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The Drazin Inverse Matrix Modification Formulae With Peirce Corners

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Abstract. Our motivation is to derive the Drazin inverse matrix modification formulae utilizing the Drazin inverses of adequate Peirce corners under some special cases, and the Drazin inverse of a special matrix with an additive perturbation. As applications, several new results for the expressions of the Drazin inverses of modified matrices $A - CB$ and $A - CD^dB$ are obtained, and some well known results in the literature, as the Sherman-Morrison-Woodbury formula and Jacobson's Lemma, are generalized.

1. introduction

A square matrix $A - CD^{-1}B$ is called, especially in the case where D is the identity matrix, a modification of A, where D is invertible matrix. It is useful that the matrix can be expressed as the sum of a matrix with a convenient structure and an additive perturbation such as a modified matrix, in various fields such as statistics, numerical analysis, optimization, etc[13]. The inverse representation of the modified matrix started from the classical Sherman-Morrison-Woodbury formula[19, 22]

$$(A - CD^{-1}B)^{-1} = A^{-1} + A^{-1}C(D - BA^{-1}C)^{-1}BA^{-1},$$

where A and D are invertible matrices, but not necessarily with the same size, and B and C are matrices with appropriate sizes such that $D - BA^{-1}C$ (and so $A - CD^{-1}B$) is invertible. Inverse matrix modification formulae of such type have been developed extensively in generalized inverses, such as the Moore-Penrose inverse [1, 15], the weighted Moore-Penrose inverse [20], the group inverse [4], the weighted Drazin inverse [6], the generalized Drazin inverse [7], and especially the Drazin inverse [17, 18, 21, 23].

In 2013, Dopazo and Martínez-Serrano [11] studied some representations of the Drazin inverse of a modified matrix, utilizing an auxiliary idempotent matrix under some special cases. In 2019, Zhang, Mosić and Tam [25] combined some equivalent statements that are about the existence of group inverses of Peirce

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corner matrices of modified matrices to obtain several new results for the Drazin inverses of modified matrices.

Some related and significant definitions of the ring theory are shown. Let R, S be both rings, and $S \subseteq R$ (with the same multiplication as R, but not assumed to have an identity initially). S is called a *corner ring* (or simply a *corner*) of R and denoted $S \prec R$, if there exists an additive subgroup $C \subseteq R$ such that

$$R = S \oplus C, \quad S \cdot C \subseteq C, \quad \text{and} \quad C \cdot S \subseteq C,$$

where any subgroup C is to be said a *complement* of the corner ring S in R and not unique. S is called a *rigid corner* of R, and denoted $S \prec_r R$, if a corner S of a ring R just exists a unique complement. Specially, Lam [14] proved that a corner ring of any ring R must exist an identity, although it may not be the identity of R.

Remark 1.1. *[14, Proposition 2.2] Let $S \prec R$, with a complement C. Recall that $e \in S$ is an identity of the ring S, if $1 = e + f$ for some $f \in C$. Particularly, the decomposition $1 = e + f$ is independent of the choice of the complement C, where e, f are complementary idempotents in R.*

R_e is defined as the *Peirce corner* of R (arising from the idempotent e) and C_e is called the *Peirce complement* of R_e such that

1. $R_e := eRe \prec R$, which is the largest subring (resp. corner) of R having e as identity element.

2. $R_e \prec_r R$ (i.e., R_e is rigid in R), with a unique complement

$$C_e := fRe \oplus eRf \oplus fRf = \{r \in R : ere = 0\},$$

where e, f are complementary idempotents in R.

Jacobson's Lemma states that if $1 - ab$ is invertible, then so is $1 - ba$ and

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a,$$

where a, b belongs any ring R (with identity).

For a square complex matrix A, there exists the unique matrix A^d, called the *Drazin inverse* of A, such that

$$AA^d = A^dA, \qquad A^dAA^d = A^d, \qquad A^k = A^{k+1}A^d,$$

where k is the index of A (i.e. the smallest non-negative integer such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$) and denoted by $\text{ind}(A)$. We also use notations $A^e = AA^d$ and $A^\pi = I - A^e$. In a special case when $\text{ind}(A) = 1$, A^d is called *group inverse* of a complex square matrix A, and denoted by $A^\#$. For interesting properties of the Drazin inverse see [2, 3, 8–10].

Our aim is to derive several new results about the Drazin inverse matrix modification formulae in terms of the Drazin inverses of appropriate Peirce corners. Precisely, combing the Peirce corner theory and some auxiliary idempotent matrix P, we establish new expressions for the Drazin inverse of arbitrary matrix under some special cases in Section 2. Utilizing some Peirce corners with auxiliary idempotent matrices P, Q, we obtain some new result for the Drazin inverse of a special matrix with an additive perturbation in Section 4. As their applications in Sections 3 and 5, we give separately the expressions of the Drazin inverses of modified matrices $A - CB$ and $A - CD^dB$, and generalize several results in the literature including the Sherman-Morrison-Woodbury formula and Jacobson's Lemma.

In this paper, $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices and I is the identity matrix of proper size. Also, we set

$$S = A - CD^dB, \quad s = A^eSA^e, \quad \bar{s} = A^\pi SA^\pi,$$

$$Z = D - BA^dC, \quad z = D^eZD^e,$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, and $D \in \mathbb{C}^{m\times m}$. We suppose that $\sum_{i=m}^{n} * = 0$ whenever $n < m$.

2. Drazin inverse matrix modification formulae with its Peirce corners

In this section, we consider new representations of Drazin inverses of matrices based on Peirce corner matrices with a general idempotent P.

We firstly stand one useful representation for the Drazin inverse of a 2×2 partitioned matrix.

Lemma 2.1. [26, Corollary 3.2] *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* A, C *are complex block matrices. If* $CA = 0$ *and* $CB = 0$, *then*

$$M^d = \begin{pmatrix} A^d + X_2C & X_1 \\ D^{2d}C & D^d \end{pmatrix},$$

where $\mathrm{ind}(A) = r$, $\mathrm{ind}(D) = t$ *and, for* $i = 1, 2$,

$$X_i = \sum_{j=0}^{t-1} A^{d(i+j+1)}BD^jD^\pi + A^\pi \sum_{j=0}^{r-1} A^jBD^{d(i+j+1)} - \sum_{j=0}^{i-1} A^{d(j+1)}BD^{d(i-j)}. \tag{1}$$

Let $\mathbb{A}$ denote a complex unital algebra, and let $M_2(\mathbb{A})$ be the 2×2 matrix algebra over $\mathbb{A}$. Given an idempotent e in $\mathbb{A}$, we consider a mapping σ from $\mathbb{A}$ to $M_2(\mathbb{A}, e)$ and the set

$$M_2(\mathbb{A}, e) = \begin{pmatrix} e\mathbb{A}e & e\mathbb{A}(1-e) \\ (1-e)\mathbb{A}e & (1-e)\mathbb{A}(1-e) \end{pmatrix} \subset M_2(\mathbb{A}).$$

Lemma 2.2. [24, Lemma 3.3] *Let* e *be an idempotent of* $\mathbb{A}$. *For any* $a \in \mathbb{A}$ *let*

$$\sigma(a) = \begin{pmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{pmatrix} \in M_2(\mathbb{A}, e).$$

Then the mapping σ *is an algebra isomorphism from* $\mathbb{A}$ *to* $M_2(\mathbb{A}, e)$ *such that*

1. $(\sigma(a))^d = \sigma(a^d)$;
2. *if* $(\sigma(a))^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *then* $a^d = \alpha + \beta + \gamma + \delta$.

We establish a expression for the Drazin inverse of an arbitrary matrix S by terms of an idempotent P and using a corresponding Peirce corner matrix. For an idempotent $P \in \mathbb{C}^{n\times n}$, we denote by $\overline{P} = I - P$.

Theorem 2.3. *Let* $S, P \in \mathbb{C}^{n\times n}$ *and let* P *be idempotent. If* $\overline{P}SPS = 0$, *then*

$$S^d = (PSP)^d + X_2SP + X_1 + (\overline{P}S\overline{P})^{2d}SP + (\overline{P}S\overline{P})^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ *and, for* $i = 1, 2$,

$$\begin{aligned} X_i \;=\; & \sum_{j=0}^{r-1} (PSP)^{d(i+j+1)}(S\overline{P})^{j+1}(\overline{P}S\overline{P})^\pi + (PSP)^\pi \sum_{j=0}^{t-1} (PS)^{j+1}(\overline{P}S\overline{P})^{d(i+j+1)} \\ & - \sum_{j=0}^{i-1} (PSP)^{d(j+1)}S(\overline{P}S\overline{P})^{d(i-j)}. \end{aligned} \tag{2}$$

Proof. Since P is idempotent, set

$$N = \begin{pmatrix} PSP & PS\overline{P} \\ \overline{P}SP & \overline{P}S\overline{P} \end{pmatrix}.$$

Combining $\overline{P}SPS = 0$, Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} S^d &= (PSP)^d + X_2\overline{P}SP + X_1 + (\overline{P}S\overline{P})^{2d}\overline{P}SP + (\overline{P}S\overline{P})^d \\ &= (PSP)^d + X_2SP + X_1 + (\overline{P}S\overline{P})^{2d}SP + (\overline{P}S\overline{P})^d, \end{aligned}$$

where, for $i = 1, 2$,

$$\begin{aligned} X_i &= \sum_{j=0}^{r-1}(PSP)^{d(i+j+1)}(PS\overline{P})(\overline{P}S\overline{P})^j(\overline{P}S\overline{P})^\pi + (PSP)^\pi\sum_{j=0}^{t-1}(PSP)^jPS\overline{P}(\overline{P}S\overline{P})^{d(i+j+1)} \\ &- \sum_{j=0}^{i-1}(PSP)^{d(j+1)}PS\overline{P}(\overline{P}S\overline{P})^{d(i-j)} \\ &= \sum_{j=0}^{r-1}(PSP)^{d(i+j+1)}(S\overline{P})^{j+1}(\overline{P}S\overline{P})^\pi + (PSP)^\pi\sum_{j=0}^{t-1}(PS)^{j+1}(\overline{P}S\overline{P})^{d(i+j+1)} \\ &- \sum_{j=0}^{i-1}(PSP)^{d(j+1)}S(\overline{P}S\overline{P})^{d(i-j)} \end{aligned}$$

as desired. □

Using Theorem 2.3, we can get the following result.

Corollary 2.4. *Let $S, P \in \mathbb{C}^{n\times n}$ and let P be idempotent. If $\overline{P}SP = 0$, then*

$$S^d = (PSP)^d + Y + (\overline{P}S)^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S) = r$ and

$$Y = \sum_{i=0}^{r-1}(PSP)^{d(i+2)}(S\overline{P})^{i+1}(\overline{P}S)^\pi + (PSP)^\pi\sum_{i=0}^{t-1}(PS)^{i+1}(\overline{P}S)^{d(i+2)} - (PSP)^dS(\overline{P}S)^d.$$

Applying transpose to Theorem 2.3, we show that the next formula for S^d holds in the case that $SPS\overline{P} = 0$.

Theorem 2.5. *Let $S, P \in \mathbb{C}^{n\times n}$ and let P be idempotent. If $SPS\overline{P} = 0$, then*

$$S^d = (PSP)^d + PSX_2' + X_1' + PS(\overline{P}S\overline{P})^{2d} + (\overline{P}S\overline{P})^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ and, for $i = 1, 2$,

$$\begin{aligned} X_i' &= (\overline{P}S\overline{P})^\pi\sum_{j=0}^{r-1}(\overline{P}S)^{j+1}(PSP)^{d(i+j+1)} + \sum_{j=0}^{t-1}(\overline{P}S\overline{P})^{d(i+j+1)}(SP)^{j+1}(PSP)^\pi \qquad (3) \\ &- \sum_{j=0}^{i-1}(\overline{P}S\overline{P})^{d(i-j)}S(PSP)^{d(j+1)}. \end{aligned}$$

By Theorem 2.5 (or Corollary 2.4), we have a simpler expression for S^d under condition $PS\overline{P} = 0$.

Corollary 2.6. *Let $S, P \in \mathbb{C}^{n\times n}$ and let P be idempotent. If $PS\overline{P} = 0$, then*

$$S^d = (PSP)^d + Y' + (S\overline{P})^d,$$

where $ind(PSP) = t$, $ind(S\overline{P}) = r$ and

$$Y' = \sum_{i=0}^{r-1}(S\overline{P})^\pi(\overline{P}S)^{i+1}(PSP)^{d(i+2)} + \sum_{i=0}^{t-1}(S\overline{P})^{d(i+2)}(SP)^{i+1}(PSP)^\pi - (S\overline{P})^dS(PSP)^d.$$

In a similar way, we present more representations of the Drazin inverse of S based on the following auxiliary result.

Theorem 2.7 follows by Theorem 2.5 interchanging P and $\overline{P}$.

Theorem 2.7. *Let $S, P \in \mathbb{C}^{n\times n}$ and let P be idempotent. If $S\overline{P}SP = 0$, then*

$$S^d = (PSP)^d + X_1'' + \overline{P}S(PSP)^{2d} + (\overline{P}S\overline{P})^d + \overline{P}SX_2'',$$

where $ind(PSP) = r$, $ind(\overline{P}S\overline{P}) = t$ and, for $i = 1, 2$,

$$\begin{aligned} X_i'' &= (PSP)^\pi \sum_{j=0}^{r-1}(PS)^{j+1}(\overline{P}S\overline{P})^{d(i+j+1)} + \sum_{j=0}^{t-1}(PSP)^{d(i+j+1)}(S\overline{P})^{j+1}(\overline{P}S\overline{P})^\pi \\ &- \sum_{j=0}^{i-1}(PSP)^{d(i-j)}S(\overline{P}S\overline{P})^{d(j+1)}. \end{aligned}$$

3. Applications to Drazin inverses of a modified matrix $A - CB$

Applying results of Section 2, we obtain some well-known representations for the Drazin inverse of a modified matrix $A - CB$.

Firstly, we observe that Theorem 2.3 implies that [11, Theorem 3.4] holds.

Corollary 3.1. [11, Theorem 3.4] *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $S = A - CB$ and let $P \in \mathbb{C}^{n\times n}$ be idempotent. If $AP = PA$, $CBPA = 0$ and $CBPC = 0$, then*

$$S^d = (PSP)^d + X_2SP + X_1 + (\overline{P}S\overline{P})^{2d}SP + (\overline{P}S\overline{P})^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ and, for $i = 1, 2$,

$$\begin{aligned} X_i &= \sum_{j=0}^{r-1}(PSP)^{d(i+j+1)}(S\overline{P})^{j+1}(\overline{P}S\overline{P})^\pi + (PSP)^\pi \sum_{j=0}^{t-1}(PS)^{j+1}(\overline{P}S\overline{P})^{d(i+j+1)} \\ &- \sum_{j=0}^{i-1}(PSP)^{d(j+1)}S(\overline{P}S\overline{P})^{d(i-j)}. \end{aligned} \tag{4}$$

Proof. Since $AP = PA$, $CBPA = 0$ and $CBPC = 0$, we have

$$\overline{P}SPS = (AP - PAP)A - (AP - PAP)CB - \overline{P}CBPA + \overline{P}CBPCB = 0.$$

By Theorem 2.3, we complete this proof. □

Corollary 2.4 recovers [11, Theorem 3.8] in the following way.

Corollary 3.2. [11, Theorem 3.8] *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $S = A - CB$ and let $P \in \mathbb{C}^{n\times n}$ be idempotent. If $PAP = AP$ and $CBP = 0$, then*

$$S^d = (AP)^d + Y + (\overline{P}S)^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ and

$$Y = \sum_{i=0}^{r-1}(AP)^{d(i+2)}(S - AP)^{i+1}(\overline{P}S)^\pi + (AP)^\pi \sum_{i=0}^{t-1}(PS)^{i+1}(\overline{P}S)^{d(i+2)} - (AP)^dS(\overline{P}S)^d.$$

Proof. Because $PAP = AP$ and $CBP = 0$, then

$$\overline{P}SP = AP - PAP - CBP + PCBP = 0,$$

$PSP = SP = AP$, $\overline{P}S\overline{P} = \overline{P}S$ and $S\overline{P} = S - AP$. Applying Corollary 2.4, we finish the proof. □

We also note that Theorem 2.7 implies that [11, Theorem 3.2] holds.

Corollary 3.3. [11, Theorem 3.2] *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $S = A - CB$ and let $P \in \mathbb{C}^{n\times n}$ be idempotent. If $A\overline{P} = A\overline{P}$, $A\overline{P}CB = 0$ and $B\overline{P}CB = 0$, then*

$$S^d = (PSP)^d + X_1 + \overline{P}S(PSP)^{2d} + (\overline{P}S\overline{P})^d + \overline{P}SX_2,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ and X_1, X_2 are represented as in (4).

Proof. The assumptions $A\overline{P} = A\overline{P}$, $A\overline{P}CB = 0$ and $B\overline{P}CB = 0$ give $AP = PA$ and

$$S\overline{P}SP = A(AP - PAP) - A\overline{P}CBP - CB(AP - PAP) + CB\overline{P}CBP = 0.$$

By Theorem 2.7, we can complete the proof. □

We utilize Corollary 2.6 to obtain [11, Theorem 3.6] as follows.

Corollary 3.4. [11, Theorem 3.6] *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$, $S = A - CB$ and let $P \in \mathbb{C}^{n\times n}$ be idempotent. If $PAP = PA$ and $PCB = 0$, then*

$$S^d = (PSP)^d + Y + (\overline{P}S\overline{P})^d,$$

where $ind(PSP) = t$, $ind(\overline{P}S\overline{P}) = r$ and

$$Y = \sum_{i=0}^{t-1} (S\overline{P})^{d(i+2)}(SP)^{i+1}(PA)^{\pi} + (S\overline{P})^{\pi} \sum_{i=0}^{r-1} (S - PA)^{i+1}(PA)^{d(i+2)} - (S\overline{P})^d S(PA)^d.$$

Proof. Since $PAP = PA$ and $PCB = 0$, we have

$$PS\overline{P} = PA - PAP - PCB + PCBP = 0,$$

$PSP = PS = PA$, $\overline{P}S\overline{P} = S\overline{P}$ and $\overline{P}S = S - PA$. By Corollary 2.6, we complete the proof. □

4. Drazin inverses of some special matrices with an additive perturbation

In this section, we consider the representation of the Drazin inverse of some special matrice with general idempotents P, Q. Similarly, we give the representation of the Drazin inverse of a Peirce corner matrix with a general idempotent P.

Lemma 4.1. [12, 16] *Let $M = \begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$ and $N = \begin{bmatrix} D & 0 \\ C & A \end{bmatrix} \in \mathbb{C}^{n\times m}$, where A and D are square matrices. Then*

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} \ \ and \ \ N^d = \begin{bmatrix} D^d & 0 \\ X & A^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{t-1} (A^d)^{i+2} C D^i D^{\pi} + A^{\pi} \sum_{i=0}^{r-1} A^i C (D^d)^{i+2} - A^d C D^d,$$

$r = \text{ind}(A)$ *and* $t = \text{ind}(D)$.

Now we present one of the main results of the section.

Theorem 4.2. *Let* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$ *and* $D \in \mathbb{C}^{m\times m}$. *If* $P \in \mathbb{C}^{n\times n}$ *and* $Q \in \mathbb{C}^{m\times m}$ *are idempotents such that* $X = (QDQ)^d QBP(PAP)^e = (QDQ)^e QBP(PAP)^d$ *and* $(QDQ)^e QBP(PAP)^e = QBP$, *then*

$$(PAP - PCQX)^d = (PAP)^d - W,$$

where $ind(QDQ - XPCQ) = t$, $ind(PAP) = r$ *and*

$$\begin{aligned} W &= \Big(\sum_{i=0}^{t-1}((PAP)^d)^{i+2}CQ(QDQ-XPCQ)^i(QDQ-XPCQ)^{\pi} - (PAP)^d C(QDQ-XPCQ)^d \\ &+ (PAP)^{\pi}\sum_{i=0}^{r-2}(PAP)^{i+1}C(QDQ-XPCQ)^{d(i+3)} \\ &- PAP(PAP)^d C(QDQ-XPCQ)^{2d} + PC(QDQ-XPCQ)^{2d}\Big)X. \end{aligned}$$

Proof. Note that

$$\begin{pmatrix} PAP & PCQ \\ QBP & QDQ \end{pmatrix}\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} = \begin{pmatrix} PAP - PCQX & PCQ \\ QBP - QDQX & QDQ \end{pmatrix}.$$

For short let us introduce the temporary notation

$$M = \begin{pmatrix} PAP & PCQ \\ QBP & QDQ \end{pmatrix} \text{ and } N = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}.$$

By $QBP = (QDQ)^e QBP(PAP)^e = QDQX$ and Cline's formula, we have

$$\begin{pmatrix} PAP - PCQX & PCQ \\ 0 & QDQ \end{pmatrix}^d = M(NM)^{2d}N.$$

A calculation yields

$$NM = \begin{pmatrix} PAP & PCQ \\ QBP - XPAP & QDQ - XPCQ \end{pmatrix}.$$

Since $QBP = (QDQ)^e QBP(PAP)^e = XPAP$, by Lemma 4.1, we have

$$\begin{aligned} \begin{pmatrix} PAP & PCQ \\ 0 & QDQ - XPCQ \end{pmatrix}^{2d} &= \begin{pmatrix} (PAP)^d & Y \\ 0 & (QDQ - XPCQ)^d \end{pmatrix}^2 \\ &= \begin{pmatrix} (PAP)^{2d} & (PAP)^d Y + Y(QDQ-XPCQ)^d \\ 0 & (QDQ-XPCQ)^{2d} \end{pmatrix}, \end{aligned}$$

where $ind(QDQ - XPCQ) = t$, $ind(PAP) = r$ and

$$\begin{aligned} Y &= \sum_{i=0}^{t-1}((PAP)^d)^{i+2}PCQ(QDQ-XPCQ)^i(QDQ-XPCQ)^{\pi} \\ &+ (PAP)^{\pi}\sum_{i=0}^{r-1}(PAP)^i PCQ(QDQ-XPCQ)^{d(i+2)} - (PAP)^d PCQ(QDQ-XPCQ)^d. \end{aligned}$$

Note that

$$\begin{aligned}(PAP)^dY + Y(QDQ-XPCQ)^d &= \sum_{i=0}^{t-1}((PAP)^d)^{i+3}PCQ(QDQ-XPCQ)^i(QDQ-XPCQ)^{\pi}\\ &- (PAP)^{2d}PCQ(QDQ-XPCQ)^d\\ &+ (PAP)^{\pi}\sum_{i=0}^{r-1}(PAP)^iPCQ(QDQ-XPCQ)^{d(i+3)}\\ &- (PAP)^dPCQ(QDQ-XPCQ)^{2d}.\end{aligned}$$

Then

$$\begin{aligned}\begin{pmatrix} PAP-PCQX & PCQ \\ 0 & QDQ\end{pmatrix}^d &= \begin{pmatrix} PAP & PCQ \\ QBP & QDQ\end{pmatrix}\begin{pmatrix} (PAP)^{2d} & (PAP)^dY+Y(QDQ-XPCQ)^d \\ 0 & (QDQ-XPCQ)^{2d}\end{pmatrix}\begin{pmatrix} I & 0 \\ -X & I\end{pmatrix}\\ &= \begin{pmatrix} (PAP)^d-UX & U \\ QBP(PAP)^{2d}-VX & V\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}U &= PAP((PAP)^dY + Y(QDQ-XPCQ)^d) + PCQ(QDQ-XPCQ)^{2d}\\ &= PAP(\sum_{i=0}^{t-1}((PAP)^d)^{i+3}PCQ(QDQ-XPCQ)^i(QDQ-XPCQ)^{\pi}\\ &- (PAP)^{2d}PCQ(QDQ-XPCQ)^d\\ &+ (PAP)^{\pi}\sum_{i=0}^{r-1}(PAP)^iPCQ(QDQ-XPCQ)^{d(i+3)}\\ &- (PAP)^dPCQ(QDQ-XPCQ)^{2d}) + PCQ(QDQ-XPCQ)^{2d},\\ &= \sum_{i=0}^{t-1}((PAP)^d)^{i+2}CQ(QDQ-XPCQ)^i(QDQ-XPCQ)^{\pi}\\ &- (PAP)^dC(QDQ-XPCQ)^d\\ &+ (PAP)^{\pi}\sum_{i=0}^{r-2}(PAP)^{i+1}C(QDQ-XPCQ)^{d(i+3)}\\ &- PAP(PAP)^dC(QDQ-XPCQ)^{2d} + PC(QDQ-XPCQ)^{2d}\end{aligned}$$

and

$$\begin{aligned}
V &= QBP((PAP)^dY + Y(QDQ - XPCQ)^d) + QDQ(QDQ - XPCQ)^{2d} \\
&= QBP(\sum_{i=0}^{t-1}((PAP)^d)^{i+3}PCQ(QDQ - XPCQ)^i(QDQ - XPCQ)^\pi \\
&- (PAP)^{2d}PCQ(QDQ - XPCQ)^d \\
&+ (PAP)^\pi \sum_{i=0}^{r-1}(PAP)^iPCQ(QDQ - XPCQ)^{d(i+3)} \\
&- (PAP)^dPCQ(QDQ - XPCQ)^{2d}) + QDQ(QDQ - XPCQ)^{2d} \\
&= QBP(\sum_{i=0}^{t-1}((PAP)^d)^{i+3}CQ(QDQ - XPCQ)^i(QDQ - XPCQ)^\pi \\
&- (PAP)^{2d}C(QDQ - XPCQ)^d \\
&+ (PAP)^\pi \sum_{i=0}^{r-1}(PAP)^iPC(QDQ - XPCQ)^{d(i+3)} \\
&- (PAP)^dC(QDQ - XPCQ)^{2d}) + QD(QDQ - XPCQ)^{2d}.
\end{aligned}$$

Therefore,

$$(PAP - PCQX)^d = (PAP)^d - UX = (PAP)^d - W,$$

where

$$\begin{aligned}
W &= UX \\
&= (\sum_{i=0}^{t-1}((PAP)^d)^{i+2}CQ(QDQ - XPCQ)^i(QDQ - XPCQ)^\pi \\
&- (PAP)^dC(QDQ - XPCQ)^d \\
&+ (PAP)^\pi \sum_{i=0}^{r-2}(PAP)^{i+1}C(QDQ - XPCQ)^{d(i+3)} \\
&- PAP(PAP)^dC(QDQ - XPCQ)^{2d} + PC(QDQ - XPCQ)^{2d})X.
\end{aligned}$$

as desired □

In a similar way as Theorem 4.2, we can verify the following result.

Theorem 4.3. *Let $S, P \in \mathbb{C}^{n\times n}$ and let P be idempotent. If $X = (\overline{P}S\overline{P})^d\overline{P}SP(PSP)^e = (\overline{P}S\overline{P})^e\overline{P}SP(PSP)^d$ and $(\overline{P}S\overline{P})^e\overline{P}SP(PSP)^e = \overline{P}SP$, then*

$$(PSP)^d = USP + VSP,$$

where $ind(\overline{P}S\overline{P}) = t$, $ind(PSP - PS\overline{P}X) = r$,

$$U = (PSP - PS\overline{P}X)^{2d}$$

and

$$\begin{aligned} V &= \sum_{i=0}^{t-1}(PSP-PS\overline{P}X)^{d(i+3)}(S\overline{P})^{i+1}(\overline{P}S\overline{P})^{\pi}-(PSP-PS\overline{P}X)^{2d}S(\overline{P}S\overline{P})^{d}\\ &+ (PSP-PS\overline{P}X)^{\pi}\sum_{i=0}^{r-1}(PSP-PS\overline{P}X)^{i}PS(\overline{P}S\overline{P})^{d(i+3)}\\ &- (PSP-PS\overline{P}X)^{d}S(\overline{P}S\overline{P})^{2d}. \end{aligned}$$

5. Applications to Drazin inverses of a modified matrix $A - CD^dB$

Applying results of Section 4, we obtain new representations of the Drazin inverse of $A - CD^dB$ and also recover some well-known results. Recall that $S = A - CD^dB$, $s = A^eSA^e$, $\overline{s} = A^\pi SA^\pi$ and $z = D^eZD^e$.

Corollary 5.1. [23, Lemma 3.2] *If $D^dBA^e = D^eBA^d$, then*

$$s^d = A^d + A^dCz^dD^dBA^e - \sum_{i=0}^{r-1}(A^d)^{i+2}CD^ez^iz^\pi D^dBA^e,$$

where $ind(z) = r$.

Proof. Since $(A^eA)^\# = A^d$ and $(D^eD)^\# = A^d$, we show this formula setting $P = A^e$ and $Q = D^e$ in Theorem 4.2. □

Using Theorem 2.3 and Corollary 5.1, we can obtain the following expression for the Drazin inverse of S.

Theorem 5.2. *If $A^\pi CD^dBA^eS = 0$ and $D^dBA^e = D^eBA^d$, then*

$$S^d = s^d + X_2SA^e + X_1 + \overline{s}^{2d}SA^e + \overline{s}^d,$$

where s^d is represented as in Corollary 5.1, $ind(s) = t$, $ind(\overline{s}) = r$ and, for $i = 1, 2$,

$$X_i = \sum_{j=0}^{r-1}s^{d(i+j+1)}(SA^\pi)^{j+1}\overline{s}^\pi + s^\pi\sum_{j=0}^{t-1}(A^eS)^{j+1}\overline{s}^{d(i+j+1)} - \sum_{j=0}^{i-1}s^{d(j+1)}S\overline{s}^{d(i-j)}.$$

Proof. Firstly, we observe that

$$A^\pi SA^eS = A^\pi(A - CD^dB)A^eS = A^\pi CD^dBA^eS = 0.$$

Applying Theorem 2.3 and Corollary 5.1, we complete the proof. □

Since $A^\pi CD^dB = 0$ implies $A^\pi CD^dBA^eS = 0$, by Theorem 5.2, we can show that [23, Lemma 3.3] holds.

Corollary 5.3. [23, Lemma 3.3] *If $A^\pi CD^dB = 0$ and $D^dBA^e = D^eBA^d$, then*

$$S^d = s^d + \sum_{j=0}^{r-1}s^{d(j+2)}SA^\pi A^j,$$

where s^d is represented as in Corollary 5.1 and $ind(A) = r$.

We consider to establish a formula for the Drazin inverse of S in terms of the Drazin inverse of the original matrices A, D and Z, and generalize the classical Sherman-Morrison-Woodbury formula for the Drazin inverse. So we add the assume that $D^{\pi}BA^{d}C = 0$, and utilize an analogous strategy as Corollary 5.3 to get

$$Z^{d} = z^{d} + \sum_{i=0}^{t-1}(z^{d})^{i+2}ZD^{i}D^{\pi},$$

where $t = \text{ind}(D)$. Note that $Z^{d}D^{e} = z^{d}$, $D^{e}Z^{d} = Z^{d}$ and $z = ZD^{e}$. Then $z^{i} = Z^{i}D^{e}$ and $z^{\pi}D^{d} = (I - zZ^{d})D^{d} = Z^{\pi}D^{d}$, implying

$$z^{i}z^{\pi}D^{d} = Z^{i}D^{e}(I - Z^{e})D^{d} = Z^{i}Z^{\pi}D^{d}.$$

Now the following corollary follows from Corollary 5.3.

Corollary 5.4. [23, Theorem 3.4] *If $A^{\pi}CD^{d}B = 0$, $D^{\pi}BA^{d}C = 0$ and $D^{d}BAA^{d} = D^{d}DBA^{d}$, then*

$$S^{d} = s^{d} + \sum_{i=0}^{k-1}(s^{d})^{i+2}SA^{i}A^{\pi},$$

where $k = \text{ind}(A)$, $s = AA^{d}SAA^{d}$,

$$s^{d} = A^{d} + A^{d}CZ^{d}D^{d}BAA^{d} - \sum_{i=0}^{r-1}(A^{d})^{i+2}CDD^{d}Z^{i}Z^{\pi}D^{d}BAA^{d},$$

and $\text{ind}(Z) = r$.

If we assume that $P = I$ and $Q = I$ in Theorem 4.2, we prove the next result which recovers a generalization of Jacobson's Lemma (see [5, Theorem 3.6]) for the case of matrices.

Corollary 5.5. *If $D^{d}BA^{e} = D^{e}BA^{d}$ and $D^{e}BA^{e} = B$, then*

$$(A - CD^{d}BA^{e})^{d} = A^{d} - W,$$

where $ind(D - D^{e}BA^{d}C) = t$, $ind(A) = r$ and

$$\begin{aligned} W &= (\sum_{i=0}^{t-1}(A^{d})^{i+2}C(D - D^{e}BA^{d}C)^{i}(D - D^{e}BA^{d}C)^{\pi} \\ &- A^{d}C(D - D^{e}BA^{d}C)^{d} + A^{\pi}\sum_{i=0}^{r-1}A^{i}C(D - D^{e}BA^{d}C)^{d(i+3)})D^{d}BA^{e}. \end{aligned}$$

As a consequence of Corollary 5.5, we get a generalization of Jacobson's Lemma proved in [5] for the Drazin inverse.

Corollary 5.6. [5, Theorem 3.6] *Let $ind(I - BC) = r$. Then*

$$(I - CB)^{d} = I - (\sum_{i=0}^{r-1}C(I - BC)^{i}(I - BC)^{\pi} - C(I - BC)^{d})B.$$

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