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Some Classical Inequalities and their Applications

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Abstract. In this paper, we define analogies of classical Hölder-McCarthy and Young type inequalities in terms of the Berezin symbols of operators on a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$. These inequalities are applied in proving of some new inequalities for the Berezin number of operators. We also define quasi-paranormal and absolute-k-quasi paranormal operators and study their properties by using the Berezin symbols.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a Hilbert space of complex-valued functions on some set Ω such that $f \to f(\lambda)$ is a continuous functional (evaluation functional) for any λ in Ω. Then, according to the Riesz's representation theorem there exists uniquely $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for all $f \in \mathcal{H}$. The function $k_\lambda(z), \lambda \in \Omega$, is called the reproducing kernel of the space $\mathcal{H}$, and $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|}$ is called the normalized reproducing kernel in $\mathcal{H}$ (see [2]). The space $\mathcal{H}$ with the reproducing kernels $k_\lambda, \lambda \in \Omega$, is called reproducing kernel Hilbert space (RKHS). For a bounded linear operator A (i.e., for $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$) its Berezin symbol $\widetilde{A}$ is defined by (Berezin [6, 7])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \ \lambda \in \Omega.$$

The Berezin number $\operatorname{ber}(A)$ of operator A is the following number:

$$\operatorname{ber}(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|.$$

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Since $\left|\widetilde{A}(\lambda)\right| \leq \|A\|$ (by the Cauchy-Schwarz inequality) for all $\lambda \in \Omega$, the Berezin number is a finite number and $\operatorname{ber}(A) \leq \|A\|$. Recall that

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

is the numerical range of operator A and

$$\begin{aligned} w(A) &:= \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H} \text{ and } \|x\| = 1\} \\ &= \sup\left\{|\mu| : \mu \in W(A)\right\} \end{aligned}$$

is the numerical radius of A (for more information, see [1, 20–22]). It is well known that

$$\operatorname{Ber}(A) \subset W(A) \text{ and } \operatorname{ber}(A) \leq w(A)$$

for any $A \in \mathcal{B}(\mathcal{H})$. More information about $\operatorname{ber}(A)$ and relations between $\operatorname{ber}(A)$, $w(A)$ and $\|A\|$ can be found in Karaev [16, 18], and also in [3–5, 9–15, 17, 19, 23–25].

In this paper, we will use some known operator inequalities to prove some new inequalities for the Berezin number of operators acting on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Some other related questions also will be studied. In general, the present paper is motivated by the paper of Garayev [16], where the McCarthy, Hölder-McCarthy and Kantorovich operator inequalities were extensively used to get some new inequalities for the Berezin number of operators and their powers. Recall that for any positive operator A (i.e., $\langle Ax, x\rangle \geq 0$ for any $x \in \mathcal{H}$, shortly $A \geq 0$), there exists a unique positive operator R such that $R^2 = A$ (denoted by $R = A^{\frac{1}{2}}$). An operator $T \in \mathcal{B}(\mathcal{H})$ can be decomposed into $T = UP$, where U is a partial isometry and $P = |T| := (T^*T)^{\frac{1}{2}}$ (moduli of operator T) with $\ker(T) = \ker(P)$ and the last condition uniquely determines U and P of the polar decomposition $T = UP$ (see Furuta [8]). In general, we will refer to the book of Furuta [8] for main definitions and notations.

2. Hölder-McCarthy Type Inequalities and Berezin number

In this section, by using the Hölder-McCarthy inequality, we prove some inequalities for the Berezin number of some operators on the RKHS $\mathcal{H}$.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then :*
1) $\operatorname{ber}(A^\mu) \geq \operatorname{ber}(A)^\mu$ *for any $\mu > 1$.*
2) $\operatorname{ber}(A^\mu) \leq \operatorname{ber}(A)^\mu$ *for any $\mu \in [0, 1]$.*
3) If A is invertible, then $\operatorname{ber}(A^\mu) \geq \operatorname{ber}(A)^\mu$ *for any $\mu < 0$.*

Proof. First we prove 2). Indeed, assume that 2) holds for some $\alpha, \beta \in [0, 1]$. Then we only have to prove 2) holds for $\frac{\alpha+\beta}{2} \in [0, 1]$ by continuity of an operator. In fact, we have for any $\lambda \in \Omega$ that

$$\begin{aligned} &\left|\left\langle A^{\frac{\alpha+\beta}{2}}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\ &= \left|\left\langle A^{\frac{\alpha}{2}}\widehat{k}_\lambda, A^{\frac{\beta}{2}}\widehat{k}_\lambda\right\rangle\right|^2 \text{ (by Cauchy-Schwarz inequality)} \\ &\leq \left\langle A^{\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle A^{\beta}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \text{ (by assumption)} \\ &\leq \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\alpha+\beta}, \end{aligned}$$

so that $\widetilde{A^{\frac{\alpha+\beta}{2}}}(\lambda) \leq \widetilde{A}(\lambda)^{\frac{\alpha+\beta}{2}}$ holds for $\frac{\alpha+\beta}{2} \in [0, 1]$. This implies the desired inequality $\operatorname{ber}(A^\mu) \leq \operatorname{ber}(A)^\mu$ *for any $\mu \in [0, 1]$.*

1) Let $\mu > 1$. *Then* $\frac{1}{\mu} \in [0,1]$. *For any* $\lambda \in \Omega$

$$\begin{aligned}\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle &= \left\langle A^{\mu \frac{1}{\mu}} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &\leq \left\langle A^{\mu} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{1}{\mu}} \text{ by 2),}\end{aligned}$$

hence $\left\langle A^{\mu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \geqslant \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\mu}$ *for any* $\mu > 1$, *which shows that* $\operatorname{ber}(A^{\mu}) \geqslant \operatorname{ber}(A)^{\mu}$ *for any* $\mu > 1$, *as desired.*

3) Since A *is invertible, we have the following for any* $\lambda \in \Omega$ *that*

$$\begin{aligned}1 = \left\| \widehat{k}_\lambda \right\|^4 &= \left| \left\langle A^{\frac{1}{2}} \widehat{k}_\lambda, A^{-\frac{1}{2}} \widehat{k}_\lambda \right\rangle \right|^2 \\ &\leq \left\| A^{\frac{1}{2}} \widehat{k}_\lambda \right\|^2 \left\| A^{-\frac{1}{2}} \widehat{k}_\lambda \right\|^2 \\ &= \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle A^{-1}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &= \widetilde{A}(\lambda) \widetilde{A^{-1}}(\lambda),\end{aligned}$$

and hence

$$1 \leq \widetilde{A}(\lambda) \widetilde{A^{-1}}(\lambda) \text{ for any } \lambda \in \Omega, \tag{1}$$

which gives us

$$\operatorname{ber}(A) \operatorname{ber}\left(A^{-1}\right) \geqslant 1,$$

or equivalently

$$\operatorname{ber}\left(A^{-1}\right) \geqslant \operatorname{ber}(A)^{-1}.$$

Case: $\mu \in (-\infty, -1)$. *Then we have the following for any* $\lambda \in \Omega$ *that*

$$\begin{aligned}\left\langle A^{\mu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle &= \left\langle A^{-|\mu|}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &\geqslant \left\langle A^{-1}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\mu} \text{ (by 1) since } |\mu| > 1) \\ &\geqslant \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{-|\mu|} \text{ (by (1))} \\ &= \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\mu}\end{aligned}$$

which implies that $\operatorname{ber}(A^{\mu}) \geqslant \operatorname{ber}(A)^{\mu}$, *as desired.*

Case: $\mu \in [-1, 0)$. *For every* $\lambda \in \Omega$ *we have*

$$\begin{aligned}\widetilde{A^{\mu}}(\lambda) = \left\langle A^{\mu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle &= \left\langle A^{-|\mu|}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &\geqslant \left\langle A^{|\mu|}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{-1} \text{ (by (1))} \\ &\geqslant \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{-|\mu|} = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\mu} = \left(\widetilde{A}(\lambda)\right)^{\mu},\end{aligned}$$

and the last inequality follows by 2) since $|\mu| \in [0,1]$ *and taking inverses of both sides. The theorem is proved.* □

Next result proves the equivalence of Hölder-McCarthy type inequality and Young type inequality.

Theorem 2.2. *For a positive operator $A \in \mathcal{B}(\mathcal{H})$ and $\mu \in [0,1]$ the following inequalities are equivalent: Hölder-McCarthy type inequality:*

$$\widetilde{A}(\lambda)^{\mu} \geqslant \widetilde{A^{\mu}}(\lambda) \text{ for all } \lambda \in \Omega. \tag{2}$$

Young type inequality:

$$[\mu A + I - \mu]^{\sim} \geqslant \widetilde{A^{\mu}}. \tag{3}$$

Proof. Let us define a scalar function

$$f(t) := \mu t + 1 - \mu - t^{\mu}$$

for positive numbers t and $\mu \in [0,1]$. Then it is easy to see that $f(t)$ is a nonnegative convex function with the minimum value $f(1) = 0$, so we have

$$\mu a + 1 - \mu \geqslant a^{\mu} \tag{4}$$

for positive a and $\mu \in [0,1]$.

(2) $\Rightarrow$ (3). Replacing a by $\widetilde{A}(\lambda) \geqslant 0$ and $\mu \in [0,1]$ in (4), we obtain

$$\mu\widetilde{A}(\lambda) + 1 - \mu \geqslant A(\lambda)^{\mu} \geqslant \widetilde{A^{\mu}}(\lambda) \text{ by (2),}$$

so we have (3).

(3) $\Rightarrow$ (2). We may assume $\mu \in (0,1]$. In (3), replace A by $k^{\frac{1}{\mu}}A$ for a positive number k, then

$$\mu k^{\frac{1}{\mu}}\widetilde{A}(\lambda) + 1 - \mu \geqslant k\widetilde{A^{\mu}}(\lambda) \tag{5}$$

for $\lambda \in \Omega$ by (3). We put $k = \widetilde{A}(\lambda)^{-\mu}$ in (5) if $\widetilde{A}(\lambda) \neq 0$, then we have

$$\mu\widetilde{A}(\lambda)^{-1}\widetilde{A}(\lambda) + 1 - \mu \geqslant \widetilde{A}(\lambda)^{-\mu}\widetilde{A^{\mu}}(\lambda),$$

that is $A(\lambda)^{\mu} \geqslant \widetilde{A^{\mu}}(\lambda)$ for all $\lambda \in \Omega$ and we get (2). If $\widetilde{A}(\lambda) = 0$, then it means that $A^{\frac{1}{2}}\widehat{k}_{\lambda} = 0$, so $A^{\mu}\widehat{k}_{\lambda} = 0$ for $\mu \in (0,1]$ by the induction and continuity of A, and thus we have (2). The theorem is proved. □

Proposition 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive invertible operator and $B \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then for any real number μ, we have*

$$\mathrm{ber}\left((BAB^{*})^{\mu}\right) = \mathrm{ber}\left(BA^{\frac{1}{2}}\left(A^{\frac{1}{2}}B^{*}BA^{\frac{1}{2}}\right)^{\mu-1}A^{\frac{1}{2}}B^{*}\right). \tag{6}$$

Proof. Let $BA^{\frac{1}{2}} = UP$ be the polar decomposition of $BA^{\frac{1}{2}}$, where U is unitary and $P = \left|BA^{\frac{1}{2}}\right|$. Then it is easy to see that:

$$\begin{aligned}(BAB^{*})^{\mu} &= \left(UP^{2}U^{*}\right)^{\mu} = BA^{\frac{1}{2}}P^{-1}P^{2\mu}P^{-1}A^{\frac{1}{2}}B^{*} \\ &= BA^{\frac{1}{2}}\left(A^{\frac{1}{2}}B^{*}BA^{\frac{1}{2}}\right)^{\mu-1}A^{\frac{1}{2}}B^{*}.\end{aligned}$$

Now (6) is immediate from this equality. □

3. Paranormal operators and related problems

Recall that an operator A on a Hilbert space H is called paranormal if $\left\|A^2x\right\| \geq \|Ax\|^2$ for every unit vector $x \in H$.

Definition 3.1. *We will say that A is a quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$, if $\left\|A^2\widehat{k}_\lambda\right\| \geq \left\|A\widehat{k}_\lambda\right\|^2$ for any $\lambda \in \Omega$.*

Definition 3.2. *An operator T belongs to class $\widetilde{\mathcal{A}}$ if $\widetilde{\left|T^2\right|} \geq \widetilde{|T|^2}$.*

Definition 3.3. *For each $k > 0$, an operator T is absolute-k-quasi-paranormal if*

$$\left\||T|^k T\widehat{k}_\lambda\right\| \geq \left\|T\widehat{k}_\lambda\right\|^{k+1} \tag{7}$$

for every $\lambda \in \Omega$.

It follows from these definitions that:
(a) If A is quasi-paranormal, then

$$\operatorname{ber}\left(\left|A^2\right|^2\right) \geq \operatorname{ber}\left(|A|^2\right)^2;$$

(b) If A belongs to class $\widetilde{\mathcal{A}}$, then

$$\operatorname{ber}\left(\left|A^2\right|\right) \geq \operatorname{ber}\left(|A|^2\right);$$

(c) If A is absolute-k-quasi-paranormal, then

$$\operatorname{ber}\left(\left||A|^k A\right|^2\right) \geq \operatorname{ber}\left(|A|\right)^{k+1}.$$

In this section, to prove some inequalities for the Berezin number of such operators, we need to other properties of these operators.

Proposition 3.4. *Every operator in $\widetilde{\mathcal{A}}$ is a quasi-paranormal operator on a RKHS.*

Proof. Suppose $A \in \widetilde{\mathcal{A}}$, i.e.,

$$\widetilde{\left|A^2\right|} \geq \widetilde{|A|^2}. \tag{8}$$

Then for every $\lambda \in \Omega$, we have $\widetilde{\left|A^2\right|}(\lambda) \geq \widetilde{|A|^2}(\lambda)$, and therefore it follows from the proof of Theorem 2.1 that

$$\begin{aligned}
\left\|A^2\widehat{k}_\lambda\right\|^2 &= \left\langle A^2\widehat{k}_\lambda, A^2\widehat{k}_\lambda\right\rangle = \left\langle \left(A^2\right)^* A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= \left\langle \left|A^2\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&\geq \left\langle \left|A^2\right| \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^2 \text{ (see the proof of Theorem 2.1, 1))} \\
&\geq \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^2 \text{ (by (8))} \\
&= \left\|A\widehat{k}_\lambda\right\|^4.
\end{aligned}$$

Hence

$$\left\|A^2\widehat{k}_\lambda\right\| \geq \left\|A\widehat{k}_\lambda\right\|^2$$

for every $\lambda \in \Omega$, so that A is quasi-paranormal, which proves the proposition. □

Definition 3.5. *For each $k > 0$, we say that an operator A belongs to class $\widetilde{\mathcal{A}}(k)$ if*

$$\left(\left(A^* |A|^{2k} A\right)^{\frac{1}{k+1}}\right)^{\sim} \geq \widetilde{|A|^2}.$$

The proof of Theorem 2.1 allows us also prove the following.

Proposition 3.6. *(a) Every quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is an absolute-k-quasi-paranormal operator for $k \geq 1$.*

(b) For each $k > 0$, every class $\widetilde{\mathcal{A}}(k)$ operator is an absolute-k-quasi-paranormal operator.

Proof. (a) Suppose that A is a quasi-paranormal operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. Then, for any $\lambda \in \Omega$ and $k \geq 1$, we have

$$\begin{aligned}
\left\| |A|^k A\widehat{k}_\lambda \right\|^2 &= \left\langle |A|^{2k} A\widehat{k}_\lambda, A\widehat{k}_\lambda \right\rangle \\
&\geq \left\langle |A|^2 A\widehat{k}_\lambda, A\widehat{k}_\lambda \right\rangle^k \left\| A\widehat{k}_\lambda \right\|^{2(1-k)} \text{ (see the proof of Theorem 2.1, 1))} \\
&= \left\| A^2 \widehat{k}_\lambda \right\|^{2k} \left\| A\widehat{k}_\lambda \right\|^{2(1-k)} \\
&\geq \left\| A\widehat{k}_\lambda \right\|^{4k} \left\| A\widehat{k}_\lambda \right\|^{2(1-k)} \text{ (by quasi-paranormality of } A) \\
&\geq \left\| A\widehat{k}_\lambda \right\|^{2(k+1)},
\end{aligned}$$

and hence

$$\left\| |A|^k A\widehat{k}_\lambda \right\| \geq \left\| A\widehat{k}_\lambda \right\|^{k+1}$$

for all $\lambda \in \Omega$ and $k \geq 1$, so that A is absolute-k-quasi-paranormal operator for $k \geq 1$.

(b) Let $A \in \widetilde{\mathcal{A}}(k)$ for $k > 0$, that is

$$\left(\left(A^* |A|^{2k} A\right)^{\frac{1}{k+1}}\right)^{\sim} \geq \widetilde{|A|^2} \text{ for } k > 0. \tag{9}$$

Then for any $\lambda \in \Omega$,

$$\begin{aligned}
\left\| |A|^k A\widehat{k}_\lambda \right\|^2 &= \left\langle A^* |A|^{2k} A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\geq \left\langle \left(A^* |A|^{2k} A\right)^{\frac{1}{k+1}} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{k+1} \\
&\geq \left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{k+1} \text{ (by (9))} \\
&= \left\| A\widehat{k}_\lambda \right\|^{2(k+1)},
\end{aligned}$$

from which

$$\left\| |A|^k A\widehat{k}_\lambda \right\| \geq \left\| A\widehat{k}_\lambda \right\|^{k+1} \text{ for all } \lambda \in \Omega,$$

so that A is absolute-k-quasi-paranormal operator for $k > 0$. This completes the proof. $\square$

As further extension of previous results, we prove the following result.

Theorem 3.7. *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ be an absolute-k-quasi-paranormal operator for $k > 0$. Then for every $\lambda \in \Omega$,*

$$F(\ell) = \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{1}{\ell+1}}$$

is increasing for $\ell > k > 0$, and the following inequality holds:

$$F(\ell) \geq \left\| A\widehat{k_\lambda} \right\|,$$

i.e., A is absolute-ℓ-quasi-paranormal operator for $\ell \geq k > 0$.

Proof. Assume that A is an absolute-k-quasi-paranormal operator on $\mathcal{H} = \mathcal{H}(\Omega)$ for $k > 0$, i.e.,

$$\left\| |A|^{k} A\widehat{k_\lambda} \right\| \geq \left\| A\widehat{k_\lambda} \right\|^{k+1} \tag{10}$$

for every $\lambda \in \Omega$. Clearly, (10) holds if and only if

$$F(k) = \left\| |A|^{k} A\widehat{k_\lambda} \right\|^{\frac{1}{k+1}} \geq \left\| A\widehat{k_\lambda} \right\|$$

for any $\lambda \in \Omega$. Then for every $\lambda \in \Omega$ and any ℓ such that $\ell \geq k > 0$, we have

$$\begin{aligned}
F(\ell) &= \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{1}{\ell+1}} = \left\langle |A|^{2\ell} A\widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle^{\frac{1}{2(\ell+1)}} \\
&\geq \left\{ \left\langle |A|^{2k} A\widehat{k_\lambda}, A\widehat{k_\lambda} \right\rangle^{\frac{1}{k}} \left\| A\widehat{k_\lambda} \right\|^{2\left(1-\frac{1}{k}\right)} \right\}^{\frac{1}{2(\ell+1)}} \\
&\geq \left\{ \left\| A\widehat{k_\lambda} \right\|^{\frac{2\ell(k+1)}{k}} \left\| A\widehat{k_\lambda} \right\|^{2\left(1-\frac{1}{k}\right)} \right\}^{\frac{1}{2(\ell+1)}} \quad (\text{by } (10)) \\
&= \left\| A\widehat{k_\lambda} \right\|,
\end{aligned}$$

and hence

$$F(\ell) = \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{1}{\ell+1}} \geq \left\| A\widehat{k_\lambda} \right\| \tag{11}$$

for every $\lambda \in \Omega$ and $\ell \geq k$, so that A is absolute-ℓ-quasi-paranormal for $\ell \geq k > 0$.

Now we prove that, $F(\ell)$ is increasing for $\ell \geq k > 0$. Indeed, for any $\lambda \in \Omega$, m and ℓ such that $m \geq \ell \geq k > 0$, we have:

$$\begin{aligned}
F(m) &= \left\| |A|^{m} A\widehat{k_\lambda} \right\|^{\frac{1}{m+1}} = \left\langle |A|^{2m} A\widehat{k_\lambda}, A\widehat{k_\lambda} \right\rangle^{\frac{1}{2(m+1)}} \\
&= \left\{ \left\langle |A|^{2\ell} A\widehat{k_\lambda}, A\widehat{k_\lambda} \right\rangle^{\frac{m}{\ell}} \left\| A\widehat{k_\lambda} \right\|^{2\left(1-\frac{m}{\ell}\right)} \right\}^{\frac{1}{2(m+1)}} \\
&= \left\{ \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{2m}{\ell}} \left\| A\widehat{k_\lambda} \right\|^{2\left(1-\frac{m}{\ell}\right)} \right\}^{\frac{1}{2(m+1)}} \\
&\geq \left\{ \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{2m}{\ell}} \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{2}{\ell+1}\left(1-\frac{m}{\ell}\right)} \right\}^{\frac{1}{2(m+1)}} \quad (\text{by } (11)) \\
&= \left\| |A|^{\ell} A\widehat{k_\lambda} \right\|^{\frac{1}{\ell+1}} = F(\ell),
\end{aligned}$$

hence $F(m) \geq F(\ell)$, that is $F(\ell)$ is increasing for $\ell \geq k > 0$. This proves the theorem. □

Corollary 3.8. *$F(\ell) \geq \sqrt{\mathrm{ber}\left(|A|^2\right)}$ for $\ell \geq k > 0$.*

The following lemma is well known (see, for instance, [8]).

Lemma 3.9. *Let a and b be positive real numbers. Then,*

$$a^{\lambda} b^{\mu} \leq \lambda a + \mu b$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

Our next result characterizes absolute-k-quasi-paranormal operators A on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 3.10. *For each $k > 0$, an operator A on $\mathcal{H}$ is absolute-k-quasi-paranormal if and only if*

$$\left(A^* |A|^{2k} A - (k+1)\alpha^k |A|^2 + k\alpha^{k+1}\right)^{\sim} \geq 0$$

holds for all $\alpha > 0$.

Proof. $\Rightarrow$. Suppose that A is absolute-k-quasi-paranormal for $k > 0$, i.e.,

$$\left\| |A|^k A\widehat{k}_\lambda \right\| \geq \left\| A\widehat{k}_\lambda \right\|^{k+1} \tag{12}$$

for every $\lambda \in \Omega$. Inequality (12) holds if and only if

$$\left\| |A|^k Ak_\lambda \right\|^{\frac{1}{k+1}} \|k_\lambda\|^{\frac{k}{k+1}} \geq \|Ak_\lambda\|$$

for all $\lambda \in \Omega$, or equivalently

$$\left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle^{\frac{1}{k+1}} \langle k_\lambda, k_\lambda \rangle^{\frac{k}{k+1}} \geq \left\langle |A|^2 k_\lambda, k_\lambda \right\rangle$$

for all $\lambda \in \Omega$. By Lemma 3.9, we have:

$$\begin{aligned} &\left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle^{\frac{1}{k+1}} \langle k_\lambda, k_\lambda \rangle^{\frac{k}{k+1}} \\ &= \left\{ \left(\frac{1}{\alpha}\right)^k \left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle \right\}^{\frac{1}{k+1}} \{\alpha \langle k_\lambda, k_\lambda \rangle\}^{\frac{k}{k+1}} \\ &\leq \frac{1}{k+1}\frac{1}{\alpha^k} \left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle + \frac{k}{k+1}\alpha \langle k_\lambda, k_\lambda \rangle \end{aligned} \tag{13}$$

for all $\lambda \in \Omega$ and $\alpha > 0$, so that (12) ensures the following inequality by (13) :

$$\frac{1}{k+1}\frac{1}{\alpha^k} \left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle + \frac{k}{k+1}\alpha \langle k_\lambda, k_\lambda \rangle \geq \left\langle |A|^2 k_\lambda, k_\lambda \right\rangle \tag{14}$$

for all $\lambda \in \Omega$ and $\alpha > 0$.

Conversely, (14) implies (12) by putting $\alpha = \left\{ \frac{\langle A^*|A|^{2k} Ak_\lambda, k_\lambda \rangle}{\langle k_\lambda, k_\lambda \rangle} \right\}^{\frac{1}{k+1}}$; in case $\left\langle A^* |A|^{2k} Ak_\lambda, k_\lambda \right\rangle = 0$, let $\alpha \to 0$. Hence (14) holds if and only if

$$\left(A^* |A|^{2k} A - (k+1)\alpha^k |A|^2 + k\alpha^{k+1}\right)^{\sim} \geq 0$$

holds for all $\alpha > 0$, which completes the proof of the theorem. □

Since absolute-1-quasi-paranormal is quasi-paranormal, the following is immediate from Theorem 3.10.

Corollary 3.11. *An operator A is quasi-paranormal if and only if*

$$\left(A^{*2}A^2 - 2\alpha A^*A + \alpha^2\right)^{\sim} \geq 0$$

holds for all $\alpha > 0$.

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