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Common Fixed Point Theorem for Modified Kannan Enriched Contraction Pair in Banach Spaces and Its Applications

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Abstract. The purpose of this paper is to introduce the class of (a, b, c)-modified enriched Kannan pair of mappings (T, S) in the setting of Banach space that includes enriched Kannan mappings, contraction and nonexpansive mappings and some other mappings. Some examples are presented to support the concepts introduced herein. We establish the existence of common fixed point of the such pair. We also show that the common fixed point problem studied herein is well posed. A convergence theorem for the Krasnoselskij iteration is used to approximate fixed points of the (a, b, c)-modified enriched Kannan pair. As an application of the results proved in this paper, the existence of a solution of integral equations is established. The presented results improve, unify and generalize many known results in the literature.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$. An element $x \in X$ is called a fixed point of T if $x = Tx$. Kannan [9] proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space. If $T : X \to X$ is a mapping such that*

$$d(Tx, Ty) \le \alpha\{d(x, Tx) + d(y, Ty)\}, \tag{1}$$

holds for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Reich [10] proved the following result:

Theorem 1.2. *Let (X, d) be a complete metric space. If $T : X \to X$ satisfies:*

$$d(Tx, Ty) \le ad(x, Tx) + bd(y, Ty) + cd(x, y), \tag{2}$$

for all $x, y \in X$, where a, b, c are nonnegative and $a + b + c < 1$. Then T has a unique fixed point.

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If in the above Theorem, we take $a = b$ and $c = 0$, then we obtain Theorem 1.1.

Browder and Petryshyn [6] introduced the notion of asymptotic regularity of a self-mapping on a metric space.

Definition 1.3. *[6] A self mapping T on a metric space (X, d) is said to be asymptotically regular if*

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0, \ \ \forall\, x \in X.$$

Górnicki [7] has obtained the following generalization of the Theorem 1.2.

Theorem 1.4. *Let (X, d) be a complete metric space and T is a continuous asymptotically regular self mapping on X such that the following condition holds:*

$$d(Tx, Ty) \leq cd(x, y) + a\{d(x, Tx) + d(y, Ty)\}, \ \ \forall\, x, y \in X, \tag{3}$$

where $c \in [0, 1)$ and $a \in [0, \infty)$. Then T has a unique fixed point $p \in X$ and $T^n x \to p$ for each $x \in X$.

We now recall some weaker forms of continuity of a self mapping on a metric space:

Definition 1.5. *[8] If T is a self mapping on a metric space (X, d), then the set*

$$O(T, x) = \{T^n x : \ n = 0, 1, 2, 3, ...\}$$

is called the orbit of T at $x \in X$.
A mapping T is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(T, x)$ for some $x \in X$, $\lim\limits_{n\to\infty} x_n = z$ implies that $\lim\limits_{n\to\infty} Tx_n = Tz$.

Definition 1.6. *[8] A self mapping T on a metric space (X, d) is called k-continuous if $\lim\limits_{n\to\infty} T^{k-1} x_n = z$ implies that $\lim_{n\to\infty} T^k x_n = Tz$, where $k \in \{1, 2, ...\}$.*

Bisht [4] improved Theorem 1.4 as follows:

Theorem 1.7. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping satisfying (3). Then T has a unique fixed point $p \in X$ provided T is either k-continuous for some $k \geq 1$ or orbitally continuous. Moreover, $T^n x \to p$ for each $x \in X$.*

Recently, Berinde and Păcurar [2] introduced the concept of a Kannan enriched contraction as follows.

Definition 1.8. *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Kannan mapping if there exist $a \in [0, \frac{1}{2})$ and $b \in [0, \infty)$ such that*

$$\left\|b(x - y) + Tx - Ty\right\| \leq a\{\|x - Tx\| + \left\|y - Ty\right\|\}, \tag{4}$$

for all $x, y \in X$,

An enriched Kannan contraction mapping in a Banach space has a unique fixed point which can be approximated by means of the Krasnoselskij iterative scheme [3].
In this paper, we introduce the pair (T, S) of self mappings on a metric space called (a, b, c)-modified enriched Kannan pair. The notion of modified enriched Kannan mappings is also introduced. The class of such mappings incudes the classes of Banach contraction mappings, nonexpansive mappings, Lipschitzian mappings, Kannan contraction, enriched Kannan contraction and mappings satisfying (3). We prove common fixed point result and establish well posedness of a common fixed point problem. Consequently, a fixed point result is obtained which is also new in the known literature on metric fixed point theory. These results extend and strengthen various known results in the literature. As an application of our result, integral equations are solved.
We introduce the following.

Definition 1.9. *Let $(X, \|\cdot\|)$ be a linear normed space and $T, S : X \to X$. If there exist $a, b \in [0, \infty)$ and $c \in [0, 1)$ such that*

$$\left\|b(x-y)+Tx-Sy\right\| \le c\left\|x-y\right\| + a\Big\{\|x-Tx\| + \left\|y-Sy\right\|\Big\}, \tag{5}$$

holds for all $x, y \in X$. Then (T, S) is called (a, b, c)-modified enriched Kannan pair.

Example 1.10. *Let $X = [-1, 1]$ be endowed with the usual norm. Suppose that $T, S : X \to X$ are mappings defined by:*

$$Tx = \begin{cases} \frac{2x}{3} & \text{if } x \ge 0 \\ 1 & \text{if } x < 0 \end{cases} \quad Sx = \begin{cases} 0 & \text{if } x \le 0 \\ -1 & \text{if } x > 0. \end{cases}$$

Then for all $b \in [0, \infty)$ such that $c = b + \frac{1}{6} < 1$, (T, S) is $(1, b, b + \frac{1}{6})$-modified enriched Kannan pair. Indeed, for all $x, y \in X$ it is easy to verify that

$$|b(x-y)+Tx-Sy| \le b|x-y| + |Tx-Sy| \le \Big(b+\frac{1}{6}\Big)|x-y| + |x-Tx| + |y-Sy|$$

If we take $S = T$ in the above definition, we obtain the following concept which is also new and first of its kind in the known literature on fixed point theory.

Definition 1.11. *Let $(X, \|\cdot\|)$ be a linear normed space and $T : X \to X$. If there exist $a, b \in [0, \infty)$ and $c \in [0, 1)$ such that*

$$\|b(x-y)+Tx-Ty\| \le c\|x-y\| + a\Big\{\|x-Tx\| + \|y-Ty\|\Big\}, \tag{6}$$

holds for all $x, y \in X$. Then the mapping T is said to be (a, b, c)- modified enriched Kannan mapping.

Remark 1.12. *A Banach contraction mapping is a $(0, 0, c)$-modified enriched Kannan mapping.*

Remark 1.13. *If we take $c = 0$ in (6), then we obtain the definition of enriched Kannan mapping.*

Remark 1.14. *A class of $(a, 0, c)$-modified enriched Kannan mappings coincides with the class of mappings satisfying (3).*

Example 1.15. *Let $X = \mathbb{R}$ be endowed with the usual norm. Define $T : X \to X$ by*

$$Tx = 100 - 2x.$$

Note that T is $(a, 2, c)$-modified enriched Kannan mapping for all $c \in [0, 1)$ and $a \in [0, \infty)$.
Indeed, for all $x, y \in X$, we have

$$\|2(x-y)+Tx-Ty\| = 0 \le c\|x-y\| + a\{\|x-Tx\| + \|y-Ty\|\}.$$

Example 1.16. *Let $X = \mathbb{R}$ be endowed with the usual norm and $T : X \to X$ be the mapping defined by $Tx = -x$ for all $x \in X$. Then T is $(a, 1, c)$-modified enriched Kannan mapping. However, T is not enriched Kannan mapping. Indeed, if we take $b = 2$ and $a = \frac{1}{4}$, then the inequality (4) is not satisfied for $x = 0$ and $y = 1$.*

Let us recall the following property.

Remark 1.17. *Let M be a convex subset of a linear space X and $T : M \to M$. Then for any $\lambda \in (0, 1)$, the set of all fixed points of a mapping $T_\lambda : M \to M$ given by $T_\lambda(x) = (1-\lambda)x + \lambda Tx$ coincides with $Fix(T)$.*

We need the following technical result in the sequel.

Lemma 1.18. *[12] Let $\{a_n\}$ be sequence of non-negative real numbers satisfying*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\beta_n, \ n \ge 0$$

where $\alpha_n \in (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\limsup_{n\to\infty} \beta_n \le 0$ or $\sum_{n=0}^{\infty} |\alpha_n\beta_n| = \infty$.
Then $\{a_n\}$ converges to zero as $n \to \infty$.

2. Main Results

We start with the following common fixed point result.

Theorem 2.1. *Suppose that $(X, \|.\|)$ is a Banach space and (T, S) is (a, b, c)-modified enriched Kannan pair. Then T and S have a unique common fixed point p. Moreover, $\lim_{n\to\infty} T_\lambda^n x = p = \lim_{n\to\infty} S_\lambda^n x$ for any $x \in X$, provided that T_λ and S_λ are asymptotically regular and T_λ and S_λ are either k-continuous for some $k \geq 1$ or orbitally continuous, where $\lambda = \frac{1}{b+1}$.*

Proof. We divide the proof into the following two cases.
Case 1. Suppose that $b > 0$. Clearly, $0 < \lambda < 1$. In this case, (5) becomes

$$\|T_\lambda x - S_\lambda y\| \leq d\|x - y\| + a\{\|x - T_\lambda x\| + \|y - S_\lambda y\|\}, \tag{7}$$

where $d = \lambda c$. Note that $d \in [0, 1)$. We now show that $\lim_{n\to\infty} \|T_\lambda^n x - S_\lambda^n x\| = 0$ holds for any $x \in X$. For $S = T$, the result is trivial. Suppose $S \neq T$ and $c = 0$. Then (7) becomes

$$\|T_\lambda x - S_\lambda y\| \leq a\{\|x - T_\lambda x\| + \|y - S_\lambda y\|\}.$$

For any $x \in X$, let $x_1 = T_\lambda^n x$ and $y_1 = S_\lambda^n x$. Then

$$\|T_\lambda^{n+1} x - S_\lambda^{n+1} y\| \leq a\{\|T_\lambda^n x - T_\lambda^{n+1} x\| + \|S_\lambda^n x - S_\lambda^{n+1} x\|\}.$$

Using the fact that T_λ and S_λ are asymptotically regular, we obtain that $\lim_{n\to\infty} \|T_\lambda^{n+1} x - S_\lambda^{n+1} y\| = 0$. Moreover,

$$\|T_\lambda^n x - S_\lambda^n y\| \leq \|T_\lambda^n x - T_\lambda^{n+1} y\| + \|T_\lambda^{n+1} x - S_\lambda^{n+1} y\| + \|S_\lambda^{n+1} x - S_\lambda^n y\|. \tag{8}$$

Using asymptotically regularity of T_λ and S_λ and (8), we have

$$\lim_{n\to\infty} \|T_\lambda^n x - S_\lambda^n x\| = 0.$$

Next, suppose that $S \neq T$ and $c \neq 0$. Define $x_1 = T_\lambda^n x$ and $y_1 = S_\lambda^n x$ for any $x \in X$. Then (7) becomes

$$\|T_\lambda^{n+1} x - S_\lambda^{n+1} y\| \leq d\|T_\lambda^n x - S_\lambda^n x\| + a\{\|T_\lambda^n x - T_\lambda^{n+1} x\| + \|S_\lambda^n x - S_\lambda^{n+1} x\|\}.$$

Let

$$\begin{aligned} a_n &= \|T_\lambda^n x - S_\lambda^n x\|, \text{ and } \alpha_n = 1 - d \\ \beta_n &= \frac{a}{1-d}\|T_\lambda^n x - T_\lambda^{n+1} x\| + \frac{a}{1-d}\|S_\lambda^n x - S_\lambda^{n+1} x\|. \end{aligned}$$

By asymptotically regularity of T_λ and S_λ, we have $\lim_{n\to\infty} \beta_n = 0$. Moreover, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Hence by Lemma 1.18, we have that

$$\lim_{n\to\infty} \|T_\lambda^n x - S_\lambda^n x\| = 0, \tag{9}$$

for any $x \in X$. Let $x_n = T_\lambda^n x$ for any $x \in X$. We now show that $\{x_n\}$ is Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) \geq k$ such that for $k = 1, 2, ...$, we obtain that

$$\|T_\lambda^{m(k)} x - T_\lambda^{n(k)} x\| \geq \epsilon. \tag{10}$$

Choosing $m(k)$, the smallest number exceeding $n(k)$ for which (10) holds, we have

$$\|T_\lambda^{m(k)-1} x - T_\lambda^{n(k)} x\| < \epsilon.$$

Note that

$$\epsilon \leq \|T_\lambda^{m(k)}x - T_\lambda^{n(k)}x\| \leq \|T_\lambda^{m(k)}x - T_\lambda^{m(k)-1}x\| + \|T_\lambda^{m(k)-1}x - T_\lambda^{n(k)}x\| < \|T_\lambda^{m(k)}x - T_\lambda^{m(k)-1}x\| + \epsilon$$

On taking limit as $k \to \infty$ on the both sides of the above inequality, we have

$$\lim_{n\to\infty} \|T_\lambda^{m(k)}x - T_\lambda^{n(k)}x\| = \epsilon. \tag{11}$$

Moreover, it follows from asymptotic regularity of T_λ and the inequality

$$\|T_\lambda^{m(k)-1}x - T_\lambda^{n(k)-1}x\| \leq \|T_\lambda^{m(k)-1}x - T_\lambda^{m(k)}x\| + \|T_\lambda^{m(k)}x - T_\lambda^{n(k)}x\| + \|T_\lambda^{n(k)}x - T_\lambda^{n(k)-1}x\|,$$

that

$$\lim_{n\to\infty} \|T_\lambda^{m(k)-1}x - T_\lambda^{n(k)-1}x\| = \epsilon. \tag{12}$$

Now, (7) implies that

$$\begin{aligned}\|T_\lambda^{m(k)}x - T_\lambda^{n(k)}x\| &\leq \|T_\lambda^{m(k)}x - S_\lambda^{n(k)}x\| + \|S_\lambda^{n(k)}x - T_\lambda^{n(k)}x\| \\ &\leq \|S_\lambda^{n(k)}x - T_\lambda^{n(k)}x\| + d\|T_\lambda^{m(k)-1}x - T_\lambda^{n(k)-1}x\| \\ &+ a\{\|T_\lambda^{m(k)-1}x - T_\lambda^{m(k)}x\| + \|S_\lambda^{n(k)-1}x - S_\lambda^{n(k)}x\|. \\ &+ d\|S_\lambda^{n(k)-1}x - T_\lambda^{n(k)-1}x\|.\end{aligned}$$

Taking limit as $k \to \infty$ on the both sides of the above inequality, and using (11), (12) and asymptotic regularity of T_λ and S_λ, we obtain that $\epsilon \leq d\epsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Assume that there exists a point $p \in X$ such that $\{x_n\}$ converges to p. Note that

$$\|S_\lambda^n x - p\| \leq \|S_\lambda^n x - T_\lambda^n x\| + \|T_\lambda^n x - p\|.$$

From (9), we have $\lim_{n\to\infty} \|S_\lambda^n x - p\| = 0$. Suppose that T_λ is k-continuous. Since $\lim_{n\to\infty} T_\lambda^{k-1}x_n = p$, k-continuity of T_λ implies that $\lim_{n\to\infty} T_\lambda^k x_n = T_\lambda p$. By the uniqueness of limit, we have $T_\lambda p = p$. On the other hand, if T_λ is orbitally continuous. Then, $\lim_{n\to\infty} x_n = p$, and orbital continuity of T_λ implies that $\lim_{n\to\infty} T_\lambda x_n = T_\lambda p$. Thus, $T_\lambda p = p$, that is, $Tp = p$. Similarly, we have $S_\lambda p = Sp = p$. Let $q \in X$ be such that $T_\lambda q = q = S_\lambda q$ and $q \neq p$. Taking $x = p$ and $y = q$ in (7), we have $\|p - q\| \leq d\|p - q\| < \|p - q\|$, a contradiction and hence p is the unique common fixed point of T_λ and S_λ.
Case 2. $b = 0$. In this case $\lambda = 1$. Following arguments similar to those given in the Case 1 with $T = T_1$ and $S = S_1$, the result follows. Indeed, the Krasnoselskij iteration reduces, in fact, to the simple Picard iteration associated with T and S. □

If we take $b = 0$ in the Theorem 2.1, we obtain Theorem 2.2 of [8] in the setting of Banach space

Corollary 2.2. *[8] Suppose that $(X, \|.\|)$ is a Banach space and (T, S) is $(a, 0, c)$-modified enriched Kannan pair. Then T and S have a unique fixed point p. Moreover, $\lim_{n\to\infty} T^n x = p = \lim_{n\to\infty} S^n x$ for any $x \in X$, provided that T and S are asymptotically regular and T and S are either k-continuous for some $k \geq 1$ or orbitally continuous.*

Example 2.3. *Let $X = [0, 1]$ be endowed with the usual norm and mappings $T, S : X \to X$ be defined by $Sx = \frac{1}{2}$, and*

$$Tx = \begin{cases} 1 - x & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

If $a = \frac{1}{6}$ and $c = \frac{1}{2}$. Then (T, S) is not $(a, 0, c)$-modified enriched Kannan pair. Indeed, $x = 1$ and $y = \frac{1}{2}$ give that

$$\|Tx - Sy\| = \frac{1}{2} > 0.42 = \frac{1}{2}\|x - y\| + \frac{1}{6}\{\|x - Tx\| + \|y - Sy\|\}$$

On the other hand, for $b = 1$, $c = \frac{1}{2}$ and $a = 1$, the pair (T, S) is $(1, 1, \frac{1}{2})$-modified enriched Kannan mapping. Moreover, for $\lambda = \frac{1}{2}$, we have

$$T_{\frac{1}{2}}x = \begin{cases} \frac{1}{2} & \text{if } x > 0 \\ \frac{1}{4} & \text{if } x = 0 \end{cases}$$

and $S_{\frac{1}{2}}x = \frac{1}{4} + \frac{x}{2}$, for all $x \in [0, 1]$. Clearly, $S_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are asymptotically regular, and orbitally continuous. All the assumptions of Theorem 2.1 are satisfied. Moreover, $x = \frac{1}{2}$ is the unique common fixed point of T and S. Also, $\lim_{n\to\infty} T^n_{\frac{1}{2}}x = \frac{1}{2} = \lim_{n\to\infty} S^n_{\frac{1}{2}}x$ for any $x \in X$.

As a special case of Theorem 2.1, we have the following generalization of Corollary 2.3. of [8].

Corollary 2.4. *Suppose that $(X, \|.\|)$ is a Banach space. If for some positive integers m and q such that (T^m, S^q) is (a, b, c)-modified enriched Kannan pair. Then T and S have a unique common fixed point p, provided that T^m_λ and S^q_λ are asymptotically regular and T^m_λ and S^q_λ are either k-continuous for some $k \geq 1$ or orbitally continuous, where $\lambda = \frac{1}{b+1}$.*

Proof. Take $\mu = T^m_\lambda$ and $\nu = S^q_\lambda$. Then (μ, ν) is (a, b, c)-modified enriched Kannan pair. By Theorem 2.1, μ and ν have a unique common fixed point p. This gives $\mu(Tp) = Tp$ and $\nu(Tp) = Tp$. By the uniqueness of the common fixed point of μ and ν, it follows that $Tp = Sp = p$. □

If we take $S = T$ in the Theorem 2.1, then we have the following result.

Corollary 2.5. *Let $(X, \|.\|)$ be Banach space and $T : X \to X$. If there exists $a, b \in [0, \infty)$ and $c \in [0, 1)$ such that T is (a, b, c)-modified enriched Kannan mapping. Then T has a unique fixed point $p \in X$ provided that T_λ is asymptotically regular and T_λ is either k-continuous for $k \geq 1$ or orbitally continuous, where $\lambda = \frac{1}{b+1}$.*

Example 2.6. *Let $X = \mathbb{R}$ be endowed with the usual norm and $T : X \to X$ be defined by $Tx = 1 - 10x$. Note that, for $c = 0.1$ and $a = \frac{1}{11}$, a mapping T does not satisfy the inequality (3). Indeed, for $x = 0$ and $y = 1$, we have*

$$\|T0 - T1\| = 10 > 1.5 = 0.1\|0 - 1\| + \frac{1}{4}(1 + 10).$$

On the other hand, for $b = 10$, $c = 0.1$ and $a = \frac{1}{11}$, T satisfies all the conditions of Theorem 2.5. Indeed, for all $x, y \in X$, we have

$$\|10(x - y) + Tx - Ty\| = 0 \leq c\|x - y\| + a\{\|x - Tx\| + \|y - Ty\|\}.$$

Moreover, for $\lambda = \frac{1}{11}$, T_λ is orbitally continuous.

If we take $b = 0$ in Corollary 2.5, then we obtain Theorem 1.7 in the setting of a Banach space.

Corollary 2.7. *[4] Suppose that $(X, \|.\|)$ is a Banach space and $T : X \to X$ is $(a, 0, c)$-asymptotically regular mapping. Then T has a unique fixed point $p \in X$ provided T is either k-continuous for some $k \geq 1$ or orbitally continuous. Moreover, $\lim_{n\to\infty} T^n x = p$ for each $x \in X$.*

The following fixed point result can be established following arguments similar to those given in the proof of Corollary 2.4.

Corollary 2.8. *Suppose that $(X, \|.\|)$ is a Banach space and $T : X \to X$. If for some positive integers m such that T^m is (a, b, c)-modified enriched Kannan mapping. Then T has a unique fixed point p, provided that T^m_λ is asymptotically regular and T^m_λ is either k-continuous for some $k \geq 1$ or orbitally continuous, where $\lambda = \frac{1}{b+1}$.*

Now we consider the mappings that are not necessarily k-continuous or orbitally continuous.

Theorem 2.9. *Let $(X, \|\cdot\|)$ be Banach space. If there exists $a, b \in [0, \infty)$ and $c \in [0, 1)$ such that $T : X \to X$ is (a, b, c)-modified enriched Kannan mapping. Then*

1. *$Fix(T) = \{p\}$;*
2. *for $\lambda = \frac{1}{b+1}$; an iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \geq 0, \tag{13}$$

converges to the point p, for any initial guess x_0 in X provided that T_λ is a continuous asymptotically regular mapping.

Proof. We divide the proof into the following two cases.
Case 1. Suppose that $b > 0$. Clearly, $0 < \lambda < 1$. In this case, an inequality (6) becomes

$$\left\|T_\lambda x - T_\lambda y\right\| \leq d \left\|x - y\right\| + a\{\|x - T_\lambda x\| + \|y - T_\lambda y\|\}, \tag{14}$$

where $d = \lambda c$. Obviously $d \in [0, 1)$ and $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, ...$ By asymptotic regularity of T_λ and (14), we have for any n and $k > 0$

$$\begin{aligned}
\|x_{n+k} - x_n\| &\leq \|x_{n+k} - x_{n+k+1}\| + \|x_{n+k+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&= \|x_{n+k} - x_{n+k+1}\| + \|x_{n+1} - x_n\| + \|T_\lambda x_{n+k} - T_\lambda x_n\| \\
&\leq \|x_{n+k} - x_{n+k+1}\| + \|x_{n+1} - x_n\| + d\|x_{n+k} - x_n\| \\
&+ a\{\|x_{n+k} - T_\lambda x_{n+k}\| + \|x_n - T_\lambda x_n\|\} \\
&= d\|x_{n+k} - x_n\| + (a+1)\{\|x_{n+k} - x_{n+k+1}\| + \|x_{n+1} - x_n\|\}.
\end{aligned}$$

That is,

$$(1-d)\|x_{n+k} - x_n\| \leq (a+1)\{\|x_{n+k} - x_{n+k+1}\| + \|x_{n+1} - x_n\|\}$$

On taking limit as $n \to \infty$ on both sides of the above inequality, a sequence $\{\|x_{n+k} - x_n\|\}$ converges to 0.

This shows that $\{x_n\}$ is a Cauchy sequence in X. Next, we assume that there exists an element $p \in X$ such that $\lim_{n\to\infty} x_n = p$. As T_λ is continuous and $x_{n+1} = T_\lambda x_n$, we obtain that $T_\lambda p = p$. Suppose that q is a fixed point of T_λ and $q \neq p$. Then, we have

$$0 < \|p - q\| = \|T_\lambda p - T_\lambda q\| \leq d\|p - q\| < \|p - q\|,$$

a contradiction. Hence $Fix(T_\lambda) = Fix(T) = \{p\}$. Because

$$\begin{aligned}
\|x_{n+1} - p\| &= \|T_\lambda^n x_0 - T_\lambda^n p\| \\
&\leq \|T_\lambda^n x_0 - T_\lambda^{n+1} x_0\| + \|T_\lambda^{n+1} x_0 - T_\lambda^{n+1} p\| \\
&\leq \|T_\lambda^n x_0 - T_\lambda^{n+1} x_0\| + d\|T_\lambda^n x_0 - T_\lambda^n p\| + a\|T_\lambda^n x_0 - T_\lambda^{n+1} x_0\|,
\end{aligned}$$

so $(1-d)\|x_{n+1} - p\| \leq (1+a)\|T_\lambda^n x_0 - T_\lambda^{n+1} x_0\| \to 0$ *as* $n \to \infty$. This shows that $T_\lambda^n x$ converges to p for any $x \in X$.
Case 2. $b = 0$. In this case $\lambda = 1$. We proceed on the similar lines as in the Case 1 but with $T(= T_1)$ instead of T_λ, when the Krasnoselskij iteration (13) reduces, in fact, to the simple Picard iteration associated with T. □

Example 2.10. *Let $X = [0, 1]$ be equipped with the usual norm and $T : X \to X$ be defined by $Tx = 1 - x$. Note that, for $c = 0.1$ and $a = \frac{1}{4}$, a mapping T does not satisfy (3). Indeed, for $x = 0$ and $y = 1$, we have*

$$\|T0 - T1\| = 1 > 0.6 = 0.1\|0 - 1\| + \frac{1}{4}(1 + 1).$$

On the other hand, T is $(\frac{1}{4}, 0.1, 1)$*-modified enriched Kannan mapping.*
Indeed, for all $x, y \in [0, 1]$*, we have*

$$\|x - y + (1 - x) - (1 - y)\| = 0 \leq 0.1\|x - y\| + \frac{1}{4}\{\|2x - 1\| + \|2y - 1\|.$$

As, T is an isometry and $X = [0, 1]$ *is bounded closed and convex set,* $T_{\frac{1}{2}}$ *is continuous asymptotically regular ([7]). Thus all the condition of Theorem 2.9 are satisfied. Moreover,* $Fix(T) = \left\{\frac{1}{2}\right\}$.

If we take $b = 0$ in Theorem 2.9, we obtain Theorem 1.4 in the setting of a Banach space.

Corollary 2.11. *[7] Suppose that* $(X, \|.\|)$ *is a Banach space and T is* $(a, 0, c)$*-modified enriched Kannan continuous asymptotically regular mapping. Then T has a unique fixed point* $p \in X$ *and* $T^n x \to p$ *for each* $x \in X$.

The concept of well-posedness of a fixed point problem has attracted the attention of several mathematicians, see for example, ([5],[11]) and references mentioned therein.

We now study the well-posedness of a common fixed point problem of mappings in Theorems 2.1 and 2.9.

Definition 2.12. *[11] A common fixed point problem of a selfmappings T and S on a metric space* (X, d) *is called well-posed if the set of common fixed point of S and T is singleton, that is,* $Sx^* = x^* = Tx^*$ *and for any sequence* $\{x_n\}$ *in X satisfying*

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0 \text{ or } \lim_{n\to\infty} d(Sx_n, x_n) = 0$$

implies that $\lim_{n\to\infty} x_n = x^*$.
A fixed point problem of self-map T on X is called well-posed if T has unique fixed point x^**(say) and for any sequence* $\{x_n\}$ *in X such that* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$*, we have* $\lim_{n\to\infty} x_n = x^*$.

From the Remark 1.17, we conclude that the fixed point problem of T is called well-posed if and only if fixed point problem of T_λ well-posed.

Theorem 2.13. *Let* $(X, \|.\|)$ *be Banach space. Suppose* (T, S) *is* (a, b, c)*-modified enriched Kannan pair as in Theorem 2.1. Then the common fixed point problem of the pair* (T, S) *is well-posed.*

Proof. It follows from Theorems 2.1 that x^* is the unique common fixed point of T and S. Suppose $\lim_{n\to\infty} \|T_\lambda x_n - x_n\| = 0$. Using (7) we have,

$$\begin{aligned}\|x_n - x^*\| &\leq \|x_n - T_\lambda x_n\| + \|T_\lambda x_n - S_\lambda x^*\| \\ &\leq \|x_n - T_\lambda x_n\| + d\,\|x_n - x^*\| + a\{\|x_n - T_\lambda x_n\| + \|x^* - S_\lambda x^*\|\,,\end{aligned}$$

that is,

$$(1 - d)\,\|x_n - x^*\| \leq (1 + a)\,\|x_n - T_\lambda x_n\|\,. \tag{15}$$

Similarly, if we assume $\lim_{n\to\infty} \|S_\lambda x_n - x_n\| = 0$. Then following arguments similar to those given above, we have

$$(1 - d)\,\|x_n - x^*\| \leq (1 + a)\,\|x_n - S_\lambda x_n\|\,. \tag{16}$$

It follows from (15) and (16) that $\lim_{n\to\infty} x_n = x^*$ provided that $\lim_{n\to\infty} \|T_\lambda x_n - x_n\| = 0$ or $\lim_{n\to\infty} \|S_\lambda x_n - x_n\| = 0$. This complete the proof. □

Corollary 2.14. *Let* $(X, \|.\|)$ *be Banach space. Suppose that T is a selfmapping on X as in the Corollary 2.5. Then the fixed point problem is well-posed.*

Corollary 2.15. *Let* $(X, \|.\|)$ *be Banach space and T a selfmapping on X as in Theorem 2.9 . Then the fixed point problem is well-posed.*

3. Application

As an application of the result proved in the previous section, we establish the existence of solution of Urysohn integral equations.
For, $d, e \in \mathbb{R}$, we denote $X = C([d,e])$ the set of all continuous functions on the interval $[d,e]$. We endow X with the norm

$$\|x\| = \max_{t\in[d,e]} |x(t)|. \tag{17}$$

It is evident that $(X, \|.\|)$ is a Banach space.
Consider the Urysohn integral equations

$$x(t) = \int_d^e K_1(t,s,x(s))ds + g(t), \tag{18}$$

$$x(t) = \int_d^e K_2(t,s,x(s))ds + h(t), \tag{19}$$

where $t \in [d,e]$, and $x, g, h \in X$.
Suppose that $K_1, K_2 : [d,e] \times [d,e] \times \mathbb{R} \to \mathbb{R}$ are such that $A_x, B_x \in X$ for each $x \in X$, where

$$\begin{aligned} A_x(t) &= \int_d^e K_1(t,s,x(s))ds, \text{ and} \\ B_x(t) &= \int_d^e K_2(t,s,x(s))ds \text{ for all } t \in [d,e]. \end{aligned}$$

Theorem 3.1. *Suppose that thre following conditions hold.*

1. *There exists $a \in [0,\infty)$ such that for every $x, y \in X$, we have*

   $$|A_x(t) - B_y(t) + g(t) - h(t)| \leq aC(x,y)$$

   where

   $$C(x,y) = |x(t) - A_x(t) - g(t)| + |y(t) - B_y(t) - h(t)|.$$

2. *For all $x, y \in X$, we have,*

   $$\begin{aligned} |A_x - A_y| &\leq \frac{1}{2}|x-y| \text{ and} \\ |B_x - B_y| &\leq \frac{1}{2}|x-y|. \end{aligned}$$

   Then the Urysohn integral equations (18) and (19) have a unique common solution.

Proof. Define $T, S : X \to X$ by

$$Tx = Ax + g, \text{ and } Sx = Bx + h.$$

Then for fixed $0 < b < 1$, we have

$$|b(x-y) + Tx - Sy| \leq |b(x-y)| + |Tx - Sy|.$$

This implies that

$$\left\|b(x-y) + Tx - Sy\right\| \leq b\left\|x-y\right\| + \left\|Tx - Sy\right\|. \tag{20}$$

Note that

$$\begin{aligned}
|Tx - Sy| &= \Big| \int_d^e K_1(t,s,x(s))ds + g(t) - \int_d^e K_2(t,s,y(s))ds - h(t) \Big| \\
&\leq |A_x(t) - B_y(t) + g(t) - h(t)| \\
&\leq a\{|x(t) - A_x(t) - g(t)| + |y(t) - B_y(t) - h(t)|\} \\
&= a\{\|x(t) - A_x(t) - g(t)\| + \|y(t) - B_y(t) - h(t)\|
\end{aligned}$$

Then (20) becomes,

$$\|b(x-y) + Tx - Sy\| \leq b\|x-y\| + a\{\|x - Tx\| + \|y - Sy\|\}.$$

Thus (T, S) is (a, b, b)-modified enriched Kannan pair. We have

$$\begin{aligned}
T_\lambda x &= (1-\lambda)x + \lambda(A_x + g) \text{ and} \\
S_\lambda x &= (1-\lambda)x + \lambda(B_x + h),
\end{aligned}$$

where $\lambda = \frac{1}{b+1}$. Since $A_x, B_x \in X$ for each $x \in X$, T_λ and S_λ are continuous maps. We now show that T_λ and S_λ are asymptotically regular. For all $x, y \in X$, we have

$$\begin{aligned}
|T_\lambda x(t) - T_\lambda y(t)| &= |(1-\lambda)x(t) + \lambda A_x(t) + \lambda g(t) - ((1-\lambda)y(t) + \lambda A_y(t) + \lambda g(t))| \\
&= |(1-\lambda)(x(t) - y(t)) + \lambda(A_x(t) - A_y(t))| \\
&\leq (1-\lambda)|x(t) - y(t)| + \lambda|A_x(t) - A_y(t)|.
\end{aligned}$$

Using condition (2), we have

$$\|T_\lambda x - T_\lambda y\| \leq (1 - \frac{\lambda}{2})\|x - y\|.$$

This shows that T_λ is contraction map on X. Similarly, we can show that S_λ is a contraction map. It follows from [7], T_λ and S_λ are asymptotically regular mappings. By Theorem 2.1, the Urysohn integral equations (18) and (19) have a unique common solution. □

Theorem 3.2. *Let $X = C[d, e]$ be the space of all continuous function on the interval $[d, e]$ equipped with norm (17) and $m < \frac{1}{2(e-d)}$.*
Consider the Volterra integral equation

$$x(t) = \int_d^t G(t,s)x(s)ds + g(t). \tag{21}$$

Suppose that the kernal G is continuous on $[d, e] \times [d, e]$ such that

$$|G(t,s)| \leq m, \ \ \forall\ (t,s) \in [d,e] \times [d,e].$$

Then the Volterra integral equation (21) has a unique solution.

Proof. Note that Volterra integral equation (21) can be written $Tx = x$ with $T : C[d, e] \to C[d, e]$ defined by

$$Tx(t) = 2\int_d^t G(t,s)x(s)ds + 2g(t) - x(t).$$

Clearly T is not a contraction map but T is $(0, 1, 2m(e-d))$-modified enriched Kannan map.
Indeed

$$\begin{aligned}
\|x - y + Tx - Ty\| &= \|x - y + 2\int_d^t G(t,s)x(s)ds + 2g - x - 2\int_d^t G(t,s)y(s)ds - 2g + y\| \\
&= 2\|\int_d^t G(t,s)(x(s) - y(s))ds\|,
\end{aligned}$$

implies that

$$\|x - y + Tx - Ty\| = 2\| \int_d^t G(t,s)(x(s) - y(s))ds\|. \tag{22}$$

$$| \int_d^t G(t,s)(x(s) - y(s))ds| \leq m(t - d)\|x - y\|.$$

Using $t - d \leq e - d$ on the right-hand side of above inequality and taking the maximum over $t \in [d, e]$ on the left hand side, we obtain that

$$\| \int_d^t G(t,s)(x(s) - y(s))ds\| \leq m(e - d)\|x - y\|. \tag{23}$$

Using (23) in (22), we have

$$\|x - y + Tx - Ty\| \leq 2m(e - d)\|x - y\| + 0\{\|x - Tx\| + \|y - Ty\|\}.$$

Since $b = 1$,

$$T_{\frac{1}{2}}x(t) = \frac{1}{2}x(t) + \frac{1}{2}Tx(t) = \int_d^t G(t,s)x(s)ds + g(t).$$

implies that $T_{\frac{1}{2}}$ is contraction map on X. It follows from [7], $T_{\frac{1}{2}}$ is asymptotically regular mapping. By Theorem 2.9, the Volterra integral equation (21) has a unique solution. □

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