p\,d\chi\right)^{\frac{1}{p}}\\
&\times\left\{\int_0^1\left[(e^{s\chi}-1)\left|\psi'(a_1)\right|^q+m\left(e^{(1-\chi)s}-1\right)\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right]d\chi\right\}^{\frac{1}{q}}\\
&=\left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(a_1)\right|^q+m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 4.7. *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : \left(0, \frac{a_2}{mk}\right] \to \Re$ *is differentiable on* $\left(0, \frac{a_2}{mk}\right)$ *with* $0 < a_1 < a_2$. *If* $\left|\psi'\right|^q$ *is* (s, m)*–exponential type convex on* $\left(0, \frac{a_2}{mk}\right]$ *for* $q > 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds:*

$$\left|\frac{\psi(a_1) + \psi\left(\frac{a_2}{k}\right)}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta\right| \leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \tag{15}$$
$$\times \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 4.1, power mean inequality and (s, m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\left|\frac{\psi(a_1) + \psi\left(\frac{a_2}{k}\right)}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{2k}\right)\left\{\int_0^1 |1 - 2\chi|\left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right| d\chi\right\}$$
$$\leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\int_0^1 |1 - 2\chi|\, d\chi\right)^{1-\frac{1}{q}}\left\{\int_0^1 |1 - 2\chi|\left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{2k}\right)\left(\int_0^1 |1 - 2\chi|\, d\chi\right)^{1-\frac{1}{q}}$$
$$\times \left\{\int_0^1 |1 - 2\chi|\left[\left(e^{s\chi} - 1\right)\left|\psi'(a_1)\right|^q + m\left(e^{(1-\chi)s} - 1\right)\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right] d\chi\right\}^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{2k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Theorem 4.8. *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : \left(0, \frac{a_2}{m}\right] \to \Re$ *is differentiable on* $\left(0, \frac{a_2}{m}\right)$ *with* $0 < a_1 < a_2$. *If* $\left|\psi'\right|^q$ *is* (s, m)*–exponential type convex on* $\left(0, \frac{a_2}{m}\right]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds:*

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{2} - \frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right| \leq \left(\frac{a_2 - ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{16}$$
$$\times \left\{\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 4.2, Hölder's inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right| \\
&\le\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|^p\,d\chi\right)^{\frac{1}{p}}\left\{\int_0^1\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right\}^{\frac{1}{q}} \\
&\le\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|^p\,d\chi\right)^{\frac{1}{p}} \\
&\times\left\{\int_0^1\left[m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q+\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q\right]d\chi\right\}^{\frac{1}{q}} \\
&=\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 4.9. *Suppose $0<k\le 1$ and a mapping $\psi:\left(0,\frac{a_2}{m}\right]\to\mathfrak{R}$ is differentiable on $\left(0,\frac{a_2}{m}\right)$ with $0<a_1<a_2$. If $\left|\psi'\right|^q$ is (s,m)–exponential type convex on $\left(0,\frac{a_2}{m}\right]$ for $q>1$, then for some fixed $s,m\in(0,1]$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right|\le\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}\tag{17}$$

Proof. From Lemma 4.2, power mean inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(ka_1)+\psi(a_2)}{2}-\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta\right| \\
&\le\left(\frac{a_2-ka_1}{2}\right)\left\{\int_0^1|2\chi-1|\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|d\chi\right\} \\
&\le\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|\,d\chi\right)^{1-\frac{1}{q}}\left\{\int_0^1|2\chi-1|\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right\}^{\frac{1}{q}} \\
&\le\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\chi-1|\,d\chi\right)^{1-\frac{1}{q}} \\
&\times\left[\int_0^1|2\chi-1|\left\{\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q+m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right\}d\chi\right]^{\frac{1}{q}} \\
&=\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 4.10. *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : \left(0, \frac{a_2}{m}\right] \to \mathfrak{R}$ *is differentiable on* $\left(0, \frac{a_2}{m}\right)$ *with* $0 < a_1 < a_2$. *If* $\left|\psi'\right|^q$ *is* (s, m)*–exponential type convex on* $\left(0, \frac{a_2}{m}\right]$ *for* $q > 1$ *and* $q^{-1} + p^{-1} = 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds:*

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right| \leq \left(\frac{a_2 - ka_1}{k+1}\right) \tag{18}$$
$$\times \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 4.3, Hölder's inequality and (s, m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1|^p \, d\chi\right)^{\frac{1}{p}} \left\{\int_0^1 \left|\psi'(k(1-\chi)a_1 + \chi a_2)\right|^q d\chi\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1|^p \, d\chi\right)^{\frac{1}{p}} \left\{\int_0^1 \left[\left(e^{(1-\chi)s} - 1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi} - 1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right] d\chi\right\}^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Theorem 4.11. *Suppose* $0 < k \leq 1$ *and a mapping* $\psi : \left(0, \frac{a_2}{m}\right] \to \mathfrak{R}$ *is differentiable on* $\left(0, \frac{a_2}{m}\right)$ *with* $0 < a_1 < a_2$. *If* $\left|\psi'\right|^q$ *is* (s, m)*–exponential type convex on* $\left(0, \frac{a_2}{m}\right]$ *for* $q > 1$, *then for some fixed* $s, m \in (0, 1]$, *the following inequality holds:*

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right| \leq \left(\frac{a_2 - ka_1}{k+1}\right) \tag{19}$$
$$\times \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. From Lemma 4.3, power mean inequality and (s, m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\left|\frac{\psi(ka_1) + \psi(a_2)}{k+1} - \frac{2}{(k+1)(a_2 - ka_1)} \int_{ka_1}^{a_2} \psi(\theta)\, d\theta\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left[\int_0^1 |2\chi - 1| \left|\psi'(k(1-\chi)a_1 + \chi a_2)\right| d\chi\right]$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1| \, d\chi\right)^{1-\frac{1}{q}} \left\{\int_0^1 |2\chi - 1| \left|\psi'(k(1-\chi)a_1 + \chi a_2)\right|^q d\chi\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\chi - 1| \, d\chi\right)^{1-\frac{1}{q}} \left[\int_0^1 |2\chi - 1| \left\{\left(e^{(1-\chi)s} - 1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi} - 1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right\} d\chi\right]^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}},$$

which completes the proof. □

Theorem 4.12. *Suppose $0 < k \leq 1$ and a mapping $\psi : \left(0, \frac{a_2}{km}\right] \rightarrow \mathfrak{R}$ is differentiable on $\left(0, \frac{a_2}{km}\right)$ with $0 < a_1 < a_2$. If $\left|\psi'\right|^q$ is (s, m)–exponential type convex on $\left(0, \frac{a_2}{km}\right]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s, m \in (0, 1]$, the following inequality holds:*

$$\left|\frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right| \leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{1}{2}\right)^{\frac{1}{p}} \left(\left|\psi'(a_1)\right|^q \left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)\right)^{\frac{1}{q}}\right]. \tag{20}$$

Proof. From Lemma 4.4, Hölder's inequality and (s, m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\left|\frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right| \leq$$
$$\left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \chi^p d\chi\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 1 d\chi\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|\psi'\left(\chi a_1 + (1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right)^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \chi^p d\chi\right)^{\frac{1}{p}} \left(\int_0^1 \left((e^{s\chi} - 1)\left|\psi'(a_1)\right|^q + m\left(e^{(1-\chi)s} - 1\right)\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right) d\chi\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 1 d\chi\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left((e^{s\chi} - 1)\left|\psi'(a_1)\right|^q + m\left(e^{(1-\chi)s} - 1\right)\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right) d\chi\right)^{\frac{1}{q}}\right]$$
$$= \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\left(\frac{e^s - s - 1}{s}\right)\left(\left|\psi'(a_1)\right|^q + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{1}{2}\right)^{\frac{1}{p}} \left(\left|\psi'(a_1)\right|^q \left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)\right)^{\frac{1}{q}}\right],$$

which completes the proof. □

Theorem 4.13. *Suppose $0 < k \leq 1$ and a mapping $\psi : \left(0, \frac{a_2}{km}\right] \rightarrow \mathfrak{R}$ is differentiable on $\left(0, \frac{a_2}{km}\right)$ with $0 < a_1 < a_2$. If $\left|\psi'\right|^q$ is (s, m)–exponential type convex on $\left(0, \frac{a_2}{km}\right]$ for $q > 1$, then for some fixed $s, m \in (0, 1]$, the following inequality holds:*

$$\left|\frac{k}{a_2 - ka_1} \int_{a_1}^{\frac{a_2}{k}} \psi(\theta)\, d\theta - \psi\left(\frac{a_1 + a_2}{2k}\right)\right| \leq \left(\frac{a_2 - ka_1}{k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \tag{21}$$
$$\left[\left(\left|\psi'(a_1)\right|^q \left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\psi'(a_1)\right|^q \left(\frac{e^s - e^{\frac{s}{2}}}{s} - \frac{1}{2}\right) + m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q \left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)\right)^{\frac{1}{q}}\right].$$

Proof. From Lemma 4.4, power mean inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{k}{a_2-ka_1}\int_{a_1}^{\frac{a_2}{k}}\psi(\theta)\,d\theta-\psi\left(\frac{a_1+a_2}{2k}\right)\right|\le\left(\frac{a_2-ka_1}{k}\right)\\
&\times\left\{\int_0^1\chi\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|d\chi+\int_{\frac12}^1\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|d\chi\right\}\\
&\le\left(\frac{a_2-ka_1}{k}\right)\left\{\left(\int_0^1\chi\,d\chi\right)^{1-\frac1q}\left(\int_0^1\chi\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right)^{\frac1q}\right.\\
&\left.+\left(\int_{\frac12}^1 1\,d\chi\right)^{1-\frac1q}\left(\int_{\frac12}^1\left|\psi'\left(\chi a_1+(1-\chi)\frac{a_2}{k}\right)\right|^q d\chi\right)^{\frac1q}\right\}\\
&\le\left(\frac{a_2-ka_1}{k}\right)\left[\left(\int_0^1\chi\,d\chi\right)^{1-\frac1q}\left(\int_0^1\chi\left\{\left(e^{\chi}-1\right)\left|\psi'(a_1)\right|^q+m\left(e^{(1-\chi)s}-1\right)\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right\}d\chi\right)^{\frac1q}\right.\\
&\left.+\left(\int_{\frac12}^1 1\,d\chi\right)^{1-\frac1q}\times\left(\int_{\frac12}^1\left\{\left(e^{s\chi}-1\right)\left|\psi'(a_2)\right|^q+m\left(e^{(1-\chi)s}-1\right)\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\right\}d\chi\right)^{\frac1q}\right]\\
&=\left(\frac{a_2-ka_1}{k}\right)\left(\frac12\right)^{1-\frac1q}\left[\left(\left|\psi'(a_1)\right|^q\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)+m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\right)^{\frac1q}\right.\\
&\left.+\left(\left|\psi'(a_1)\right|^q\left(\frac{e^s-e^{\frac s2}}{s}-\frac12\right)+m\left|\psi'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^{\frac s2}-s-2}{2s}\right)\right)^{\frac1q}\right],
\end{aligned}$$

which completes the proof. □

Theorem 4.14. *Suppose $0<k\le 1$ and a mapping $\psi:\left(0,\frac{a_2}{m}\right]\to\Re$ is differentiable on $\left(0,\frac{a_2}{m}\right)$ with $0<a_1<a_2$. If $\left|\psi'\right|^q$ is (s,m)–exponential type convex on $\left(0,\frac{a_2}{m}\right]$ for $q>1$ and $q^{-1}+p^{-1}=1$, then for some fixed $s,m\in(0,1]$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)\right|\le(a_2-ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac1p}\left(\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(ka_1)\right|^q+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right)^{\frac1q}\right.\\
&\left.+\left(\frac12\right)^{\frac1p}\left(\left|\psi'(ka_1)\right|^q\left(\frac{2e^{\frac s2}-s-2}{2s}\right)+m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{2e^s-2e^{\frac s2}-s}{2s}\right)\right)^{\frac1q}\right]. \qquad (22)
\end{aligned}$$

Proof. From Lemma 4.5, Hölder's inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta-\psi\left(\frac{ka_1+a_2}{2}\right)\right|\le\\
&(a_2-ka_1)\left[\left(\int_0^1\chi^p d\chi\right)^{\frac1p}\left(\int_0^1\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right)^{\frac1q}+\left(\int_{\frac12}^1 1d\chi\right)^{\frac1p}\left(\int_{\frac12}^1\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right)^{\frac1q}\right]
\end{aligned}$$

$$\le (a_2 - ka_1)\left[\left(\int_0^1 \chi^p d\chi\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 1 d\chi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}}\right]$$
$$= (a_2 - ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{e^s-s-1}{s}\right)\left(\left|\psi'(ka_1)\right|^q + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\left|\psi'(ka_1)\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right) + m\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)\right)^{\frac{1}{q}}\right],$$

which completes the proof. □

Theorem 4.15. *Suppose $0 < k \le 1$ and a mapping $\psi : \left(0, \frac{a_2}{m}\right] \to \Re$ is differentiable on $\left(0, \frac{a_2}{m}\right)$ with $0 < a_1 < a_2$. If $\left|\psi'\right|^q$ is (s,m)–exponential type convex on $\left(0, \frac{a_2}{m}\right]$ for $q > 1$, then for some fixed $s, m \in (0,1]$, the following inequality holds:*

$$\left|\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta - \psi\left(\frac{ka_1+a_2}{2}\right)\right| \le (a_2 - ka_1)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \tag{23}$$
$$\times\left\{\left(\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\left|\psi'(ka_1)\right|^q + m\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\left|\psi'(ka_1)\right|^q + m\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 4.5, power mean inequality and (s,m)–exponential type convexity of $\left|\psi'\right|^q$, we have

$$\left|\frac{1}{a_2 - ka_1}\int_{ka_1}^{a_2}\psi(\theta)\,d\theta - \psi\left(\frac{ka_1+a_2}{2}\right)\right| \le (a_2 - ka_1)$$
$$\times\left\{\int_0^1 |-\chi|\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|d\chi + \int_{\frac{1}{2}}^1\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|d\chi\right\}$$
$$\le (a_2 - ka_1)\left\{\left(\int_0^1 \chi\, d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1 \chi\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 1\, d\chi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\psi'(k(1-\chi)a_1+\chi a_2)\right|^q d\chi\right)^{\frac{1}{q}}\right\}$$
$$\le (a_2 - ka_1)\left\{\left(\int_0^1 \chi\, d\chi\right)^{1-\frac{1}{q}}\left(\int_0^1 \chi\left(\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 1\, d\chi\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{1}{2}}^1\left(\left(e^{(1-\chi)s}-1\right)\left|\psi'(ka_1)\right|^q + m\left(e^{s\chi}-1\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)d\chi\right)^{\frac{1}{q}}\right\}$$
$$= (a_2 - ka_1)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\left|\psi'(ka_1)\right|^q + m\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\left|\psi'(ka_1)\right|^q + m\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)\left|\psi'\left(\frac{a_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\},$$

which completes the proof. □

5. Applications

Let consider the following two special means for positive real numbers $a_1 \neq a_2$:

1. The arithmetic mean:
$$\mathcal{A}(a_1, a_2) = \frac{a_1 + a_2}{2},$$
2. The generalized logarithmic mean:
$$\mathcal{L}_l(a_1, a_2) = \left[\frac{a_2^{l+1} - a_1^{l+1}}{(l+1)(a_2 - a_1)}\right]^{\frac{1}{l}}; \ \ l \in \Re \setminus \{-1, 0\}.$$

Dragomir et al. [3], have proved that for $s \in (0,1)$, where $1 \leq l \leq \frac{1}{s}$, the function $\psi(x) = x^{ls}, \ x > 0$ is s–convex function. Then from Proposition 2.5, it is also s–exponential convex function for some fixed $s \in [\ln 2.5, 1)$.

Using Sect. 4, we have

Proposition 5.1. *Let $0 < a_1 < a_2$, $0 < k \leq 1$ and $q > 1$ such that $p^{-1} + q^{-1} = 1$. Then for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq \frac{1}{s}$, we have*
$$\left|\mathcal{A}\left(a_1^{ls}, \left(\frac{a_2}{k}\right)^{ls}\right) - \frac{k}{a_2 - ka_1}\mathcal{L}_{ls}^{ls}\left(a_1, \frac{a_2}{k}\right)\right| \leq \frac{ls(a_2 - ka_1)}{k\sqrt[p]{2}} \tag{24}$$
$$\times \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}} \mathcal{A}^{\frac{1}{q}}\left(a_1^{(ls-1)q}, \left(\frac{a_2}{k}\right)^{(ls-1)q}\right).$$

Proof. *Consider the s–exponential convex function $\psi(x) = x^{ls}, \ x > 0$ and using Theorem 4.6, we have the required result.* □

Proposition 5.2. *Let $0 < a_1 < a_2$, $0 < k \leq 1$ and $q > 1$. Then for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq \frac{1}{s}$, we have*
$$\left|\mathcal{A}\left(a_1^{ls}, \left(\frac{a_2}{k}\right)^{ls}\right) - \frac{k}{a_2 - ka_1}\mathcal{L}_{ls}^{ls}\left(a_1, \frac{a_2}{k}\right)\right| \leq \frac{ls(a_2 - ka_1)}{4^{\left(1-\frac{1}{q}\right)}k} \tag{25}$$
$$\times \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^{\frac{1}{q}} \mathcal{A}^{\frac{1}{q}}\left(a_1^{(ls-1)q}, \left(\frac{a_2}{k}\right)^{(ls-1)q}\right).$$

Proof. *Using Theorem 4.7, we get the required result.* □

Proposition 5.3. *Let $0 < a_1 < a_2$, $0 < k \leq 1$ and $q > 1$ such that $p^{-1} + q^{-1} = 1$. Then for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq \frac{1}{s}$, we have*
$$\left|\mathcal{A}\left((ka_1)^{ls}, a_2^{ls}\right) - \mathcal{L}_{ls}^{ls}(ka_1, a_2)\right| \leq \frac{ls(a_2 - ka_1)}{\sqrt[p]{2}} \tag{26}$$
$$\times \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}} \mathcal{A}^{\frac{1}{q}}\left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).$$

Proof. *Using Theorem 4.8, we obtain the required result.* □

Proposition 5.4. *Let $0 < a_1 < a_2$, $0 < k \leq 1$ and $q > 1$. Then for some fixed $s \in [\ln 2.5, 1)$, where $1 \leq l \leq \frac{1}{s}$, we have*
$$\left|\mathcal{A}\left((ka_1)^{ls}, a_2^{ls}\right) - \mathcal{L}_{ls}^{ls}(ka_1, a_2)\right| \leq \frac{ls(a_2 - ka_1)}{4^{\left(1-\frac{1}{q}\right)}} \tag{27}$$
$$\times \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^{\frac{1}{q}} \mathcal{A}^{\frac{1}{q}}\left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q}\right).$$

Proof. *Using Theorem 4.9, we have the required result.* □

Proposition 5.5. *Let* $0 < a_1 < a_2$, $0 < k \leq 1$ *and* $q > 1$ *such that* $p^{-1} + q^{-1} = 1$. *Then for some fixed* $s \in [\ln 2.5, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\left|\frac{2}{k+1}\left(\mathcal{A}\left((ka_1)^{ls}, a_2^{ls}\right) - \mathcal{L}_{ls}^{ls}(ka_1, a_2)\right)\right| \leq \sqrt[q]{2}\frac{ls(a_2 - ka_1)}{k+1} \times \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}}\mathcal{A}^{\frac{1}{q}}\left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q}\right). \tag{28}$$

Proof. *Using Theorem 4.10, we get the required result.* □

Proposition 5.6. *Let* $0 < a_1 < a_2$, $0 < k \leq 1$ *and* $q > 1$. *Then for some fixed* $s \in [\ln 2.5, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\left|\frac{2}{k+1}\left(\mathcal{A}\left((ka_1)^{ls}, a_2^{ls}\right) - \mathcal{L}_{ls}^{ls}(ka_1, a_2)\right)\right| \leq \frac{ls(a_2 - ka_1)}{2(k+1)} \times \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^{\frac{1}{q}}\mathcal{A}^{\frac{1}{q}}\left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q}\right). \tag{29}$$

Proof. *Using Theorem 4.11, we obtain the required result.* □

Proposition 5.7. *Let* $0 < a_1 < a_2$, $0 < k \leq 1$ *and* $q > 1$ *such that* $p^{-1} + q^{-1} = 1$. *Then for some fixed* $s \in [\ln 2.5, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned}\left|\mathcal{L}_{ls}^{ls}(ka_1, a_2) - \mathcal{A}^{ls}(ka_1, a_2)\right| &\leq ls(a_2 - ka_1) \\ &\times \left\{\sqrt[q]{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}}\mathcal{A}^{\frac{1}{q}}\left((ka_1)^{(ls-1)q}, a_2^{(ls-1)q}\right)\right. \\ &\left. + \left(\frac{1}{2}\right)^{\frac{1}{p}}\left((ka_1)^{(ls-1)q}\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right) + a_2^{(ls-1)q}\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{30}$$

Proof. *Using Theorem 4.14, we have the required result.* □

Proposition 5.8. *Let* $0 < a_1 < a_2$, $0 < k \leq 1$ *and* $q > 1$. *Then for some fixed* $s \in [\ln 2.5, 1)$, *where* $1 \leq l \leq \frac{1}{s}$, *we have*

$$\begin{aligned}\left|\mathcal{L}_{ls}^{ls}(ka_1, a_2) - \mathcal{A}^{ls}(ka_1, a_2)\right| &\leq ls(a_2 - ka_1) \\ &\times \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left((ka_1)^{(ls-1)q}\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right) + a_2^{(ls-1)q}\left(\frac{(2s-2)e^s - s^2 + 2}{2s^2}\right)\right)^{\frac{1}{q}}\right. \\ &\left. + \left((ka_1)^{(ls-1)q}\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right) + a_2^{(ls-1)q}\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{31}$$

Proof. *Using Theorem 4.15, we get the required result.* □

At the end, let consider some applications of the integral inequalities obtained above, to find new bounds for the trapezoidal and midpoint formula.

For $a_2 > 0$, let $\mathcal{U} : 0 = \chi_0 < \chi_1 < \ldots < \chi_{n-1} < \chi_n = a_2$ be a partition of $[0, a_2]$.

We denote, respectively,

$$\mathcal{T}(\mathcal{U}, \psi) = \sum_{i=0}^{n-1}\left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2}\right)h_i, \quad \mathcal{M}(\mathcal{U}, \psi) = \sum_{i=0}^{n-1}\psi\left(\frac{\chi_i + \chi_{i+1}}{2}\right)h_i,$$

and

$$\int_0^{a_2} \psi(x)dx = \mathcal{T}(\mathcal{U},\psi) + \mathcal{R}(\mathcal{U},\psi), \quad \int_0^{a_2} \psi(x)dx = \mathcal{M}(\mathcal{U},\psi) + \mathcal{R}^*(\mathcal{U},\psi),$$

where $\mathcal{R}(\mathcal{U},\psi)$ and $\mathcal{R}^*(\mathcal{U},\psi)$ are the remainders terms and $h_i = \chi_{i+1} - \chi_i$ for $i = 0, 1, 2, \ldots, n-1$.

Using above notations, we are in position to prove the following error estimations.

Proposition 5.9. *Suppose a mapping $\psi : (0, a_2] \to \Re$ is differentiable on $(0, a_2)$ with $a_2 > 0$. If $\left|\psi'\right|^q$ is s–exponential type convex on $(0, a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:*

$$\begin{aligned} \left|\mathcal{R}(\mathcal{U},\psi)\right| &\leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}} \\ &\times \sum_{i=0}^{n-1} h_i^2 \Big[\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q \Big]^{\frac{1}{q}}. \end{aligned} \tag{32}$$

Proof. Using the Theorem 4.6 on subinterval $[\chi_i, \chi_{i+1}]$ of closed interval $[0, a_2]$, for all $i = 0, 1, 2, \ldots, n-1$ and $m = 1$, we have

$$\begin{aligned} \left|\left(\frac{\psi(\chi_i) + \psi(\chi_{i+1})}{2}\right) h_i - \int_{\chi_i}^{\chi_{i+1}} \psi(x)dx\right| &\leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{e^s - s - 1}{s}\right)^{\frac{1}{q}} \\ &\times h_i^2 \Big[\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q \Big]^{\frac{1}{q}}. \end{aligned} \tag{33}$$

Summing inequality (33) over i from 0 to $n-1$ and using the property of modulus, we obtain the desired inequality (32). □

Proposition 5.10. *Suppose a mapping $\psi : (0, a_2] \to \Re$ is differentiable on $(0, a_2)$ with $a_2 > 0$. If $\left|\psi'\right|^q$ is s–exponential type convex on $(0, a_2]$ for $q > 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:*

$$\begin{aligned} \left|\mathcal{R}(\mathcal{U},\psi)\right| &\leq \left(\frac{1}{2}\right)^{2-\frac{1}{q}} \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)^{\frac{1}{q}} \\ &\times \sum_{i=0}^{n-1} h_i^2 \Big[\left|\psi'(\chi_i)\right|^q + \left|\psi'(\chi_{i+1})\right|^q \Big]^{\frac{1}{q}}. \end{aligned} \tag{34}$$

Proof. Applying the same technique as in Proposition 5.9 but using Theorem 4.7. □

Proposition 5.11. *Suppose a mapping $\psi : (0, a_2] \to \Re$ is differentiable on $(0, a_2)$ with $a_2 > 0$. If $\left|\psi'\right|^q$ is s–exponential type convex on $(0, a_2]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s \in (0, 1]$, the remainder term satisfies the following error estimation:*

$$\left|\mathcal{R}^*(\mathcal{U},\psi)\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{e^s-s-1}{s}\right)^{\frac{1}{q}} \tag{35}$$
$$\times \sum_{i=0}^{n-1} h_i^2\Big[\left|\psi'(\chi_i)\right|^q+\left|\psi'(\chi_{i+1})\right|^q\Big]^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1} h_i^2\left[\left|\psi'(\chi_i)\right|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)+\left|\psi'(\chi_{i+1})\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right]^{\frac{1}{q}}.$$

Proof. Applying the same technique as in Proposition 5.9 but using Theorem 4.12. □

Proposition 5.12. *Suppose a mapping $\psi : (0,a_2] \to \Re$ is differentiable on $(0,a_2)$ with $a_2 > 0$. If $\left|\psi'\right|^q$ is s–exponential type convex on $(0,a_2]$ for $q > 1$, then for some fixed $s \in (0,1]$, the remainder term satisfies the following error estimation:*

$$\left|\mathcal{R}^*(\mathcal{U},\psi)\right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \tag{36}$$
$$\times\Bigg[\sum_{i=0}^{n-1} h_i^2\left\{\left|\psi'(\chi_i)\right|^q\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)+\left|\psi'(\chi_{i+1})\right|^q\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\right\}^{\frac{1}{q}}$$
$$+\sum_{i=0}^{n-1} h_i^2\left\{\left|\psi'(\chi_i)\right|^q\left(\frac{e^s-e^{\frac{s}{2}}}{s}-\frac{1}{2}\right)+\left|\psi'(\chi_{i+1})\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right\}^{\frac{1}{q}}\Bigg].$$

Proof. Applying the same technique as in Proposition 5.9 but using Theorem 4.13. □

6. Conclusion

In this article, authors showed new generalizations of Hermite–Hadamard type inequality for the new class of functions, the so–called (s,m)–exponential type convex function ψ and for the products of two (s,m)–exponential type convex functions ψ and ϕ. We have obtained refinements of the (H–H) inequality for functions whose first derivative in absolute value at certain power are (s,m)–exponential type convex and founded new bounds for special means and for the error estimates for the trapezoidal and midpoint formula. We hope that our new ideas and techniques may inspired many researchers in this fascinating field.

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