Filomat 35:8 (2021), 2663–2678
https://doi.org/10.2298/FIL2108663S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Hilbert-Schmidt Numerical Radius of Block Operators

**Satyajit Sahoo**[a], **Mohammad Sababheh**[b]

[a]*P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar-751004, India*
[b]*Department of Basic Sciences, Princess Sumaya University for Technology, Amman 11941, Jordan*

**Abstract.** The main goal of this article is to present new inequalities for the recently defined generalized numerical radius of block operators.

## 1. Introduction

In the sequel, $\mathcal{L}(\mathcal{H})$ will denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, endowed with an inner product $\langle \cdot, \cdot \rangle$.

If $T \in \mathcal{L}(\mathcal{H})$, the *numerical range* $W(T)$ of $T$ is the complex set

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Among the most well studied norms on $\mathcal{L}(\mathcal{H})$ are the usual operator norm $\|\cdot\|$ and the numerical radius norm $w(\cdot)$. These two norms are defined respectively by

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \text{ and } w(T) = \sup\{|z| : z \in W(T)\}.$$

The most basic relation between $w(\cdot)$ and $\|\cdot\|$ is the well-known inequality

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|, \tag{1}$$

for every $T \in \mathcal{L}(\mathcal{H})$. Thus, the two norms are equivalent.

Computing the numerical radius of an arbitrary $T \in \mathcal{L}(\mathcal{H})$ is not an easy task. However, the operator norm computations are much easier, in general. This urges the need to find bounds of $w(\cdot)$ in terms of $\|\cdot\|$.

We refer the reader to [3, 5–7, 10–12] as a recent list of papers dealing with the numerical radius; where new bounds, refinements and generalizations have been given. Let tr denote the trace functional and let $\|\cdot\|_2$ denote the Hilbert-Schmidt norm on $\mathcal{L}(\mathcal{H})$. We say that $A \in C_1$ (the trace class) if $\mathrm{tr}|A|$ is finite, and $A \in C_2$ (Hilbert-Schmidt class) if $\|A\|_2 = (\mathrm{tr}A^*A)^{\frac{1}{2}}$ is finite. The Cauchy-Schwartz inequality asserts

$$|\mathrm{tr}AB| \le \|A\|_2\|B\|_2, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 47A12; Secondary 47A30, 15A60, 47A63.
*Keywords*. Numerical raidus, generalized numerical radius, block operators
Received: 05 July 2020; Revised: 09 November 2020; Accepted: 03 April 2021
Communicated by Dragan S. Djordjević
*Email addresses:* satyajitsahoo2010@gmail.com (Satyajit Sahoo), sababheh@psut.edu.jo;sababheh@yahoo.com (Mohammad Sababheh)

when $A, B \in C_2$, which in turns implies that $AB \in C_1$.
As one of the most recent advancements of the study of the numerical radius is the introduction of a new definition of the so called the generalized numerical radius [1]. The motivation of this definition is as follows: It is noted in [13] that

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}T)\|;\ T \in \mathcal{L}(\mathcal{H}), \tag{3}$$

where the real and imaginary parts of an operator $T$ are defined as $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$, respectively. In view of this, the authors in [1] introduced the following definition.

**Definition 1.1.** *Let $T \in \mathcal{L}(\mathcal{H})$ and let $N$ be any norm on $\mathcal{L}(\mathcal{H})$. Then the generalized numerical radius of $T$, induced by the norm $N$, is defined by $w_N(T) = \sup_{\theta\in\mathbb{R}} N(\Re(e^{i\theta}T))$.*

When $N(\cdot)$ is the Hilbert-Schmidt norm $\|\cdot\|_2$, the norm $w_N(\cdot)$ is denoted by $w_2(\cdot)$. That is,

$$w_2(T) = \sup_{\theta\in\mathbb{R}} \|\Re(e^{i\theta}T)\|_2. \tag{4}$$

The authors in [1] showed some properties of $w_N(\cdot)$ that come along with those of $w(\cdot)$. For example, they showed that if $N(\cdot)$ is weakly unitarily invariant, then so is $w_N(\cdot)$, in the sense that for every $A, U \in \mathcal{L}(\mathcal{H})$ such that $U$ is unitary, we have

$$w_N(UAU^*) = w_N(A), \tag{5}$$

and self-adjoint, in the sense that $w_N(A^*) = w_N(A)$. Further, it is shown in the same reference that if $A \in C_2$, then

$$w_2(A) = \sqrt{\frac{1}{2}\|A\|_2^2 + \frac{1}{2}|trA^2|}, \tag{6}$$

which implies

$$w_2(A) = \frac{1}{\sqrt{2}}\|A\|_2, \quad \text{if } A^2 = 0. \tag{7}$$

In 2012, Saddi [9] introduced the *A-numerical radius*, as follows. Let $A \in \mathcal{L}(\mathcal{H})$ be positive. Then $A$ defines a positive semi-definite sesquilinear form

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}; \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Now, given $T \in \mathcal{L}(\mathcal{H})$, we define the $A$-numerical radius of $T$ by

$$w_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}, \tag{8}$$

By setting $A = I$ in (8), we reach the usual definition of the numerical radius. In 2019, Zamani [14] came up with the following new formula for computing the $A$-numerical radius of $T \in \mathcal{L}_A(\mathcal{H})$:

$$w_A(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\right\|_A = \sup_{\theta\in\mathbb{R}} \left\|\Im_A(e^{i\theta}T)\right\|_A, \tag{9}$$

where $\Re_A(T) = \frac{T+T^{\#_A}}{2}$ and $\Im_A(T) = \frac{T-T^{\#_A}}{2i}$ for $T \in \mathcal{L}_A(\mathcal{H})$. If we set $A = I$ in (9), we get (3). Here the set of all operators which admit $A$-adjoints is denoted by $\mathcal{L}_A(\mathcal{H})$. Let $\mathbb{A}$ be an $n \times n$ diagonal operator matrix whose diagonal entries are positive operator $A$. For $n = 2$, the operator matrix is of the form $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Throughout this paper, $A$ is always assumed to be a positive operator, when we refer to $w_A(\cdot)$. In some cases we also assume $A$ be strictly positive. Zamani [14] extended the renowned inequality (1) to the context of $A$-numerical radius setting by showing

$$\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A. \tag{10}$$

Furthermore, if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. For more details about $A$-numerical radius one may refer to [7, 8, 14].

In this article, we further explore the properties of $w_2(\cdot)$, where we present several inequalities for $w_2(\cdot)$ for block operators, similar to some known inequalities about $w_{\mathbb{A}}$. We remark that our analysis of $w_2$ is due to the fact that the Hilbert-Schmidt norm is one of easiest norms to deal with. In other words, looking at the Schatten norms $\|\cdot\|_p$, it is customary to investigate $\|\cdot\|_\infty, \|\cdot\|_2$ and $\|\cdot\|_1$. The first norm implies the usual numerical radius, while the second implies $w_2$. Unfortunately, $w_1$ is not as easy to deal with as $w_2$. This justifies our tendency to investigate $w_2$, rather than any other norm.

The following lemmas will be needed to accomplish our results.

**Lemma 1.1.** ([1], Theorem 8) *Let $A \in C_2$. Then*

$$\frac{1}{\sqrt{2}}\|A\|_2 \leq w_2(A) \leq \|A\|_2. \tag{11}$$

**Lemma 1.2.** ([2], Theorem 4)*Let $A, B \in C_2$. Then*

$$\frac{\max(w_2(A+B), w_2(A-B))}{\sqrt{2}} \leq w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{w_2(A+B) + w_2(A-B)}{\sqrt{2}}. \tag{12}$$

**Lemma 1.3.** ([2], Lemma 2) *Let $T_1, T_2 \in C_2$. Then*

(i) $w_2\left(\begin{bmatrix} 0 & T_1 \\ e^{i\theta}T_2 & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix}\right)$ *for every $\theta \in \mathbb{R}$.*

(ii) $w_2\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_2\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right)$.

(iii) $w_2\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = \sqrt{2}w_2(T_2)$.

(iv) $w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) \leq \sqrt{w_2^2(T_1+T_2) + w_2^2(T_1-T_2)}$.

**Lemma 1.4.** ([2], Theorem 1) *Let $T_1, T_2 \in C_2$. Then*

$$w_2\left(\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}\right) \leq \sqrt{w_2^2(T_1) + w_2^2(T_2)}.$$

## 2. Main Results

Due to the theme of the results, we will split our main results into two subsections. For the reader convenience, we will present the known results for $w_{\mathbb{A}}$, then show our $w_2$ version. This should make it easier for the reader to follow and compare.

### *2.1. The Hilbert-Schmidt numerical radius of* $2 \times 2$ *block operators*

The pinching inequalities is one of the most important inequalities of operator matrices. Very recently, Rout et al. [8] established some pinching type $A$-numerical radius inequalities (see Lemma 2.1). For usual pinching type numerical radius inequalities one may see [5, Lemma 3.1]. Our first aim of this section is to establish certain pinching type Hilbert-schmidt numerical radius inequalities for $2 \times 2$ operator matrices. Because of the similarity, we recall the following result about $w_{\mathbb{A}}$, where we extend this to the context of $w_2$ next.

**Lemma 2.1.** [8, Lemma 2.2] *Let* $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$. *Then*

(i) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

(ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

We begin with the following lemma; where we show the $w_2$ version of Lemma 2.1.

**Lemma 2.2.** *Let* $T_1, T_2, T_3, T_4 \in C_2$. *Then*

(i) $w_2\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

(ii) $w_2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

*Proof.* Let $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ and $U = \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$. Then $U$ is a unitary operator on $\mathcal{H} \bigoplus \mathcal{H}$. Further,

$$T + U^*TU = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} + \begin{bmatrix} I & O \\ O & -I \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\begin{bmatrix} I & O \\ O & -I \end{bmatrix} = \begin{bmatrix} 2T_1 & O \\ O & 2T_4 \end{bmatrix}.$$

So, we have
(i)

$$\begin{aligned} w_2\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) &= \frac{1}{2} w_2(T + U^*TU) \\ &\leq \frac{1}{2}[w_2(T) + w_2(U^*TU)] \\ &= \frac{1}{2}[w_2(T) + w_2(T)] \quad (\text{by } w_2(U^*TU) = w_2(T)) \\ &= w_2(T) = w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right). \end{aligned}$$

(ii)

$$\begin{aligned} w_2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) &= \frac{1}{2} w_2(T - U^*TU) \\ &\leq \frac{1}{2}[w_2(T) + w_2(U^*TU)] \\ &= \frac{1}{2}[w_2(T) + w_2(T)] \\ &= w_2(T) = w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right). \end{aligned}$$

This completes the proof. □

The following result establishes upper and lower bounds for the $\mathbb{A}$-numerical radius of a particular type of $2 \times 2$ operator matrix.

**Theorem 2.1.** *[8, Theorem 2.6] Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\max\{w_A(T_1), w_A(T_2)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq w_A(T_1) + w_A(T_2). \tag{13}$$

Extending this to the context of $w_2$, we have the following result.

**Theorem 2.2.** *Let $T_1, T_2 \in C_2$. Then*

$$\sqrt{2}\max\{w_2(T_1), w_2(T_2)\} \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq \sqrt{2}\big(w_2(T_1) + w_2(T_2)\big). \tag{14}$$

*Proof.* Using Lemma 1.3 and Lemma 2.2, we obtain

$$\sqrt{2}w_2(T_1) = w_2\left(\begin{bmatrix} T_1 & O \\ O & -T_1 \end{bmatrix}\right) \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right),$$

and

$$\sqrt{2}w_2(T_2) = w_2\left(\begin{bmatrix} O & T_2 \\ -T_2 & O \end{bmatrix}\right) \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right).$$

Therefore,

$$\sqrt{2}\max\{w_2(T_1), w_2(T_2)\} \leq w_2\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right).$$

On the other hand Lemma 1.3 implies

$$w_2\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq w_2\left(\begin{bmatrix} T_1 & O \\ O & -T_1 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} O & T_2 \\ -T_2 & O \end{bmatrix}\right) = \sqrt{2}w_2(T_1) + \sqrt{2}w_2(T_2),$$

which completes the proof. □

The reader is encouraged to look at the usual numerical radius version of the inequality (14) [5, Theorem 3.2]), which reads as follows.

**Corollary 2.1.** *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Then*

$$\max\{w(T_1), w(T_2)\} \leq w\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq w(T_1) + w(T_2). \tag{15}$$

A particular case of the inequality (14) is the following.

**Remark 2.1.** *If we choose $T_2 = T_1$ in inequality (14), then*

$$\sqrt{2}w_2(T_1) \leq w_2\left(\begin{bmatrix} T_1 & T_1 \\ -T_1 & -T_1 \end{bmatrix}\right) \leq 2\sqrt{2}w_2(T_1).$$

The following identity is proved by Rout et al. for $\mathbb{A}$-numerical radius setting.

**Proposition 2.1.** *[8, Lemma 2.9] Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) = \max\{w_A(T_1 + iT_2), w_A(T_1 - iT_2)\}.$$

The following result provides the $w_2$ version of Proposition 2.1.

**Proposition 2.2.** *Let $T_1, T_2 \in C_2$. Then*

$$w_2\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) \leq \sqrt{w_2^2(T_1 + iT_2) + w_2^2(T_1 - iT_2)}.$$

*Proof.* Let $T = \begin{bmatrix} iT_2 & -T_1 \\ T_1 & iT_2 \end{bmatrix}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$. So, $U^* = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix}$. It is not difficult to show that $U$ is a unitary operator on $\mathcal{H} \bigoplus \mathcal{H}$.

Then, $U^*TU = \begin{bmatrix} -i(T_1 - T_2) & O \\ O & i(T_1 + T_2) \end{bmatrix}$. Using the fact that $w_2(T) = w_2(U^*TU)$, we get

$$\begin{aligned} w_2(T) = w_2(U^*TU) &= w_2\left(\begin{bmatrix} -i(T_1 - T_2) & O \\ O & i(T_1 + T_2) \end{bmatrix}\right) \\ &\leq \sqrt{w_2^2(-i(T_1 - T_2)) + w_2^2(i(T_1 + T_2))} \text{ (by Lemma 1.4)} \\ &= \sqrt{w_2^2(T_1 - T_2) + w_2^2(T_1 + T_2)}. \end{aligned}$$

Replacing $T_2$ by $-iT_2$ in the identity, we have

$$w_2\left(\begin{bmatrix} T_2 & -T_1 \\ T_1 & T_2 \end{bmatrix}\right) \leq \sqrt{w_2^2(T_1 + iT_2) + w_2^2(T_1 - iT_2)}.$$

□

For the purpose of insight to the next result, we present the following inequality for $w_{\mathbb{A}}$.

**Theorem 2.3.** [8, Theorem 2.11] *Let $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$\begin{aligned} w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq \frac{1}{2}\max\Big\{&w_A(T_1 + T_4 + i(T_2 - T_3)), w_A(T_1 + T_4 - i(T_2 - T_3))\Big\} \\ &+ \frac{1}{2}(w_A(T_4 - T_1) + w_A(T_2 + T_3)). \end{aligned}$$

The following theorem provides an upper bound for the Hilbert-Schmidt numerical radius of a block operator matrix of the form $\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$; as the $w_2$ version of Theorem 2.3.

**Theorem 2.4.** *$T_1, T_2, T_3, T_4 \in C_2$. Then*

$$\begin{aligned} w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \leq \frac{1}{2}\Big\{&\sqrt{w_2^2((T_3 - T_2) + i(T_1 + T_4)) + w_2^2((T_3 - T_2) - i(T_1 + T_4))} \\ &+ \sqrt{2}\big(w_2(T_2 + T_3) + w_2(T_4 - T_1)\big)\Big\}. \end{aligned}$$

*Proof.* Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. It can be shown that $U$ is a unitary operator on $\mathcal{H} \bigoplus \mathcal{H}$. Using the identity

$w_2(T) = w_2(U^*TU)$, we have

$$\begin{aligned} w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) &= w_2\left(U^* \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} U\right) \\ &= \frac{1}{2} w_2\left(\begin{bmatrix} T_1+T_2+T_3+T_4 & -T_1+T_2-T_3+T_4 \\ -T_1-T_2+T_3+T_4 & T_1-T_2-T_3+T_4 \end{bmatrix}\right) \\ &= \frac{1}{2} w_2\left(\begin{bmatrix} T_1+T_4 & T_2-T_3 \\ T_3-T_2 & T_1+T_4 \end{bmatrix} + \begin{bmatrix} T_2+T_3 & T_4-T_1 \\ T_4-T_1 & -T_3-T_2 \end{bmatrix}\right) \\ &\le \frac{1}{2}\left\{ w_2\left(\begin{bmatrix} T_1+T_4 & -(T_3-T_2) \\ T_3-T_2 & T_1+T_4 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} T_2+T_3 & T_4-T_1 \\ T_4-T_1 & -(T_3+T_2) \end{bmatrix}\right)\right\} \\ &\le \frac{1}{2}\Big\{ \sqrt{w_2^2((T_3-T_2)+i(T_1+T_4)) + w_2^2((T_3-T_2)-i(T_1+T_4))} \\ &\qquad + \sqrt{2}\Big(w_2(T_2+T_3) + w_2(T_4-T_1)\Big)\Big\}, \end{aligned}$$

where we have used Lemma 2.2 and Lemma 1.3 to obtain the last inequality. This completes the proof. □

Further, we have the following result for $w_{\mathbb{A}}$, whose $w_2$ version is shown next.

**Theorem 2.5.** [Theorem 2.13, [8]] *Let $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \le \max\{w_A(T_1), w_A(T_4)\} + \frac{w_A(T_2+T_3) + w_A(T_2-T_3)}{2}.$$

**Theorem 2.6.** *Let $T_1, T_2, T_3, T_4 \in C_2$. Then*

$$w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \le \sqrt{w_2^2(T_1) + w_2^2(T_4)} + \frac{w_2(T_2+T_3) + w_2(T_2-T_3)}{\sqrt{2}}.$$

*Proof.* Using similar argument to the proof of Theorem 2.4, we have

$$\begin{aligned} w_2\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) &= \frac{1}{2} w_2\left(\begin{bmatrix} T_1+T_4 & T_4-T_1 \\ T_4-T_1 & T_1+T_4 \end{bmatrix} + \begin{bmatrix} T_2+T_3 & T_2-T_3 \\ T_3-T_2 & -T_3-T_2 \end{bmatrix}\right) \\ &\le \frac{1}{2}\left[ w_2\left(\begin{bmatrix} T_1+T_4 & T_4-T_1 \\ T_4-T_1 & T_1+T_4 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} T_2+T_3 & T_2-T_3 \\ -(T_2-T_3) & -(T_2+T_3) \end{bmatrix}\right)\right] \\ &\le \frac{1}{2}\Big\{ \sqrt{w_2^2(T_1+T_4+T_4-T_1) + w_2^2(T_1+T_4-T_4+T_1)} \\ &\qquad + \sqrt{2}(w_2(T_2+T_3) + w_2(T_2-T_3))\Big\} \quad \text{(by Lemma 1.3 and Theorem 2.2)} \\ &= \sqrt{w_2^2(T_1) + w_2^2(T_4)} + \frac{w_2(T_2+T_3) + w_2(T_2-T_3)}{\sqrt{2}}. \end{aligned}$$

This completes the proof. □

A refinement of (10) was shown in [8], as follows.

**Proposition 2.3.** [8, Theorem 3.2] *Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) \le w_A(T_1) + w_A(T_2) - \frac{1}{2}|w_A(T_1+T_2) - w_A(T_1-T_2)|.$$

*In particular,*

$$\frac{\|T_1\|_A}{2} + \frac{\|\mathfrak{R}_A(T_1)\|_A - \|\mathfrak{I}_A(T_1)\|_A}{2} \le w_A(T_1).$$

Now, we present two Hilbert-Schmidt numerical radius inequalities simulating Proposition 2.3. For these results we need the following identity that for any two real numbers $a$ and $b$, we have

$$\frac{a+b}{2} = \max(a,b) - \frac{|a-b|}{2}. \tag{16}$$

The following result is our first lower bound for $w_2(\cdot)$ of equation (11). For usual numerical radius, related bounds can be found in [5].

**Theorem 2.7.** *Let $A, B \in C_2$. Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \frac{|w_2(A+B) - w_2(A-B)|}{\sqrt{2}} \leq \sqrt{2}(w_2(A) + w_2(B)). \tag{17}$$

*In particular if $B = A^*$, then*

$$\frac{\|A\|_2}{2} + \frac{|\,\|\Re(A)\|_2 - \|\Im(A)\|_2|}{2} \leq w_2(A). \tag{18}$$

*Proof.* By using inequality (12), we have

$$\begin{aligned} w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{w_2(A+B) + w_2(A-B)}{\sqrt{2}} \\ &= \sqrt{2}\left(\frac{w_2(A+B) + w_2(A-B)}{2}\right) \text{ (now use(16))} \\ &= \sqrt{2}\left[\max\left(w_2(A+B), w_2(A-B)\right) - \frac{|w_2(A+B) - w_2(A-B)|}{2}\right] \\ &\leq \sqrt{2}\left[w_2(A) + w_2(B) - \frac{|w_2(A+B) - w_2(A-B)|}{2}\right]. \end{aligned}$$

Thus, we get

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \frac{|w_2(A+B) - w_2(A-B)|}{\sqrt{2}} \leq \sqrt{2}\Big(w_2(A) + w_2(B)\Big). \tag{19}$$

Letting $B = A^*$ in inequality (19), we have

$$\begin{aligned} \sqrt{2}\|A\|_2 &= w_2\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right) \\ &\leq \sqrt{2}\left[w_2(A) + w_2(A^*) - \frac{|w_2(A+A^*) - w_2(A-A^*)|}{2}\right] \\ &= \sqrt{2}\Big[2w_2(A) - \Big|\|\Re(A)\|_2 - \|\Im(A)\|_2\Big|\Big]. \end{aligned}$$

Thus, we have shown

$$\frac{\|A\|_2}{2} + \frac{\Big|\|\Re(A)\|_2 - \|\Im(A)\|_2\Big|}{2} \leq w_2(A),$$

which completes the proof. □

The usual numerical radius version Theorem 2.7 can be stated as follows.

**Corollary 2.2.** [5, Theorem 4.1] *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) \leq w(T_1) + w(T_2) - \frac{1}{2}|w(T_1+T_2) - w(T_1-T_2)|.$$

*In particular,*

$$\frac{\|T_1\|}{2} + \frac{\|\Re(T_1)\| - \|\Im(T_1)\|}{2} \leq w(T_1).$$

It should be mentioned here that the inequalities in Corollary 2.2 provide a refinement of (1).

Another refinement of (10) proved in [8] can be stated as follows.

**Proposition 2.4.** [8, Theorem 3.3] *Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) + \frac{\|T_1\|_A + \|T_2\|_A}{2} + \frac{1}{2}\left|w_A(T_1 + T_2) - \frac{\|T_1\|_A + \|T_2\|_A}{2}\right| + \frac{1}{2}\left|w_A(T_1 - T_2) - \frac{\|T_1\|_A + \|T_2\|_A}{2}\right| \leq 2(w_A(T_1) + w_A(T_2)).$$

*In particular,*

$$\frac{\|T_1\|_A}{2} + \frac{1}{4}\left|\|\Re(T_1)\|_A - \frac{\|T_1\|_A}{2}\right| + \frac{1}{4}\left|\|\Im(T_1)\|_A - \frac{\|T_1\|_A}{2}\right| \leq w_A(T_1).$$

The $w_2(\cdot)$ version of Proposition 2.4 is shown next.

**Theorem 2.8.** *Let $A, B \in C_2$. Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \left(\frac{\|A\|_2 + \|B\|_2}{2}\right) + \frac{|\sqrt{2}w_2(A + B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} + \frac{|\sqrt{2}w_2(A - B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} \leq 2\sqrt{2}(w_2(A) + w_2(B)). \tag{20}$$

*In particular, if $B = A^*$, then*

$$\frac{(\sqrt{2} + 1)\|A\|_2}{4\sqrt{2}} + \frac{\left|\sqrt{2}\|\Re(A)\|_2 - \frac{\|A\|_2}{2}\right|}{4\sqrt{2}} + \frac{\left|\sqrt{2}\|\Im(A)\|_2 - \frac{\|A\|_2}{2}\right|}{4\sqrt{2}} \leq w_2(A).$$

*Proof.* By using inequality (12) and identity (16), we have

$$\begin{aligned}
&w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \left(\frac{\|A\|_2 + \|B\|_2}{2}\right) \\
&\leq \frac{w_2(A + B) + w_2(A - B)}{\sqrt{2}} + \left(\frac{\|A\|_2 + \|B\|_2}{2}\right) \\
&= \sqrt{2}\left(\frac{w_2(A + B) + w_2(A - B)}{2}\right) + \left(\frac{\|A\|_2 + \|B\|_2}{2}\right) \\
&= \frac{\sqrt{2}w_2(A + B) + \frac{\|A\|_2+\|B\|_2}{2}}{2} + \frac{\sqrt{2}w_2(A - B) + \frac{\|A\|_2+\|B\|_2}{2}}{2} \\
&= \max\left(\sqrt{2}w_2(A + B), \frac{\|A\|_2 + \|B\|_2}{2}\right) - \frac{|\sqrt{2}w_2(A + B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} \\
&+ \max\left(\sqrt{2}w_2(A - B), \frac{\|A\|_2 + \|B\|_2}{2}\right) - \frac{|\sqrt{2}w_2(A - B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} \\
&\leq \max\left(\sqrt{2}(w_2(A) + w_2(B)), \frac{\|A\|_2 + \|B\|_2}{2}\right) - \frac{|\sqrt{2}w_2(A + B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} \\
&+ \max\left(\sqrt{2}(w_2(A) + w_2(B)), \frac{\|A\|_2 + \|B\|_2}{2}\right) - \frac{|\sqrt{2}w_2(A - B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} \\
&= 2\sqrt{2}(w_2(A) + w_2(B)) - \frac{|\sqrt{2}w_2(A + B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} - \frac{|\sqrt{2}w_2(A - B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2}.
\end{aligned}$$

So,

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + \left(\frac{\|A\|_2 + \|B\|_2}{2}\right) + \frac{|\sqrt{2}w_2(A+B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2} + \frac{|\sqrt{2}w_2(A-B) - \frac{\|A\|_2+\|B\|_2}{2}|}{2}$$
$$\leq 2\sqrt{2}(w_2(A) + w_2(B)). \tag{21}$$

Letting $B = A^*$ in (21), we obtain

$$\begin{aligned}\sqrt{2}\|A\|_2 =& w_2\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right) \\ &\leq 2\sqrt{2}(w_2(A) + w_2(A^*)) - \left(\frac{\|A\|_2 + \|A^*\|_2}{2}\right) \\ &\quad - \frac{|\sqrt{2}w_2(A+A^*) - \frac{\|A\|_2+\|A^*\|_2}{2}|}{2} - \frac{|\sqrt{2}w_2(A-A^*) - \frac{\|A\|_2+\|A^*\|_2}{2}|}{2} \\ &= 4\sqrt{2}w_2(A) - \|A\|_2 - \left|\sqrt{2}\|\Re(A)\|_2 - \frac{\|A\|_2}{2}\right| - \left|\sqrt{2}\|\Im(A)\|_2 - \frac{\|A\|_2}{2}\right|.\end{aligned}$$

So,

$$\frac{(\sqrt{2}+1)\|A\|_2}{4\sqrt{2}} + \frac{\left|\sqrt{2}\|\Re(A)\|_2 - \frac{\|A\|_2}{2}\right|}{4\sqrt{2}} + \frac{\left|\sqrt{2}\|\Im(A)\|_2 - \frac{\|A\|_2}{2}\right|}{4\sqrt{2}} \leq w_2(A).$$

□

The reader should compare Theorem 2.8 with the usual numerical radius version [5, Theorem 4.2], which reads as follows.

**Corollary 2.3.** [5, Theorem 4.2] *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$. Then*

$$\begin{aligned} w\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) + \frac{\|T_1\| + \|T_2\|}{2} + \frac{1}{2}\left|w(T_1+T_2) - \frac{\|T_1\| + \|T_2\|}{2}\right| \\ + \frac{1}{2}\left|w(T_1 - T_2) - \frac{\|T_1\| + \|T_2\|}{2}\right| \leq 2(w(T_1) + w(T_2)).\end{aligned}$$

*In particular,*

$$\frac{\|T_1\|}{2} + \frac{1}{4}\left|\|\Re(T_1)\| - \frac{\|T_1\|}{2}\right| + \frac{1}{4}\left|\|\Im(T_1)\| - \frac{\|T_1\|}{2}\right| \leq w(T_1).$$

### 2.2. *Some bounds for $n \times n$ block operators*

In the rest of this paper, motivated by some methods from [10] we present several inequalities for the Hilbert-Schmidt numerical radius for $n \times n$ block operators.

The following identity is proved by Rout et al. [7]. By setting $\mathbb{A} = I$ in Theorem 2.9, one may get the usual version of numerical radius equality.

**Theorem 2.9.** [7, Theorem 3.5] *Let $X_i \in \mathcal{L}_A(\mathcal{H})$, $i = 1, 2, \ldots, n$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} X_1 & & \cdots & O \\ O & X_2 & & O \\ \vdots & & \ddots & \vdots \\ O & & \cdots & X_n \end{bmatrix}\right) = \max\{w_A(X_1), \ldots, w_A(X_n)\},$$

*where $\mathbb{A}$ is an $n \times n$ diagonal operator matrix whose diagonal entries are positive operator $A$.*

Lemma 2.3 provides an estimate for Hilbert-Schmidt numerical radius of an $n \times n$ diagonal operator matrix, in a way similar to Theorem 2.9

**Lemma 2.3.** *Let $A_{ii} \in C_2, 1 \leq i \leq n$. Then*

$$w_2\left(\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}\right) \leq \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})}.$$

*Proof.* Calculation shows that

$$\left\|\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}\right\|_2^2 = \sum_{i=1}^{n} \|A_{ii}\|_2^2.$$

and

$$\left| tr \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}^2 \right| = |\mathrm{tr}A_{11}^2 + \mathrm{tr}A_{22}^2 + \cdots + \mathrm{tr}A_{nn}^2| \leq |\mathrm{tr}A_{11}^2| + |\mathrm{tr}A_{22}^2| + \cdots + |\mathrm{tr}A_{nn}^2|.$$

It follows from (6) that

$$w_2\left(\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}\right) = \sqrt{\frac{1}{2}\left\|\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}\right\|_2^2 + \frac{1}{2}\left|\mathrm{tr}\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}^2\right|}$$

$$\leq \sqrt{\frac{1}{2}\left(\sum_{i=1}^{n} \|A_{ii}\|_2^2\right) + \frac{1}{2}\sum_{i=1}^{n} |\mathrm{tr}A_{ii}^2|} = \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})}.$$

□

The following result is an upper bound for the $\mathbb{A}$-numerical radius of a general $n \times n$ operator matrix which was proved by Rout et al.[7]. By setting $\mathbb{A} = I$, one may get the usual version of numerical radius inequality.

**Theorem 2.10.** [7, Theorem 3.6] *Let $A_{ij} \in \mathcal{L}_A(\mathcal{H})$, $1 \leq i, j \leq n$ and $T = [A_{ij}]$. Then*

$$w_{\mathbb{A}}(T) \leq \max\{w_A(A_{ii}) : 1 \leq i \leq n\} + \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \|A_{ij}\|_A,$$

*where $\mathbb{A}$ is an $n \times n$ diagonal operator matrix whose diagonal entries are strictly positive operator $A$.*

Extending Theorem 2.10 to the Hilbert-Schmidt numerical radius, we have the following.

**Theorem 2.11.** *Let $A_{ij} \in C_2$ for $i, j = 1, 2, \ldots, n$, and let $T = [A_{ij}]$. Then*

$$w_2(T) \leq \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})} + \frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sqrt{\sum_{\substack{i=1, \\ j \neq i}}^{n} \|A_{ji}\|_2^2}.$$

*Proof.*

$$w_2\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right)$$

$$\leq w_2\left(\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

$$+ w_2\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right) + \cdots + w_2\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn-1} & 0 \end{bmatrix}\right)$$

Noting that

$$\begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2$$

$$= \cdots = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn-1} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Using Lemma 2.3 and (7) we have

$$w_2(T) \leq \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})} + \frac{1}{\sqrt{2}} \left\| \begin{bmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right\|_2$$

$$+ \frac{1}{\sqrt{2}} \left\| \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right\|_2 + \cdots + \frac{1}{\sqrt{2}} \left\| \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn-1} & 0 \end{bmatrix} \right\|_2$$

$$= \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})} + \frac{1}{\sqrt{2}} \sqrt{\sum_{i=2}^{n} \|A_{1i}\|_2^2} + \frac{1}{\sqrt{2}} \sqrt{\sum_{\substack{i=1, \\ i\neq 2}}^{n} \|A_{2i}\|_2^2} + \cdots + \frac{1}{\sqrt{2}} \sqrt{\sum_{\substack{i=1, \\ i\neq n}}^{n} \|A_{ni}\|_2^2}$$

$$= \sqrt{\sum_{i=1}^{n} w_2^2(A_{ii})} + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \sqrt{\sum_{\substack{i=1, \\ j\neq i}}^{n} \|A_{ji}\|_2^2}.$$

This completes the proof. □

On the other hand, the following result presents an upper bound of the usual numerical radius of an arbitrary block operator.

**Proposition 2.5.** [4, Theorem 2.9] *Let $A_{ij} \in \mathcal{L}(\mathcal{H}),\ 1 \le i, j \le n$ and $T = [A_{ij}]$. Then*

$$w(T) \le \max\{w(A_{ii}) : 1 \le i \le n\} + \frac{1}{2}\sum_{j=1}^{n} \sqrt{\Big\| \sum_{\substack{i=1,\\ j\neq i}}^{n} A_{ji}A_{ji}^{*} \Big\|}.$$

Theorem 2.12 below provides the Hilbert-Schmidt numerical radius version of this result.

**Theorem 2.12.** *Let $A_{ij} \in C_2$ for $i, j = 1, 2, \ldots, n$, and let $T = [A_{ij}]$. Then*

$$w_2(T) \le \sum_{i=1}^{n} \left( \sqrt{w_2^2(A_{ii}) + \frac{1}{2}\sum_{\substack{j=1,\\ j\neq i}}^{n} \|A_{ij}\|_2^2} \right).$$

*Proof.* Let

$$U_k = \left[\begin{array}{c|c} J_{k\times k} & 0_{k\times n-k} \\ \hline 0_{n-k\times k} & I_{n-k\times n-k} \end{array}\right],$$

where

$$J_{k\times k} = \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & & I & 0 \\ 0 & \ldots & & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}.$$

Then $U_k$ is unitary and

$$\begin{aligned} w_2(T) &\le w_2\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\ &\quad + \cdots + w_2\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right) \\ &= w_2\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w_2\left(U_2^{*}\begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}U_2\right) \\ &\quad + \cdots + w_2\left(U_n^{*}\begin{bmatrix} A_{nn} & A_{nn-1} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}U_n\right). \end{aligned}$$

Since $w_2(\cdot)$ is weak unitary, then

$$w_2(T) \le w_2\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$
$$+ \cdots + w_2\left(\begin{bmatrix} A_{nn} & A_{nn-1} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$
$$= \sqrt{w_2^2(A_{11}) + \frac{1}{2}\sum_{j=2}^{n} \|A_{1j}\|_2^2} + \sqrt{w_2^2(A_{22}) + \frac{1}{2}\sum_{\substack{j=1,\\ j\neq 2}}^{n} \|A_{2j}\|_2^2} + \cdots + \sqrt{w_2^2(A_{nn}) + \frac{1}{2}\sum_{\substack{j=1,\\ j\neq n}}^{n} \|A_{nj}\|_2^2}$$
$$= \sum_{i=1}^{n}\left(\sqrt{w_2^2(A_{ii}) + \frac{1}{2}\sum_{\substack{j=1,\\ j\neq i}}^{n} \|A_{ij}\|_2^2}\right).$$

This completes the proof. □

For comparison, we refer the reader to [4, 10] for the usual numerical radius version of such results:

**Proposition 2.6.** [4, Corollary 2.6] *Let $A_{ij} \in \mathcal{L}(\mathcal{H})$ where $1 \le i, j \le n$ and $T = [A_{ij}]$. Then*

$$w(T) \le \frac{1}{2}\sum_{i=1}^{n}\left(w(A_{ii}) + \sqrt{w^2(A_{ii}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \|A_{ij}\|^2}\right).$$

We conclude this article with Hilbert-Schmidt numerical radius of an off-diagonal block operator matrix. The $\mathbb{A}$-numerical radius inequality for an $n \times n$ off-diagonal matrix which can be stated as follows.

**Proposition 2.7.** [7, Theorem 3.4] *Let $A_i \in \mathcal{L}_A(\mathcal{H})$, $i = 1, 2, \ldots, n$ and $T = \begin{bmatrix} O & \cdots & O & A_1 \\ \vdots & & A_2 & O \\ O & \ldots & & \vdots \\ A_n & O & \cdots & O \end{bmatrix}$. If $n$ is even, then*

$$w_{\mathbb{A}}(T) \le \frac{1}{2}\sum_{i=1}^{n} \|A_i\|_A,$$

*and if $n$ is odd, then*

$$w_{\mathbb{A}}(T) \le w_A\left(A_{\frac{n+1}{2}}\right) + \frac{1}{2}\sum_{\substack{i=1\\ i\neq \frac{n+1}{2}}}^{n} \|A_i\|_A,$$

*where $\mathbb{A}$ is an $n \times n$ diagonal operator matrix whose diagonal entries are strictly positive operator $A$.*

**Theorem 2.13.** *Let $A_i \in C_2,\ i = 1, 2, \ldots, n$ and $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \ldots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$. If $n$ is even, then*

$$w_2(T) \leq \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \|A_i\|_2.$$

*On the other hand, if $n$ is odd, then*

$$w_2(T) \leq w_2\Big(A_{\frac{n+1}{2}}\Big) + \frac{1}{\sqrt{2}} \sum_{i \neq \frac{n+1}{2}}^{n} \|A_i\|_2.$$

*Proof.* Let $T = T_1 + T_2 + T_3 + \cdots + T_n$, where

$$T_1 = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & 0 & 0 \\ 0 & \ldots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & A_2 & 0 \\ 0 & \ldots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, T_n = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & 0 & 0 \\ 0 & \ldots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}.$$

Then $T_i^2 = 0$ for all $i = 1, 2, \ldots, n$. If $n$ is an even number then

$$w_2(T) = w_2\Big(\sum_{i=1}^{n} T_i\Big) \leq \sum_{i=1}^{n} w_2(T_i) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \|T_i\|_2 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \|A_i\|_2. \tag{22}$$

On the other hand, we have

$$w_2(T) = w_2\Big(\sum_{i=1}^{n} T_i\Big) \leq w_2\Big(T_{\frac{n+1}{2}}\Big) + \sum_{i \neq \frac{n+1}{2}}^{n} w_2(T_i) = w_2\Big(A_{\frac{n+1}{2}}\Big) + \frac{1}{\sqrt{2}} \sum_{i \neq \frac{n+1}{2}}^{n} \|A_i\|_2, \tag{23}$$

if $n$ is an odd number. □

**Remark 2.2.** *For $n = 2, A_1 = A, A_2 = B$, we get*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{\sqrt{2}}\Big(\|A\|_2 + \|B\|_2\Big). \tag{24}$$

The usual numerical radius version of the inequality (24)(one can see [5, Theorem 2.3]) is

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\| + \|B\|}{2}.$$

## References


[1] A. Abu-Omar and F. Kittaneh, A generalization of the numerical radius, Linear Algebra Appl. 569 (2019) 323–334.
[2] A. Aldalabih and F. Kittaneh, Hilbert-Schmidt numerical radius inequalities for operator matrices, Linear Algebra Appl. 581 (2019) 72–84.
[3] M. Bakherad, K. Shebrawi, Upper bounds for numerical radius inequalities involving off-diagonal operator matrices, Ann. Funct. Anal. 9 (2018) 297–309.
[4] H. Gulelfen, F. Kittaneh, On numerical radius inequalities for operator matrices, Numer. Funct. Anal. Optim. 40 (2019) 1231-1241.

[5] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for certain $2 \times 2$ operator matrices, Integral Equations Operator Theory 71 (2011) 129–147.

[6] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical radius inequalities for $2 \times 2$ operator matrices, Studia Mathematica 210 (2012) 99–115.

[7] N. C. Rout, S. Sahoo, D. Mishra, Some $A$-numerical radius inequalities for semi-Hilbertian space operators, Linear Multilinear Algebra 69 (2021), 980-996.

[8] N. C. Rout, S. Sahoo, D. Mishra, On $\mathbb{A}$-numerical radius inequalities for $2 \times 2$ operator matrices, Linear Multilinear Algebra (2020) https://doi.org/10.1080/03081087.2020.1810201

[9] A. Saddi, A-normal operators in semi Hilbertian spaces, AJMAA 9 (2012) 1–12.

[10] S. Sahoo, N. Das, and D. Mishra, Numerical radius inequalities for operator matrices, Adv. Oper. Theory 4 (2019) 197–214.

[11] S. Sahoo, N. C. Rout, M. Sababheh, Some extended numerical radius inequalities, Linear Multilinear Algebra 69 (2021), 907-920.

[12] S. Sahoo, N. Das, and D. Mishra, Berezin number and numerical radius inequalities for operators on Hilbert spaces, Adv. Oper. Theory 5 (2020), 714-727.

[13] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, Studia Math. 178(1) (2007) 83–89.

[14] A. Zamani, A-Numerical radius inequalities for semi-Hilbertian space operators, Linear Algebra Appl. 578 (2019) 159–183.