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Generalized Jacobson's Lemma for Generalized Drazin Inverses

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Abstract. We present new generalized Jacobson's lemma for generalized Drazin inverses. This extends the main results on g-Drazin inverse of Yan, Zeng and Zhu (Linear & Multilinear Algebra, **68**(2020), 81–93).

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. An element $a \in R$ has g-Drazin inverse in case there exists $x \in R$ such that

$$x = xax, x \in comm^2(a), a - a^2x \in R^{qnil}.$$

The preceding x is unique if it exists, we denote it by a^d. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in R^{-1} \text{ for every } x \in comm(a)\}$, where R^{-1} stands for the set of all invertible elements of R. As it is known, $a \in R$ has g-Drazin inverse if and only if there exists an idempotent $p \in comm^2(a)$ such that $a + p \in R$ is invertible and $ap \in R^{qnil}$ (see [4, Lemma 2.4]).

For any $a, b \in R$, Jacobson's Lemma for invertibility states that $1 - ab \in R^{-1}$ if and only if $1 - ba \in R^{-1}$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$ (see [7, Lemma 1.4]). Let $a, b \in R^d$. Zhuang et al. proved the Jacobson's Lemma for g-Drazin inverse. That is, it was proved that $1 - ab \in R^d$ if and only if $1 - ba \in R^d$ and

$$(1 - ba)^d = 1 + b(1 - ab)^d a$$

(see [15, Theorem 2.3]). Jacobson's Lemma plays an important role in matrix and operator theory. Many papers discussed this lemma for g-Drazin inverse in the setting of matrices, operators, elements of Banach algebras or rings. Mosić generalized Jacobson's Lemma for g-Drain inverse to the case that $bdb = bac, dbd = acd$ (see [7, Theorem 2.5]). Recently, Yan et al. extended Jacobson's Lemma to the case $dba = aca, dbd = acd$ (see [12, Theorem 3.3]). This condition was also considered for bounded linear operators in [10, 11, 13].

The motivation of this paper is to extend the main results of Yan et al. (see [12]) to a wider case. The Drazin inverse of $a \in R$, denoted by a^D, is the unique element a^D satisfying the following three equations

$$a^D = a^D a a^D, a^D \in comm(a), a^k = a^{k+1}a$$

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for some $k \in \mathbb{N}$. The smallest integer k is called the Drazin index of a, and is denoted by $i(a)$. Moreover, we prove the generalized Jacobson's lemma for the Drazin inverse.

Throughout the paper, all rings are associative with an identity. R^D and R^d denote the sets of all Drazin and g-Drazin invertible elements in R respectively. We use R^{nil} to denote the set of all nilpotents of the ring R. $\mathbb{C}$ stands for the field of all complex numbers.

2. Generalized Jacobson's lemma

We come now to the main result of this paper which will be the tool in our following development.

Theorem 2.1. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $\alpha = 1 - bd \in R^d$ if and only if $\beta = 1 - ac \in R^d$. In this case,

$$\beta^d = \Big[1 - d\alpha^{\pi}(1 - \alpha(1 + bd))^{-1}bac\Big](1 + ac) + d\alpha^d bac.$$

Proof. Let $p = \alpha^{\pi}, x = \alpha^d$. Then $1 - p\alpha(1 + bd) \in R^{-1}$. Let

$$y = \Big[1 - dp(1 - p\alpha(1 + bd))^{-1}bac\Big](1 + ac) + dxbac.$$

We shall prove that $\beta^d = y$.

Step 1. $y\beta y = y$. We see that

$$\begin{aligned} y\beta &= 1 - (ac)^2 - dp(1 - p\alpha(1 + bd))^{-1}bac\Big[1 - (ac)^2\Big] + dxbac(1 - ac) \\ &= 1 - [dbac - dxbac(1 - ac)] - dp(1 - p\alpha(1 + bd))^{-1}[bac - bac(ac)^2 \\ &= 1 - [dbac - dx(bac - bacac)] - dp(1 - p\alpha(1 + bd))^{-1}[ba - bacdba]c \\ &= 1 - [dbac - dx(1 - bd)bac] - dp(1 - p\alpha(1 + bd))^{-1}[1 - (bd)^2]bac \\ &= 1 - dpbac - dp(1 - p\alpha(1 + bd))^{-1}p\alpha(1 + bd)bac \\ &= 1 - dp(1 - p\alpha(1 + bd))^{-1}\Big[(1 - p\alpha(1 + bd)) + p\alpha(1 + bd)\Big]bac \\ &= 1 - dp(1 - p\alpha(1 + bd))^{-1}bac. \end{aligned}$$

Since $acdbd = a(cdbd) = a(cacd) = (ac)^2d = (dbac)d = dbacd$, we have $(bacd)(bd) = (bd)(bacd)$, and so $(bacd)\alpha = \alpha(bacd)$. Hence, $(bacd)x = x(bacd)$, and then

$$\begin{aligned} &dp(1 - p\alpha(1 + bd))^{-1}bacdxbac(1 - ac) \\ &= d(1 - p\alpha(1 + bd))^{-1}pxbacdbac(1 - ac) = 0. \end{aligned}$$

Therefore we have

$$\begin{aligned} y\beta y &= y - dp(1 - p\alpha(1 + bd))^{-1}bacy \\ &= y - dp(1 - p\alpha(1 + bd))^{-1}bac\Big[1 - dp(1 - p\alpha(1 + bd))^{-1}bac(1 + ac) \\ &= y - dp(1 - p\alpha(1 + bd))^{-1}bac(1 + ac) + dp(1 - p\alpha(1 + bd))^{-2}(bacd)bac(1 + ac) \\ &= y - dp(1 - p\alpha(1 + bd))^{-1}(1 + bd)bac + dp(1 - p\alpha(1 + bd))^{-2}(bd)^2(1 + bd)bac \\ &= y - dp(1 - p\alpha(1 + bd))^{-2}\Big[p - p\alpha(1 + bd) - p(bd)^2(1 + bd)bac = y. \end{aligned}$$

Step 2. $\beta-\beta y\beta \in R^{qnil}$. Since $y = y\beta y$, we see that $(1-y\beta)^2 = 1-y\beta$. Hence,

$$\begin{aligned}\beta-\beta y\beta &= \beta(1-y\beta)\\ &= \beta dp(1-p\alpha(1+bd))^{-1}(bacd)p(1-p\alpha(1+bd))^{-1}bac\\ &= \beta d(bacd)p(1-p\alpha(1+bd))^{-2}bac\\ &= (1-ac)d(bacd)p(1-p\alpha(1+bd))^{-2}bac\\ &= d\alpha bacdp(1-p\alpha(1+bd))^{-2}bac.\end{aligned}$$

Let $z \in comm(\beta-\beta y\beta)$. Then

$$zd\alpha bacdp(1-p\alpha(1+bd))^{-2}bac = d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz.$$

We will suffice to prove

$$1+d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz \in R^{-1}.$$

Clearly, we have

$$p = (bd)^2p[1-p\alpha(1+bd)]^{-1} = (bd)^4p[1-p\alpha(1+bd)]^{-2}.$$

Hence, we get

$$\begin{array}{rcl}(baczdbacd)\alpha p &=& (baczd)\alpha bacdp[1-p\alpha(1+bd)]^{-2}(bd)^4\\ &=& (baczd)\alpha bacdp[1-p\alpha(1+bd)]^{-2}bac(db)^2d\\ &=& bac\Big[zd\alpha bacdp(1-p\alpha(1+bd))^{-2}bac\Big]acdbd\\ &=& bac\Big[d\alpha bacdp(1-p\alpha(1+bd))^{-2}bacz\Big]acdbd\\ &=& bdbdbacd\Big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\Big]acdbd\\ &=& bacdbdbd\Big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\Big]a(cdbd)\\ &=& bdbdbdbd\Big[\alpha p(1-p\alpha(1+bd))^{-2}bacz\Big]a(cacd)\\ &=& bdbdbdbd\alpha p[1-p\alpha(1+bd)]^{-2}baczdbacd\\ &=& \alpha p(baczdbacd).\end{array}$$

Step 3. $y \in comm^2(\beta)$. Let $s \in comm(\beta)$. Then $s\beta = \beta s$, and so $s(ac) = (ac)s$.
Claim 1. $s(dxbac) = (dxbac)s$. We easily check that

$$(bacsdbd)\alpha = bacs\beta dbd = bac\beta sdbd = \alpha(bacsdbd).$$

Hence

$$(bacsdbd)x = x(bacsdbd),$$

and then

$$\begin{array}{rcl}s(dpbac) &=& sd(bd)^4p[1-p\alpha(1+bd)]^{-2}bac\\ &=& s(ac)^2dbdbdp[1-p\alpha(1+bd)]^{-2}bac\\ &=& d(bacsdbd)bdp[1-p\alpha(1+bd)]^{-2}bac\\ &=& dbdp[1-p\alpha(1+bd)]^{-2}(bacsdbd)bac\\ &=& dbdp[1-p\alpha(1+bd)]^{-2}bacs(ac)^3\\ &=& dbdp[1-p\alpha(1+bd)]^{-2}(bd)^3bacs\\ &=& d(bd)^4p[1-p\alpha(1+bd)]^{-2}bacs\\ &=& (dpbac)s.\end{array}$$

Since $sdbac = s(ac)^2 = (ac)^2s = dbacs$, we have

$$sd\alpha xbac = d\alpha xbacs,$$

and so

$$sdxbac - sdbdxbac = dxbacs - dbdxbacs.$$

On the other hand, we have

$$\begin{aligned}
s(dbdpbac) &= sd(bd)^5p[1-p\alpha(1+bd)]^{-2}bac\\
&= s(ac)^4dbdp[1-p\alpha(1+bd)]^{-2}bac\\
&= dbdbd(bacsdbd)p[1-p\alpha(1+bd)]^{-2}bac\\
&= dbdbdp[1-p\alpha(1+bd)]^{-2}(bacsdbd)bac\\
&= dbdbdp[1-p\alpha(1+bd)]^{-2}bacs(ac)^3\\
&= dbdbdp[1-p\alpha(1+bd)]^{-2}(bd)^3bacs\\
&= dbd(bd)^4p[1-p\alpha(1+bd)]^{-2}bacs\\
&= (dbdpbac)s.
\end{aligned}$$

Since $sdbdbac = s(ac)^3 = (ac)^3s = dbdbacs$, we have

$$dbd\alpha xbacs = sdbd\alpha xbac.$$

Then we have

$$\begin{aligned}
dbdbd\alpha xbacs &= ac(dbd\alpha xbacs)\\
&= ac(sdbd\alpha xbac)\\
&= sacdbd\alpha xbac\\
&= sdbdbd\alpha xbac,
\end{aligned}$$

and so

$$dbd(1+bd)\alpha xbacs = sdbd(1+bd)\alpha xbac,$$

and then

$$dbdxbacs - dbd(bd)^2xbacs = sdbdxbac - sdbd(bd)^2xbac.$$

One easily checks that

$$\begin{aligned}
d(bd)^3xbacs &= dx(bd)^3bacs\\
&= dxb(ac)^4s\\
&= dx(bacsdbd)bac\\
&= d(bacsdbd)xbac\\
&= (ac)^2sdbdxbac\\
&= sd(bd)^3xbac.
\end{aligned}$$

This implies that $dbdxbacs = sdbdxbac$, and therefore $s(dxbac) = (dxbac)s$.

Claim 2. $sdp(1-p\alpha(1+bd))^{-1}bac(1+ac) = dp(1-p\alpha(1+bd))^{-1}bac(1+ac)s$. Set $t = dp(1-p\alpha(1+bd))^{-1}bac(1+ac)$. Then we have

$$\begin{aligned}
st &= sdp(1-p\alpha(1+bd))^{-1}bac(1+ac)\\
&= sd(bd)^4p[1-p\alpha(1+bd)]^{-3}bac(1+ac)\\
&= (ac)^3sdbdp[1-p\alpha(1+bd)]^{-3}bac(1+ac)\\
&= dbdp[1-p\alpha(1+bd)]^{-3}bs(ac)^4(1+ac)
\end{aligned}$$

Also we have

$$\begin{aligned}
ts &= dp(1-p\alpha(1+bd))^{-1}bac(1+ac)s\\
&= dp[1-p\alpha(1+bd)]^{-3}(bd)^4bsac(1+ac)\\
&= dbdp[1-p\alpha(1+bd)]^{-3}b(ac)^3sac(1+ac)\\
&= dbdp[1-p\alpha(1+bd)]^{-3}bs(ac)^4(1+ac)
\end{aligned}$$

Hence, $st = ts$, and so $y \in comm^2(\beta)$. Therefore $y = \beta^d$, as required.

$\Longleftarrow$ Since $1-ac \in R^d$, it follows by Jacobson's Lemma that $1-ca \in R^d$. Applying the preceding discussion, we obtain that $1-bd \in R^d$, as desired. $\square$

Corollary 2.2. *([12, Theorem 3.1]) Let R be a ring, and let $a,b,c,d \in R$ satisfying*

$$acd = dbd, dba = aca.$$

Then $1-bd\in R^d$ *if and only if* $1-ac\in R^d$. *In this case,*

$$(1-bd)^d = 1-d(1-bd)^{\pi}(1-(1-bd)^{\pi}(1-bd)(1+bd))^{-1}bac(1+ac)+$$
$$d(1-ac)^d bac.$$

Proof. By hypothesis, we easily check that

$$\begin{array}{c}(ac)^2=(aca)c=(dba)c=(db)(ac),\\(db)^2=(dbd)b=(acd)b=(ac)(db);\\b(ac)a=b(aca)=b(dba)=b(db)a,\\c(ac)d=c(acd)=c(dbd)=c(db)d.\end{array}$$

Then the result follows by Theorem 2.1. □

We now generalize [12, Corollary 3.5] as follows.

Corollary 2.3. *Let R be a ring, and let $a,b,c\in R$ satisfying*

$$\begin{array}{c}(aba)b=(aca)b, b(aba)=b(aca),\\(aba)c=(aca)c, c(aba)=c(aca).\end{array}$$

Then $1-ba\in R^d$ *if and only if* $1-ac\in R^d$. *In this case,*

$$(1-ba)^d=\Big[1-a(1-ba)^{\pi}(1-(1-ba)^{\pi}(1-ba)(1+ba))^{-1}bac\Big](1+ac)+a(1-ac)^d bac.$$

Proof. By hypothesis, we verify that

$$\begin{array}{c}(ac)^2=(aca)c=(aba)c=(ab)(ac),\\(ab)^2=(aba)b=(aca)b=(ac)(db);\\b(ac)a=b(aca)=b(aba)=b(ab)a,\\c(ac)a=c(aca)=c(aba)=c(ab)a.\end{array}$$

This completes the proof by Theorem 2.1. □

It is convenient at this stage to derive the following.

Theorem 2.4. *Let R be a ring, let $n\in\mathbb{N}$, and let $a,b,c,d\in\mathcal{A}$ satisfying*

$$\begin{array}{c}(ac)^2=(db)(ac), (db)^2=(ac)(db);\\b(ac)a=b(db)a, c(ac)d=c(db)d.\end{array}$$

Then $(1-bd)^n\in R^d$ *if and only if* $(1-ac)^n\in R^d$.

Proof. $\Rightarrow$ Let $\alpha=(1-ac)^n$. Then

$$\begin{array}{rcl}\alpha&=&\sum\limits_{i=0}^{n}(-1)^i\begin{pmatrix}n\\i\end{pmatrix}(ac)^i\\&=&1-a\sum\limits_{i=1}^{n}(-1)^i\begin{pmatrix}n\\i\end{pmatrix}c(ac)^{i-1}\\&=&1-ac',\end{array}$$

where $c'=\sum\limits_{i=1}^{n}(-1)^i\begin{pmatrix}n\\i\end{pmatrix}c(ac)^{i-1}$. Let $\beta=(1-ba)^n$. Then

$$\begin{array}{rcl}\beta&=&\sum\limits_{i=0}^{n}(-1)^i\begin{pmatrix}n\\i\end{pmatrix}(bd)^i\\&=&1-\Big(\sum\limits_{i=1}^{n}(-1)^i\begin{pmatrix}n\\i\end{pmatrix}(bd)^{i-1}b\Big)d\\&=&1-b'd,\end{array}$$

where $b' = \sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}b$. We directly compute that

$$\begin{array}{rl} & (ac')^2 \\ = & \Big[a\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)c(ac)^{i-1}\Big]^2 = \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^i\Big]^2 \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^i\Big]ac\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^{i-1}\Big]; \\ & (db')(ac') \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)d(bd)^{i-1}b\Big]\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^i\Big] \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(db)^i\Big]ac\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^{i-1}\Big]. \end{array}$$

Since $(ac)^i(ac) = (db)^i(ac)$ for any $i \in \mathbb{N}$, we have $(ac')^2 = (db')(ac')$. Moreover, we check that

$$\begin{array}{rl} & (db')^2 \\ = & \Big[d\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}b\Big]^2 = \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(db)^i\Big]^2 \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(db)^i\Big]db\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(db)^{i-1}\Big]; \\ & (ac')(db') \\ = & \Big[a\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)c(ac)^{i-1}\Big]\Big[d\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)b(db)^{i-1}\Big] \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(ac)^i\Big]db\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(db)^{i-1}\Big]. \end{array}$$

Since $(ac)^i(db) = (db)^i(db)$ for any $i \in \mathbb{N}$, we have $(db')^2 = (ac')(db')$. Furthermore, we verify that

$$\begin{array}{rl} b'(ac')a = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}b\Big]\Big[a\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)c(ac)^{i-1}a\Big] \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}\Big]\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)b(ac)^ia\Big]; \\ b'(db')a = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}b\Big]\Big[d\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}b\Big]a \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)(bd)^{i-1}\Big]\Big[\sum\limits_{i=1}^{n}(-1)^i\left(\begin{array}{c} n \\ i \end{array}\right)b(db)^ia\Big]. \end{array}$$

Since $b(ac)a = b(db)a$ we see that $b(ac)^2a = b(dbac)a = bdb(ac)a = bdb(db)a = b(dbdb)a = b(db)^2a$. By induction, we have $b(ac)^ia = b(db)^ia$ for any $n \in \mathbb{N}$. Therefore $b'(ac')a = b'(db')a$. Also we have

$$\begin{array}{rl} c'(ac')d = & \Big[\sum\limits_{i=1}^{n}(-1)^ic(ac)^{i-1}\Big]\Big[a\sum\limits_{i=1}^{n}(-1)^ic(ac)^{i-1}d\Big] \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i(ca)^{i-1}\Big]\Big[\sum\limits_{i=1}^{n}(-1)^ic(ac)^id\Big]; \\ c'(db')d = & \Big[\sum\limits_{i=1}^{n}(-1)^ic(ac)^{i-1}\Big]\Big[d\sum\limits_{i=1}^{n}(-1)^i(bd)^{i-1}b\Big]d \\ = & \Big[\sum\limits_{i=1}^{n}(-1)^i(ca)^{i-1}\Big]\Big[\sum\limits_{i=1}^{n}(-1)^ic(db)^id\Big]. \end{array}$$

Since $c(ac)d = c(db)d$, by induction, we get $c(ac)^id = c(db)^id$ for any $n \in \mathbb{N}$. This implies that $c'(ac')d = c'(db')d$. In light of Theorem 2.1, $1 - db' \in R^d$ if and only if $1 - ac' \in R^d$, as desired. $\square$

Corollary 2.5. *Let R be a ring, let $n \in \mathbb{N}$, and let $a, b, c \in R$ satisfying*

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

Then $(1 - ba)^n \in R^d$ if and only if $(1 - ac)^n \in R^d$.

Proof. This is obvious by Theorem 2.4. □

3. Drazin inverse

As it is known, $a \in R^D$ if and only if there exists $x \in R$ such that $x = xax, x \in comm^2(a), a - a^2x \in R^{nil}$, and so $a^D = a^d$. For the generalized Jacobson's Lemma for Drazin inverse, we have

Theorem 3.1. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2 = (db)(ac), (db)^2 = (ac)(db);$$
$$b(ac)a = b(db)a, c(ac)d = c(db)d.$$

Then $1 - bd \in R^D$ if and only if $1 - ac \in R^D$. In this case,

$$\begin{array}{rcl} (1-ac)^D & = & \Big[1 - d(1-bd)^{\pi}(1-(1-bd)(1+bd))^{-1}bac\Big](1+ac) \\ & + & d(1-bd)^D bac, \\ i(1-bd) & \leq & i(1-ac)+1. \end{array}$$

Proof. Set $\alpha = 1 - bd$ and $\beta = 1 - ac$. Let $p = \alpha^{\pi}, x = \alpha^D$. In view of Theorem 2.1, $\beta \in R^d$ and

$$\beta^d = \Big[1 - dp(1 - p\alpha(1 + bd))^{-1}bac\Big](1 + ac) + dxbac.$$

We shall prove that $\beta^D = \beta^d$.

We will suffice to check $\beta - \beta\beta^d\beta \in R^{nil}$. As in the proof of Theorem 2.1, we have

$$\begin{array}{rcl} \beta - \beta\beta^d\beta & = & \beta(1 - \beta^d\beta) \\ & = & d\alpha bacdp(1 - p\alpha(1 + bd))^{-2}bac. \end{array}$$

In light of [6, Lemma 3.1], we will suffice to prove

$$bacd\alpha bacdp(1 - p\alpha(1 + bd))^{-2} \in R^{nil}.$$

Similarly to the discussion in Theorem 2.1, we see that $bacd \in comm(\alpha)$, and so $bacd, \alpha p$ and $(1 - p\alpha(1 + bd))^{-2}$ commute with each other. Set $n = i(\alpha)$. Then

$$\begin{array}{rl} & \Big[bacd\alpha bacdp(1 - p\alpha(1 + bd))^{-2}\Big]^n \\ = & (bacd)^{2n}(1 - p\alpha(1 + bd))^{-2n}(\alpha - \alpha^2\alpha^d)^n \\ = & 0; \end{array}$$

hence,

$$\begin{array}{l} (\beta - \beta\beta^d\beta)^{n+1} \\ = d\alpha bacdp(1 - p\alpha(1 + bd))^{-2}\Big[bacd\alpha bacdp(1 - p\alpha(1 + bd))^{-2}\Big]^n bac \\ = 0. \end{array}$$

Thus we have $\beta - \beta\beta^d\beta \in R^{nil}$, and so $\beta^D = \beta^d$. Moreover, we have $i(\beta) \leq i(\alpha) + 1$, as desired. □

As an immediate consequence of Theorem 3.1, we now derive

Corollary 3.2. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$acd = dbd, dba = aca.$$

Then $1 - bd \in R^D$ if and only if $1 - ac \in R^D$. In this case,

$$\begin{array}{rcl}(1-ac)^D & = & \Big[1 - d\alpha^{\pi}(1-\alpha(1+bd))^{-1}bac\Big](1+ac)\\ & + & d(1-bd)^D bac,\\ i(1-bd) & \leq & i(1-ac)+1.\end{array}$$

Corollary 3.3. *Let R be a ring, and let $a, b, c \in R$ satisfying*

$$\begin{array}{c}(aba)b = (aca)b, b(aba) = b(aca),\\ (aba)c = (aca)c, c(aba) = c(aca).\end{array}$$

Then $1 - ba \in R^D$ if and only if $1 - ac \in R^D$. In this case,

$$\begin{array}{rcl}(1-ac)^D & = & \Big[1 - a(1-ba)^{\pi}(1-(1-ba)(1+ba))^{-1}bac\Big](1+ac)\\ & + & a(1-ba)^D bac,\\ i(1-ba) & \leq & i(1-ac)+1.\end{array}$$

The group inverse of $a \in R$ is the unique element $a^{\#} \in R$ which satisfies $aa^{\#} = a^{\#}a, a = aa^{\#}a, a^{\#} = a^{\#}aa^{\#}$. We denote the set of all group invertible elements of R by $R^{\#}$. As it is well known, $a \in R^{\#}$ if and only if $a \in R^D$ and $i(a) = 1$. We are now ready to prove:

Theorem 3.4. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$\begin{array}{c}(ac)^2 = (db)(ac), (db)^2 = (ac)(db);\\ b(ac)a = b(db)a, c(ac)d = c(db)d.\end{array}$$

Then $1 - bd$ has group inverse if and only if $1 - ac$ has group inverse. In this case,

$$\begin{array}{rl} & (1-ac)^{\#}\\ = & \Big[1 - d\alpha^{\pi}(1-\alpha(1+bd))^{-1}bac\Big](1+ac) + d(1-bd)^{\#}bac.\end{array}$$

Proof. Since $1 - bd \in R^{\#}$, we have $1 - bd \in R^D$. In light of Theorem 3.1, $1 - ac \in R^D$. Then

$$\beta^D = [1 - d\alpha^{\pi}(1-\alpha(1+bd)^{-1}bac](1+ac) + d(1-bd)^{\#}bc.$$

Let $\alpha = 1 - bd$ and $\beta = 1 - ac, \beta^D = (1-ac)^D$. Let $p = 1 - \alpha\alpha^D$. Since $\alpha \in R^{\#}$, we have $\alpha p = \alpha - \alpha^2\alpha^D = 0$. As in the proof of Theorem 3.1, we have

$$\begin{array}{rl} & \beta - \beta\beta^D\beta\\ = & d\alpha(bacd)p(1-p\alpha(1+bd))^{-2}bac\\ = & d(bacd)\alpha p(1-p\alpha(1+bd))^{-2}bac\\ = & 0.\end{array}$$

Obviously, $\beta^D \in comm(\beta)$ and $\beta^D = \beta^D\beta\beta^D$. Therefore

$$\begin{array}{l}\beta^{\#} = \beta^D\\ = \Big[1 - d\alpha^{\pi}(1-\alpha(1+bd))^{-1}bac\Big](1+ac) + d(1-bd)^{\#}bac.\end{array}$$

This completes the proof. □

Corollary 3.5. *Let R be a ring, and let $a, b, c \in R$ satisfying*

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

Then $1 - ba$ *has group inverse if and only if* $1 - ac$ *has group inverse. In this case,*

$$\begin{aligned}&(1-ac)^{\#}\\ =\ &\left[1-a(1-ba)^{\pi}(1-(1-ba)(1+ba))^{-1}bac\right](1+ac)+a(1-ba)^{\#}bac,\end{aligned}$$

Proof. This is obvious by Theorem 3.4. □

Corollary 3.5 is a nontrivial generalization of [7, Corollary 2.4] as the following example follows.

Example 3.6.

Let $R = M_2(\mathbb{C})$. Choose

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$$

Then we see that

$$(aba)b = (aca)b, b(aba) = b(aca),$$
$$(aba)c = (aca)c, c(aba) = c(aca).$$

But $aba = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0 = aca$. In this case,

$$(1-ac)^{\#} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (1-ba)^{\#} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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References

[1] N. Castro-González; C. Mendes-Araújo and P. Parricio, Generalized inverses of a sum in rings, *Bull. Aust. Math. Soc.*, **82**(2010), 156–164.
[2] G. Corach, Extensions of Jacobson's lemma, *Comm. Algebra*, **41**(2013), 520-531.
[3] D. Cvetković-Ilić and R. Harte, On Jacobson's lemma and Drazin invertibility, *Applied Math. Letters*, **23**(2010), 417–420.
[4] J.J. Koliha, A generalized Drazin inverse, *Glasgow Math. J.*, **38**(1996), 367–381.
[5] X. Mary, Weak inverses of products - Cline's formula meets Jacobson lemma, *J. Algebra Appl.*, **17**(2018), DOI: 10.1142/S021949881850069X.
[6] V.G. Miller and H. Zguitti, New extensions of Jacobson's lemma and Cline's formula, *Rend. Circ. Mat. Palermo, II. Ser.*, Published online: 09 February 2017, Doi: 10.1007/s12215-017-0298-6.
[7] D. Mosić, Extensions of Jacobson's lemma for Drazin inverses, *Aequat. Math.*, **91**(2017), 419–428.
[8] D. Mosić, Generalized inverses, Faculty of Science and Mathematics, University of Nis, Nis 2018.
[9] K. Yang and X. Fang, Common properties of the operator products in spectral theory, *Ann. Funct. Anal.*, **6**(2015), 60–69.
[10] K. Yang and X. Fang, Common properties of the operator products in local spectral theory, *Acta Math. Sin. Engl. Ser.*, **31**(2015), 1715–1724.
[11] K. Yan; Q. Zeng and Y. Zhu, On Drazin spectral equation for the operator products, *Complex Analysis and Operator Theory*, (2020) 14:12 https://doi.org/10.1007/s11785-019-00979-y.
[12] K. Yan; Q. Zeng and Y. Zhu, Generalized Jacobson's lemma for Drazin inverses and its applications, *Linear and Multilinear Algebra*, **68**(2020), 81–93.
[13] Q. Zeng; Z. Wu and Y. Wen, New extensions of Cline's formula for generalized inverses, *Filomat*, **31**(2017), 1973–1980.
[14] Q.P. Zeng and H.J. Zhong, New results on common properties of the products *AC* and *BA*, *J. Math. Anal. Appl.*, **427**(2015), 830–840.
[15] G. Zhuang; J. Chen and J. Cui, Jacobson's lemma for the generalized Drazin inverse, *Linear Algebra Appl.*, **436**(2012), 742–746.