Approximation by Sampling-Type Nonlinear Discrete Operators in $\varphi$-Variation

İsmail Aslan

Abstract. In the present paper, our purpose is to obtain a nonlinear approximation by using convergence in $\varphi$-variation. Angeloni and Vinti prove some approximation results considering linear sampling-type discrete operators. These types of operators have close relationships with generalized sampling series. By improving Angeloni and Vinti’s one, we aim to get a nonlinear approximation which is also generalized by means of summability process. We also evaluate the rate of approximation under appropriate Lipschitz classes of $\varphi$-absolutely continuous functions. Finally, we give some examples of kernels, which fulfill our kernel assumptions.

1. Introduction

Sampling-type operators have numerous applications in speech processing, geophysics, medicine and etc (see [4, 9, 20–28, 42]). These operators are dealing with the generalized sampling series. In this study, we concentrate on the paper [2], where Angeloni and Vinti have some convergence results concerning sampling-type discrete operators. Our goal is to obtain more general approximations than their studies. To this end, we construct a nonlinear form of the operators

$$T_w (f; x) = \sum_{k \in \mathbb{Z}} f(x - \frac{k}{w}) l_{k,w} \quad (x \in \mathbb{R} \text{ and } w \in \mathbb{N}) \quad (1)$$

given in [7, 8] and we improve them via Bell-type summability method [18, 19]. Note that, Bell’s method is considerably general and beside the classical convergence, it includes Cesàro convergence, almost convergence and so on (see [30, 32, 33, 36]). Although there are many works about usages of Bell’s methods on positive linear operators [10, 29, 34, 35, 40, 44, 46], there are only a few works on nonlinear cases [11–14] in approximation theory.

Assume that $A = \{A^v \} = \{(a_{n,v}^w) \} (w, n, v \in \mathbb{N})$ is a family of infinite matrices of real or complex numbers. Then for a given sequence $(x_w)_{w \in \mathbb{N}}$, the double sequence $t_n^v := \sum_{w=1}^{\infty} a_{n,v}^w x_w$ is called $A$–transform of $(x_w)$ provided that it is convergent for all $n, v \in \mathbb{N}$. In addition, it is called “$(x_w)$ is $A$–summable to $L$” if

$$\lim_{n \to \infty} \sum_{w=1}^{\infty} a_{n,v}^w x_w = L \quad (\text{uniformly in } v) \quad [18].$$

2020 Mathematics Subject Classification. Primary 26A45, 40A25, 41A25, 47H99, 40C05.

Keywords. Approximation in $\varphi$-variation, discrete operators, generalized sampling series, rate of approximation, summability process

Received: 12 July 2020; Revised: 07 October 2020; Accepted: 07 December 2020

Communicated by Hemen Dutta

This research is supported financially by the Scientific and Technological Research Council of Turkey (TÜBİTAK 3501 Career Development Program), Project ID: 119F262.

Email address: ismail-aslan@hacettepe.edu.tr (İsmail Aslan)
This approximation is denoted by \( \mathcal{A} - \lim x = L \). \( \mathcal{A} \) is called regular if \( \lim_{x \to x^0} x_n = L \) implies \( \mathcal{A} - \lim x = L \). A characterization of regularity of \( \mathcal{A} \) is also given by Bell in [19]: \( \mathcal{A} \) is regular if and only if

\[(a_1) \text{ for every } \omega \in \mathbb{N}, \lim_{n \to \infty} a_{nw}^\omega = 0 \text{ (uniformly in } v)\]
\[(a_2) \lim_{n} \sum_{w=1}^{\infty} a_{nw}^\omega = 1 \text{ (uniformly in } v)\]
\[(a_3) \text{ for each } n, v \in \mathbb{N}, \sum_{w=1}^{\infty} a_{nw}^\omega \leq M. \]

The variation of a function was first given by Jordan in [31] and then it was developed, e.g., in [37, 45, 47, 48]. Afterwards, taking these generalizations into account, Musielak and Orlicz introduced \( \varphi \)-variation [41], which is known as the Musielak Orlicz \( \varphi \)-variation. This concept is a strict generalization of classical Jordan variation and retains many properties of it. For other applications about \( \varphi \)-variation, see [1, 4, 6–8, 17, 39]. We also refer to [5, 15], which are related to the topic of this paper.

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a \( \varphi \)-function, that is, \( \varphi \) is continuous, nondecreasing such that \( \varphi (0) = 0, \varphi (x) > 0 \) for all \( x > 0 \) and \( \lim_{x \to \infty} \varphi (x) = \infty \).

Throughout the paper, we assume that \( \mathcal{A} \) is regular with nonnegative real entries and \( \varphi \) is a convex \( \varphi \)-function together with the following limit condition

\[
\lim_{x \to 0} \frac{\varphi (x)}{x} = 0. \tag{+}
\]

Note that, this limit condition is needed to have the following inclusion \( BV (\mathbb{R}) \subset BV^\varphi (\mathbb{R}) \), i.e., the inclusion is strict in general (for further information, see Remark 4.5. in [1]).

Suppose that \( \mathcal{P} = \{ \alpha_i \}_{i=0}^{m} \) is an increasing sequence in \( \mathbb{R} \). Then \( \varphi \)-variation of a given measurable function \( f : \mathbb{R} \to \mathbb{R} \) is defined by

\[
V_\varphi [f] = \sup_{\mathcal{P}} \sum_{i=1}^{m} \varphi \left( |f (x_i) - f (x_{i-1})| \right) \tag{41}.
\]

In addition, \( f \) is called bounded \( \varphi \)-variation, if there exists a \( \lambda > 0 \) such that \( V_\varphi [\lambda f] < \infty \). By \( BV^\varphi (\mathbb{R}) \), we denote the space of all functions of bounded \( \varphi \)-variation.

One significant property of \( \varphi \)-variation is that,

\[
V_\varphi \left( \sum_{i=1}^{n} f_i \right) \leq \sum_{i=1}^{n} V_\varphi [f_i] \tag{2}
\]

holds for every measurable function \( f_i : \mathbb{R} \to \mathbb{R} (i = 1, \ldots, n) \) (see [41]).

By \( AC_p (\mathbb{R}) \), we denote the space of all \( \varphi \)-absolutely continuous functions on \( \mathbb{R} \), namely, the space of all functions of bounded \( \varphi \)-variation such that there exists a \( \lambda > 0 \) for which for all \( \varepsilon > 0 \) and for all bounded interval \( I = [a, b] \subset \mathbb{R} \), there exists a \( \delta > 0 \) satisfying that

\[
\sum_{i=1}^{n} \varphi \left( \lambda \left| f (\beta_i) - f (\alpha_i) \right| \right) < \varepsilon
\]

holds for any collections of non-overlapping intervals \( [\alpha_i, \beta_i] \subset I \), whenever

\[
\sum_{i=1}^{n} \varphi (\beta_i - \alpha_i) < \delta.
\]

Now that we have given some basic concepts, we can define our operator as follows. Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded function. Then consider the following operator

\[
T_{n, \nu} (f ; x) = \sum_{i=1}^{\infty} a_{i, \nu} \sum_{\nu \in \mathbb{Z}} H_{i, \nu} (f \left( x - \frac{k}{H} \right)) l_{i, \nu} \quad (x \in \mathbb{R} \text{ and } n, \nu \in \mathbb{N}), \tag{3}
\]
where $H_\nu : \mathbb{R} \to \mathbb{R}$, $H_\nu (0) = 0$ and $H_\nu$ is a $\psi$-Lipschitz kernel ($|H_\nu (x) - H_\nu (y)| \leq K \psi (|x - y|)$ for all $x, y \in \mathbb{R}$). Here, $\psi$ is a $\varphi$-function and $l_{k,v} \in L^1 (Z)$ is a family of discrete kernels for every $v \in N$. Then, it is not hard to see that (3) is well-defined for all real-valued bounded functions $f$.

In this work, by using $\varphi$-absolutely continuous functions, we investigate the existence of $\mu > 0$ such that the following limit holds

$$\lim_{n \to \infty} V_\varphi [\mu (T_{n,v} (f) - f)] = 0 \quad \text{(uniformly in $v \in \mathbb{N}$)},$$

where $T_{n,v} (f)$ is defined above.

Then, we will check the rate of approximation under some Lipschitz classes of $\varphi$-absolutely continuous functions. By using the relation between them, we also get the following result

$$\lim_{n \to \infty} V_\varphi [\mu (S_{n,v} (f) - f)] = 0 \quad \text{(uniformly in $v \in \mathbb{N}$)},$$

where

$$S_{n,v} (f; x) = \sum_{w=1}^{\infty} a_{nw} \sum_{k \in Z} H_\nu \left( f \left( \frac{k}{w} \right) \right) \chi \left( w - k \right), \quad (4)$$

namely, $S_{n,v} (f)$ is $\mathcal{A}$-transform of nonlinear generalized sampling series. Furthermore, we give an application of Theorem 2.4 and Theorem 3.1 at the end of the paper.

2. Convergence in $\varphi$-Variation

In this section, we prove our main approximation theorem using convergence in $\varphi$-variation.

We require the following conditions:

$(l_1)$ $\sup_{n \in \mathbb{N}} \| l_{k,v} \|_1 = A < \infty$ for some constant $A > 0$,

$(l_2)$ $\mathcal{A} - \lim \left( \sum_{k \in Z} |l_{k,v}| \right) = 1,$

$(l_3)$ $\exists r > 0$ such that $\mathcal{A} - \lim \left( \sum_{k \in Z} |l_{k,v}| \right) = 0,$

$(h)$ For every $\gamma > 0$, there exists a $\lambda > 0$ such that, for every (proper) bounded interval $J \subset \mathbb{R}$, $\mathcal{A} - \lim V_\varphi [\lambda G_{w, J}] = 0$ uniformly in $J \subset \mathbb{R}$, where $G_w (u) = H_\nu (u) - u$ and $V_\varphi [\lambda G_{w, J}]$ denotes the $\varphi$-variation of $\lambda G_\nu$ on the interval $J$.

It can be easily seen that taking $\mathcal{A} = [l]$, the identity matrix, then $(l_1) - (l_3)$ turn into (A1)-(A2) given in [2]. Here, condition $(h)$ is a natural condition due to the nonlinearity of the kernel. For the examples of $H_\nu$ in case of $\mathcal{A} = [l]$, see [1, 8]. At the end of the paper, we give a specific kernel satisfying $(l_1) - (l_3)$ and $(h)$.

The following growth condition on $\psi$ corresponding to $\psi$–Lipschitz condition of $H_\nu$ is also needed.

**Definition 2.1.** Let $\varphi, \eta, \psi$ be a $\varphi$-function. If for all $\gamma \in (0, 1)$, there exists a constant $C_\gamma$ such that

$$\varphi (C_\gamma \psi (|g|)) \leq \eta (\gamma |g|) \quad (5)$$

for every measurable function $g : \mathbb{R} \to \mathbb{R}$, then $(\varphi, \eta, \psi)$ is called properly directed.

Throughout the paper, we will assume that $(\varphi, \eta, \psi)$ is properly directed. In the nonlinear setting, this condition is common (see [1, 7, 16, 17, 38, 43]) and some examples of the triple $(\varphi, \eta, \psi)$ can be found in [1].
Lemma 2.2. Let $f \in BV_\eta(\mathbb{R})$. If $(i_1)$ is satisfied, then $T_{n,\nu}$ maps from $BV_\eta(\mathbb{R})$ into $BV_\nu(\mathbb{R})$, namely, there exists a $\mu > 0$ such that

$$V_\nu[\mu T_{n,\nu}f] \leq V_\eta[\lambda f]$$

holds, where $\lambda > 0$ is sufficiently small for which $V_\eta[\lambda f] < \infty$.

Proof. Let $[x_i]_{i\in\{1,\ldots,m\}}$ be an increasing sequence in $\mathbb{R}$. For all $\mu > 0$, it is not hard to see from Jensen’s inequality that

$$\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - T_{n,\nu} (f; x_{i-1}) \right| \right)$$

$$\leq \sum_{i=1}^{m} \varphi \left( \mu \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w| \left| H_w (f (x_i - \frac{k}{\nu})) - H_w (f (x_{i-1} - \frac{k}{\nu})) \right| \right)$$

$$\leq \frac{1}{\alpha_{n,\nu} A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w| \left| H_w (f (x_i - \frac{k}{\nu})) - H_w (f (x_{i-1} - \frac{k}{\nu})) \right| \mu.\alpha_{n,\nu}$$

where $\alpha_{n,\nu} = \sum_{w=1}^{\infty} |a_{n,\nu}^w| < \infty$ by $(a_3)$. Then, using Jensen’s inequality one more time and taking supremum, we get the following inequality,

$$\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - T_{n,\nu} (f; x_{i-1}) \right| \right)$$

$$\leq \frac{1}{\alpha_{n,\nu} A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w| \left| H_w (f (x_i - \frac{k}{\nu})) - H_w (f (x_{i-1} - \frac{k}{\nu})) \right| \mu.\alpha_{n,\nu}$$

Since $H_w$ is $\psi$-Lipschitz, then there holds

$$\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - T_{n,\nu} (f; x_{i-1}) \right| \right)$$

$$\leq \frac{1}{\alpha_{n,\nu} A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w| \left| H_w (f (x_i - \frac{k}{\nu})) - H_w (f (x_{i-1} - \frac{k}{\nu})) \right| \mu.\alpha_{n,\nu}$$

where $K$ is $\psi$-Lipschitz constant of $H_w$. Now, from (5) for every $\lambda \in (0, 1)$ for which $V_\eta[\lambda f] < \infty$, there exists a constant $C_3 \in (0, 1)$ such that

$$\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - T_{n,\nu} (f; x_{i-1}) \right| \right)$$

$$\leq \frac{1}{\alpha_{n,\nu} A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w| \left| H_w (f (x_i - \frac{k}{\nu})) - H_w (f (x_{i-1} - \frac{k}{\nu})) \right| \mu.\alpha_{n,\nu}$$

holds for all $0 < \mu \leq C_3/(\alpha_{n,\nu} A)$. Since

$$V_\eta[\lambda f (- \frac{k}{\nu})] = V_\eta[\lambda f],$$

we derive from $(i_1)$ that

$$\sum_{i=1}^{m} \varphi \left( \mu \left| T_{n,\nu} (f; x_i) - T_{n,\nu} (f; x_{i-1}) \right| \right)$$

$$\leq \frac{V_\eta[\lambda f]}{\alpha_{n,\nu} A} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} |k_w|$$

$$\leq V_\eta[\lambda f].$$

Consequently, if we take supremum over $[x_i]_{i\in\{1,\ldots,m\}}$, the proof is done. \qed
Lemma 2.3. Let \( f \in AC_{\eta} (\mathbb{R}) \). If \((l_1)\) is satisfied, then \( T_{n,v} (f) \in AC_{\psi} (\mathbb{R}) \) for all \( n, v \in \mathbb{N} \).

Proof. Assume that \( \varepsilon > 0 \) be given and let \( \delta := \delta (\varepsilon) > 0 \) corresponds to \( \eta \)-absolute continuity of \( f \) where \( \{(\alpha_i, \beta_i)\}_{i=1}^m \) be a finite nonoverlapping intervals of \( I = [a, b] \subset \mathbb{R} \) such that \( \sum_{i=1}^m \psi (\beta_i - \alpha_i) < \delta \). Then, applying Jensen’s inequality we may clearly see that

\[
\sum_{i=1}^m \varphi (\mu |T_{n,v} (f; \beta_i) - T_{n,v} (f; \alpha_i)|)
\leq \sum_{i=1}^m \varphi \left( \mu \sum_{w=1}^\infty a_{i,w}^v \sum_{k \in \mathbb{Z}} |h_{k,w}| |H_w (f (\beta_i - \frac{k}{w})) - H_w (f (\alpha_i - \frac{k}{w}))| \right)
\leq \frac{1}{a_{n,v} A} \sum_{w=1}^\infty a_{i,w}^v \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \varphi \left( \mu a_{i,w} A |H_w (f (\beta_i - \frac{k}{w})) - H_w (f (\alpha_i - \frac{k}{w}))| \right)
\leq \frac{1}{a_{n,v} A} \sum_{w=1}^\infty a_{i,w}^v \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \varphi \left( \mu a_{i,w} A \psi |f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w})| \right).
\]

Since \( (\varphi, \eta, \psi) \) is properly directed, then for every \( \lambda \in (0, 1) \) there exists a \( C_{\lambda} > 0 \) such that

\[
\sum_{i=1}^m \varphi (\mu |T_{n,v} (f; \beta_i) - T_{n,v} (f; \alpha_i)|)
\leq \frac{1}{a_{n,v} A} \sum_{w=1}^\infty a_{i,w}^v \sum_{k \in \mathbb{Z}} |h_{k,w}| \sum_{i=1}^m \eta \left( \lambda |f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w})| \right)
\]
holds for all \( 0 < \mu \leq C_{\lambda} / (a_{n,v} A) \). Moreover, seeing that \( f \) is \( \eta \)-absolutely continuous, then there exists a \( \gamma > 0 \) such that

\[
\sum_{i=1}^m \eta \left( \gamma |f (\beta_i - \frac{k}{w}) - f (\alpha_i - \frac{k}{w})| \right) < \varepsilon
\]
whenever

\[
\sum_{i=1}^m \eta \left( (\beta_i - \frac{k}{w}) - (\alpha_i - \frac{k}{w}) \right) = \sum_{i=1}^m \eta (\beta_i - \alpha_i) < \delta.
\]

Using the previous expression together with \((l_1)\) and \((a_3)\), we finally get

\[
\sum_{i=1}^m \varphi (\mu |T_{n,v} (f; \beta_i) - T_{n,v} (f; \alpha_i)|) < \frac{1}{a_{n,v} A} \sum_{w=1}^\infty a_{i,w}^v \sum_{k \in \mathbb{Z}} |h_{k,w}| \varepsilon
\]
\[
\leq \varepsilon
\]
for all \( 0 < \lambda \leq \gamma \). \( \Box \)

Now, we state our main approximation theorem.

Theorem 2.4. Assume that \((l_1) - (l_3)\) and \((h)\) hold. Then, there exists a \( \mu > 0 \) such that for a given \( f \in AC_{\psi} (\mathbb{R}) \cap BV_{\psi} (\mathbb{R}) \), we have

\[
\lim_{n \to \infty} V_\psi [\mu (T_{n,v} (f) - f)] = 0 \text{ (uniformly in } \nu \in \mathbb{N})\] (6)
Proof. Let \( \{x_i\}_{i \in \{1, \ldots, m\}} \) be an increasing sequence in \( \mathbb{R} \). Then, for all \( \mu > 0 \)

\[
I = \sum_{i=1}^{m} \varphi \left( \mu \left| \mathcal{T}_{n,v}(f; x_i) - f(x_i) - \mathcal{T}_{n,v}(f; x_{i-1}) + f(x_{i-1}) \right| \right)
\]

\[
= \sum_{i=1}^{m} \varphi \left( \mu \left| \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} a_{nmw} \sum_{k \in \mathbb{Z}} k_{kw} \left( H_w(f(x_i - \frac{k}{n})) - f(x_i - \frac{k}{n}) \right) \right| \right)
\]

holds. Now, using the convexity of \( \varphi \), one can observe the following,

\[
I \leq \frac{1}{3} \sum_{i=1}^{m} \varphi \left( 3\mu \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \left| H_w(f(x_i - \frac{k}{n})) - f(x_i - \frac{k}{n}) \right| \right)
\]

\[
- H_w(f(x_i - \frac{k}{n})) + f(x_i - \frac{k}{n}) \right)
\]

\[
+ \frac{1}{3} \sum_{i=1}^{m} \varphi \left( 3\mu \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \left| f(x_i - \frac{k}{n}) - f(x_i) - f(x_i - \frac{k}{n}) + f(x_i - \frac{k}{n}) \right| \right)
\]

\[
+ \frac{1}{3} \sum_{i=1}^{m} \varphi \left( 3\mu \left| f(x_i) - f(x_i) \right| \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} - 1 \right)
\]

\[
= I_1 + I_2 + I_3.
\]

In \( I_1 \), using two times Jensen’s inequality we immediately get

\[
I_1 \leq \frac{1}{3} \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \left| 3\mu \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \varphi \left( 3\mu a_{nmw} A \left| H_w(f(x_i - \frac{k}{n})) - f(x_i - \frac{k}{n}) \right| \right) \right.
\]

\[
- H_w(f(x_i - \frac{k}{n})) + f(x_i - \frac{k}{n}) \right)
\]

It is known from (a3) that \( a_{nmw} := \sum_{n=1}^{\infty} a_{nwm} \leq M \) for sufficiently large \( n \in \mathbb{N} \). Then, from the convexity of \( \varphi \)

\[
I_1 \leq \frac{1}{3MA} \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \sum_{i=1}^{m} \varphi \left( 3\mu MA \left| H_w(f(x_i - \frac{k}{n})) - f(x_i - \frac{k}{n}) \right| \right)
\]

\[
- H_w(f(x_i - \frac{k}{n})) + f(x_i - \frac{k}{n}) \right)
\]

\[
\leq \frac{1}{3MA} \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} k_{kw} \left| V_p \left[ 3\mu MA \left( H_w(f - \frac{k}{n}) - f(\cdot - \frac{k}{n}) \right) \right] \right.
\]

\[
- V_p \left[ 3\mu MA \left( H_w(f - f) \right) \right] \right)
\]

yields. Now, using the fact that

\[
V_p \left[ 3\mu MA \left( H_w(f - \frac{k}{n}) - f(\cdot - \frac{k}{n}) \right) \right] = V_p \left[ 3\mu MA \left( H_w(f - f) \right) \right],
\]

then holds

\[
I_1 \leq \frac{1}{3M} \sum_{n=1}^{\infty} a_{nwm} V_p \left[ 3\mu MA \left( H_w(f - f) \right) \right].
\]

Considering (b1) together with Lemma 1 in [8], we observe that for all \( \gamma > 0 \), there exists a \( \lambda > 0 \) such that

\[
\forall \varepsilon > 0, \text{ there exists a number } n_0 \text{ satisfying that }
\]

\[
I_1 < \frac{V_p \left[ y \right]}{3M} \varepsilon.
\]
for all $n > n_0$ and $0 < \mu \leq \frac{1}{\delta I}$.  

About $l_2$, using the convexity of $\varphi$, Jensen’s inequality and $(a_3)$, there holds

$$l_2 \leq \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in Z} |h_{k,n}| \varphi \left( 3 \mu MA \left[ f \left( x_i - \frac{k}{w} \right) - f (x_i) - f \left( x_{i-1} - \frac{k}{w} \right) + f (x_{i-1}) \right] \right)$$

$$\leq \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in Z} |h_{k,n}| V_{\varphi} \left[ 3 \mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f (\cdot) \right) \right]$$

(7)

for sufficiently large $n \in \mathbb{N}$. Here, one can observe the $\varphi$-modulus of smoothness of $f \in AC_\varphi (\mathbb{R})$ by Subsection 2.4. in [41], that is, if $\varphi$ satisfies (+), then

$$\lim_{\lambda \to 0^+} \sup_{|t| < \lambda^k} V_{\varphi} \left[ \lambda (f (\cdot - t) - f (\cdot)) \right] = 0$$

for some $\lambda > 0$ if and only if $f \in AC_\varphi (\mathbb{R})$. So, one can find a $\lambda_1 > 0$ such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$V_{\varphi} \left[ \lambda_1 (f (\cdot - t) - f (\cdot)) \right] < \varepsilon$$

(8)

whenever $|t| < \delta$. Now, from (2) we can divide the sum in (7) as follows

$$l_2 \leq \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in Z} |h_{k,n}| V_{\varphi} \left[ 3 \mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f (\cdot) \right) \right]$$

$$+ \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in cr} |h_{k,n}| V_{\varphi} \left[ 3 \mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f (\cdot) \right) \right]$$

$$+ \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in cr} |h_{k,n}| V_{\varphi} \left[ 3 \mu MA \left( f \left( \cdot - \frac{k}{w} \right) - f (\cdot) \right) \right]$$

$$:= l_1^2 + l_2^2 + l_3^2$$

where $r > 0$ is given in $(l_3)$ and $w_1$ is such that

$$\frac{k}{w} < \frac{r}{w} < \frac{1}{\delta}$$

for all $w > w_1$. 

In $l_2^1$, since $V_{\varphi} \left[ 6 \mu MA f \left( \cdot - \frac{k}{w} \right) \right] = V_{\varphi} \left[ 6 \mu MA f \right]$, it can easily be observed from $(l_1)$ that

$$l_1^1 \leq \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M \sum_{k \in Z} |h_{k,n}| V_{\varphi} \left[ 6 \mu MA f \right]$$

$$\leq \frac{1}{3 MA} \sum_{n=1}^{\infty} a_n^M V_{\varphi} \left[ 6 \mu MA f \right] .$$

Then, there holds

$$l_1^1 \leq \frac{w_1 V_{\varphi} \left[ 6 \mu MA f \right]}{3 M} \varepsilon$$

for all $0 < \mu \leq \bar{\mu}/(6 MA)$ and for sufficiently large $n \in \mathbb{N}$. 

From (8), $(l_1)$ and $(a_3)$ we obtain

$$l_2^2 \leq \frac{\varepsilon}{3}$$

for all $0 < \mu \leq \lambda_1/(6 MA)$. 

From $(l_3)$, we get

$$l_3^2 \leq \frac{V_{\varphi} \left[ 6 \mu MA f \right]}{3 M} \varepsilon$$
for sufficiently large \( n \in \mathbb{N} \).

On the other hand, since
\[
\left| \sum_{k \in \mathbb{Z}} a_{nw}^\nu \sum_{k} l_{kw} - 1 \right| < 1 \quad \text{for sufficiently large } n \in \mathbb{N}, \text{ by the convexity of } \varphi
\]

holds. Then from \((l_2)\), we get
\[
l_3 < \frac{1}{3} V_{\varphi} [3 \mu f] \sum_{k} a_{nw}^\nu \sum_{k} l_{kw} - 1
\]

for sufficiently large \( n \in \mathbb{N} \). Finally, taking supremum over \( \{x_i\}_{i=1}^m \) in the first inequality, we complete the proof. \( \square \)

3. Order of Approximation

In this section, we examine the order of approximation. For this reason, we first consider the following Lipschitz class
\[
\mathcal{V}_{\varphi}^{\text{Lip}}(\alpha) = \{ f \in AC_{\varphi}(\mathbb{R}) : \exists \rho > 0 \text{ s.t. } V_{\varphi} \left[ \rho \left| f \left( t - t \right) - f \left( t \right) \right| \right] = O(|t|^{\alpha}) \text{ as } t \to 0 \}
\]

for any \( \alpha > 0 \) (see also [3]).

For a given nonnegative regular method \( \mathcal{A} = \{(a_{nw}^\nu)\}_{n \in \mathbb{N}} \) and \( \alpha > 0 \), we take into account the following orders of approximations:

\[
\left( \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{k} l_{kw} - 1 \right) = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}, \quad (9)
\]

there exists a number \( r > 0 \) such that
\[
\sum_{w=1}^{\infty} a_{nw}^\nu \sum_{|k| < r} l_{kw} = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}, \quad (10)
\]

and for each \( w \in \mathbb{N} \),
\[
a_{nw}^\nu = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}. \quad (12)
\]

**Theorem 3.1.** Assume that \((9)-(12)\) and \((l_1)\) hold. Assume further that for every \( \gamma > 0 \), there exists a \( \lambda > 0 \) such that
\[
\sum_{w=1}^{\infty} a_{nw}^\nu V_{\varphi} [AG_{nw}] / \varphi (y m (J)) = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{ and uniformly in every proper bounded interval } J \subset \mathbb{R}). \quad (13)
\]

Then, there exists a \( \mu > 0 \) such that
\[
V_{\varphi} [\mu (T_{n,v} (f) - f)] = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{)}
\]

for all \( f \in \mathcal{V}_{\varphi}^{\text{Lip}}(\alpha) \cap BV_{\varphi}(\mathbb{R}) \).
Proof. By the proof of Theorem 2.4, we may easily obtain the following inequality

\[
V_p \left[ \mu \left( T_{n,v} (f - f) \right) \right] \leq \frac{1}{3M} \sum_{w=1}^{\infty} \sum_{k \in \mathbb{Z}} a_{mw}^w \left| l_{k,w} \right| V_p \left[ 3\mu AM \left( H_w (f - f) \right) \right] \\
+ \frac{1}{3M} \sum_{w=1}^{\infty} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right] \\
+ \frac{V_w [3\mu f]}{3} \sum_{k \in \mathbb{Z}} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} - 1 \right| 
=: I_1 + I_2 + I_3
\]

for sufficiently large \( n \in \mathbb{N} \). Considering (13) in [8], there exists a constant \( L > 0 \) such that

\[
I_1 = \frac{1}{3M} \sum_{w=1}^{\infty} a_{mw}^w V_p \left[ 3\mu AM \left( H_w (f - f) \right) \right] \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| \\
\leq \frac{L}{3M} V_p \left[ \gamma f \right] n^{-a} \\
= O \left( n^{-a} \right) \quad \text{as } n \to \infty \quad \text{(uniformly in } v) 
\]

for sufficiently small \( \mu > 0 \).

In \( I_2 \), since \( f \in V_p Lip (\alpha) \), there exist \( \rho, N, \delta > 0 \) s.t. \( V_p \left[ \rho \left| f \left( \cdot - t \right) - f \left( \cdot \right) \right| \right] \leq N \left| t \right|^{\alpha} \) if \( |t| < \delta \). Moreover, for a given \( \tilde{t} > 0 \), we can find a number \( w' \) such that

\[
\frac{k}{w} < \frac{\tilde{t}}{w} < \delta
\]

for every \( w > w' \). Taking these arguments into account, we divide \( I_2 \) as follows,

\[
I_2 = \frac{1}{3M} \sum_{w=1}^{w'} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right] \\
+ \frac{1}{3M} \sum_{w'=w+1}^{\infty} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right] \\
+ \frac{1}{3M} \sum_{w'=w+1}^{\infty} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| V_p \left[ 3\mu AM \left( f \left( \cdot - \frac{k}{w} \right) - f \left( \cdot \right) \right) \right] \\
=: I_1^2 + I_2^2 + I_3^2.
\]

Then, it follows from (10) that

\[
I_2^2 \leq \frac{N}{3M} \sum_{w'=w+1}^{w'} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| l_{k,w} \right| \left| \frac{k}{w} \right|^{a} \\
\leq \frac{N}{3M} \sum_{w'=w+1}^{\infty} a_{mw}^w \sum_{k \in \mathbb{Z}} \left| \frac{k}{w'} \right|^{\alpha} \\
= O \left( n^{-a} \right) \quad \text{as } n \to \infty \quad \text{(uniformly in } v) 
\]

for all \( 0 < \mu \leq \frac{\rho}{3MA} \). On the other hand, for \( I_3^2 \) it is not hard to see from (2) that

\[
I_3^2 \leq \frac{1}{3M} \sum_{w=1}^{w} a_{mw}^w V_p \left[ 6\mu AM f \right] 
\]

and therefore, from (12)

\[
I_3^2 = O \left( n^{-a} \right) \quad \text{as } n \to \infty \quad \text{(uniformly in } v) 
\]
holds. About \( J_2 \), from (2) and (11), we observe the following

\[
J_2^3 \leq \frac{V_\phi \left[ 6 \mu AMf \right]}{3MA} \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{|k|\geq r} |l_{kw}| = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu).}
\]

Finally, directly from (9) we get

\[
J_3 = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } \nu).
\]

Now, we investigate a special case of the operator \( (3) \), where \( l_{kw}, w \equiv \chi(k) \) and \( \chi : \mathbb{R} \to \mathbb{R} \), namely, \( l_{kw} \) is not depending on \( w \). Then, \( (3) \) reduces to

\[
\bar{T}_n,\nu (f; x) = \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} \chi(k) \left(H_w \left(f \left(x - \frac{k}{w}\right)\right)\right),
\]

which is (in some cases) equivalent to \( \mathcal{A} \)-transform of nonlinear generalized sampling series given in (4).

Under these considerations, \((l_1)\) and \((l_2)\) turn into the following assumptions

\[
\begin{align*}
(l_1') &: \chi \in l^1(\mathbb{Z}) \\
(l_2') &: \sum_{k \in \mathbb{Z}} |\chi(k)| = 1
\end{align*}
\]

where on the other hand \((l_3)\) is clearly not satisfied. But these two conditions are still enough to verify the following theorem.

Theorem 3.2. Let \( f \in AC_\phi (\mathbb{R}) \cap BV_\eta (\mathbb{R}) \). If \((l_1')\) and \((l_2')\) hold, then there exists a \( \mu > 0 \) such that

\[
\lim_{n \to \infty} V_\phi \left[ \mu \left( \bar{T}_{n,\nu} (f) - f \right) \right] = 0 \text{ (uniformly in } \nu \in \mathbb{N}).
\]

Proof. Considering \((l_2')\) in the proof of Theorem 2.4, then for every \( \mu > 0 \)

\[
V_\phi \left[ \mu \left( \bar{T}_{n,\nu} (f) - f \right) \right] \leq \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} |\chi(k)| V_\phi \left[ 3 \mu MA (H_w \circ f - f) \right] + \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^\nu \sum_{k \in \mathbb{Z}} |\chi(k)| V_\phi \left[ 3 \mu MA \left(f \left(\cdot - \frac{k}{w}\right) - f (\cdot)\right)\right] + \frac{1}{3} V_\phi \left[ 3 \mu f \right] \sum_{w=1}^{\infty} a_{nw}^\nu - 1
\]

holds, where \( \bar{A} = ||\chi||_0 \). From \((h)\), \((l_1')\), and Lemma 1 in [8], one can clearly see that

\[
L_1 < \frac{V_\phi \left[ \gamma f \right]}{3M} \epsilon
\]

for sufficiently large \( n \in \mathbb{N} \) and for all \( 0 < \mu \leq \lambda/(3MA) \) where \( \lambda \) and \( \gamma \) correspond to Lemma 1 in [8]. On the other hand, since \( \chi \in l^1(\mathbb{Z}) \), for all \( \epsilon > 0 \) there exists a \( r > 0 \) such that

\[
\sum_{|k| \geq r} |\chi(k)| < \epsilon.
\]
Hence, if we divide $L_2$ into two parts as follows,

\[
L_2 = \frac{1}{3MA} \sum_{n=1}^{\infty} a_{mn} \sum_{|k|<F} |\chi(k)| V_f [3\mu MA f \left( \cdot - \frac{k}{l} \right) - f(\cdot)] \\
+ \frac{1}{3MA} \sum_{n=1}^{\infty} a_{mn} \sum_{|k|<F} |\chi(k)| V_f [3\mu MA f \left( \cdot - \frac{k}{l} \right) - f(\cdot)] \\
=: L_1^1 + L_2
\]

then, there holds

\[
L_2 < \frac{V_f [6\mu MA f]}{3A} \epsilon
\]

For $L_2$, using $\varphi$-modulus of smoothness of the function $f \in AC_{\varphi}(R)$, we obviously see that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

\[
\frac{\delta}{\varphi} < \frac{\epsilon}{\delta} < \delta
\]

for all $\varphi > \delta$, which implies

\[
V_f [3\mu MA f (\cdot - t) - f(\cdot))] < \epsilon.
\]

Then, dividing $L_2$ as follows,

\[
L_2 = \frac{1}{3MA} \sum_{n=1}^{\infty} a_{mn} \sum_{|k|<F} |\chi(k)| V_f [3\mu MA f \left( \cdot - \frac{k}{l} \right) - f(\cdot)] \\
+ \frac{1}{3MA} \sum_{n=1}^{\infty} a_{mn} \sum_{|k|<F} |\chi(k)| V_f [3\mu MA f \left( \cdot - \frac{k}{l} \right) - f(\cdot)]
\]

we may easily obtain

\[
L_2 < \left( \frac{\varphi V_f [6\mu MA f]}{3M} + \frac{1}{3} \right) \epsilon.
\]

Finally, using (a2) we conclude

\[
L_3 < \frac{V_f [3\mu f]}{3} \epsilon
\]

for sufficiently large $n \in N$, which completes the proof. \(\square\)

**Remark 3.3.** Note that, the operators $\mathcal{F}$ and $\mathcal{S}$ are different in general but, in some cases, they coincide.

**Corollary 3.4.** Assume that $f \in B^1_{\mu, \nu}(R) \cap BV_{\varphi}(R)$ and $\psi (|f|) \in B^1_{\mu, \nu}(R)$ (the Paley-Wiener Space $B^p_{\mu, \nu}(R) = \{ f \in L^p(R) : f$ has an extension to whole $C$ s.t. $|f(z)| \leq \exp (\pi \varphi |z|) \| f \| \text{ for every } z \in C \}$ for some $w > 0$, where $\| \cdot \|$ denotes supremum norm. If $\chi \in B^\infty_{\mu, \nu}(R)$ and $(t_1), (t_2), (h)$ are satisfied, then there exists a $\mu > 0$ such that

\[
\lim_{n \to \infty} V_f [\mu (S_{n, \nu}(f) - f)] = 0 \text{ (uniformly in } \nu \in N).
\]

**Proof.** First of all, we should say that since $|H_{\varphi}(f)| \leq K \psi (|f|)$ and $\psi (|f|) \in B^1_{\mu, \nu}(R)$, then $H_{\varphi}(f) \in B^1_{\mu, \nu}(R)$. From Proposition 4.3. in [2] and (+), we may easily see that $B^1_{\mu, \nu}(R) \subset AC_{\varphi}(R)$. Therefore, using the similar arguments on Lemma 4.2. in [2], we deduce that

\[
S_{n, \nu}(f) = \mathcal{F}_{n, \nu}(f)
\]

for all $n, \nu \in N$. Consequently, by the Theorem 3.2 the proof completes. \(\square\)

An example of $\chi \in B^\infty_{\mu, \nu}(R)$ satisfying $(t_1)$ and $(t_2)$ can be found in Example 4.5. in [2].
4. Conclusions and Applications

We remark that operator (3) can be written as

\[ T_{n,\nu} (f; x) = \sum_{w=1}^{\infty} a_{nw} T_{w} (f; x) \]

where \( T_{w} (f; x) \) is introduced by

\[ T_{w} (f; x) = \sum_{k \in \mathbb{Z}} H_{w} \left( f \left( x - \frac{k}{w} \right) \right) l_{k,w}. \]

Using certain methods, some significant results of Theorem 2.4 are given below:

- If we take \( A = \{ C_{1} \} \), Cesàro matrix [30], where \( C_{1} = [c_{nw}] \) is such that

  \[ c_{nw} = \begin{cases} \frac{1}{n} & \text{if } 1 \leq w \leq n \\ 0 & \text{otherwise,} \end{cases} \]

  then we get

  \[ \lim_{n \to \infty} V_{\phi} \left[ \frac{T_{1} (f) + T_{2} (f) + \cdots + T_{n} (f)}{n} - f \right] = 0 \]

  for all \( f \in AC_{\phi} (\mathbb{R}) \).

- Putting \( A = F \), the almost convergence matrix [36], where \( F = [c_{nu}^{\nu}] \) is such that

  \[ c_{nu}^{\nu} = \begin{cases} \frac{1}{\nu^2} & \text{if } \nu \leq w \leq n + \nu - 1 \\ 0 & \text{otherwise,} \end{cases} \]

  then we get

  \[ \lim_{n \to \infty} V_{\phi} \left[ \frac{T_{\nu} (f) + T_{\nu+1} (f) + \cdots + T_{n+\nu-1} (f)}{n} - f \right] = 0 \text{ uniformly in } \nu \]

  for all \( f \in AC_{\phi} (\mathbb{R}) \).

- If \( A = \{ I \} \), the identity matrix, then we get

  \[ \lim_{n \to \infty} V_{\phi} \left[ T_{n} (f) - f \right] = 0, \]

  where \( T_{n} \) is nonlinear form of (1).

- If one take \( H_{w} (u) = u \), then \( T_{n} \) reduces to linear case given in (1) and the previous estimations hold for the operator (1).

- On the other hand, all the previous results are still valid for the generalized sampling series \( S_{n,\nu} (f) \) given in (4).

Now, we will investigate the existence of kernels which satisfy \((l_{1}) - (l_{3}), (h)\) and conditions \((9) - (13)\). Let \( A = F = [F^{\nu}] \), \( \alpha = 1/2 \) and \( l_{k,w}, H_{w} \) and \( \psi \) are defined by

\[ l_{k,w} := \begin{cases} \frac{1}{2^{w|\nu| - 1}} & : \quad w = m^2 \ (m \in \mathbb{N}) \\ \frac{2^{w} - 1}{2^{w|\nu|} (2^{w+1})} & : \quad w \neq m^2 \ (m \in \mathbb{N}), \end{cases} \]
\( H_w(u) := u + \tanh \left( \frac{u}{w} \right) \) and \( \psi(|u|) := |u| \). Then, if \( w = m^2 (m \in \mathbb{N}) \), we have
\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 2 \left( \frac{2^w + 1}{2^w - 1} \right) \leq 6
\]
and if \( w \neq m^2 \), we have
\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 1,
\]
which implies \((l_1)\) for \( A = 6 \).

For \((l_2)\) and \((9)\), consider the following inequality
\[
|\sum_{w = v}^{n+1-1} \frac{1}{n} \sum_{k \in \mathbb{Z}} l_{k,w} - 1| \leq \sum_{w = v}^{n+1-1} \frac{1}{n} \left| \sum_{k \in \mathbb{Z}} l_{k,w} - 1 \right| \\
= \sum_{w = v}^{n+1-1} \frac{1}{n} \left( 2 \left( \frac{2^w + 1}{2^w - 1} \right) - 1 \right) \\
\leq \frac{5}{n} \left( \sqrt{n + v - 1} - \sqrt{v + 1} \right) \\
= \frac{5(n-1)}{n \left( \sqrt{n + v - 1} + \sqrt{v + 1} \right)} + \frac{5}{n} \\
\leq \frac{5}{\sqrt{n + v - 1} + \sqrt{v + 1}} + \frac{5}{n} \\
\leq \frac{10}{\sqrt{n}} = O\left( \frac{1}{\sqrt{n}} \right) \quad \text{(uniformly in } v),
\]
which proves \((l_2)\) and \((9)\).

For \((l_3)\) and \((11)\), if \( w = m^2 \), then
\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 4 \left( \frac{2^w}{2^w - 1} \right) \frac{1}{2^w}
\]
and if \( w \neq m^2 \),
\[
\sum_{k \in \mathbb{Z}} |l_{k,w}| = 2 \left( \frac{2^w - 1}{2^w + 1} \right) \left( \frac{2^w}{2^w - 1} \right) \frac{1}{2^w}
\]
hold. Therefore, we get the following expression
\[
\sum_{w = v}^{n+1-1} \frac{1}{n} \sum_{k \in \mathbb{Z}} |l_{k,w}| \leq \frac{4}{n} \sum_{w = v}^{n+1-1} \frac{2^w}{2^w - 1} \frac{1}{2^w} \\
\leq \frac{8}{n} \sum_{w = v}^{n+1-1} \frac{1}{2^w} \\
\leq \frac{8}{n} \frac{1}{2^v} \\
= \frac{8}{n} \left( \frac{2^v}{2^v - 1} \right),
\]
which shows (l₃) is satisfied for r = 1. Furthermore, by the fact that for all r ≥ 1
\[ \left( \frac{2^r}{2^r - 1} \right) \leq 2 \]
and so, we conclude
\[ \sum_{n=0}^{n+1-1} \frac{1}{n} \sum_{i \leq r} |h_{2w}| \leq \frac{16}{n} \leq \frac{16}{\sqrt{n}} \]
\[ = O\left( \frac{1}{\sqrt{n}} \right) \text{ (uniformly in } v \text{).} \]

For the condition (10), we may clearly get
\[ \frac{1}{n} \sum_{n=0}^{n+1-1} \frac{1}{w} \leq \frac{2(\sqrt{n+1} - \sqrt{n})}{n} \]
\[ \leq \frac{2(n-1)}{n \sqrt{w(n+1)+1}} \]
\[ \leq \frac{2}{\sqrt{n}} \]
\[ = O\left( \frac{1}{\sqrt{n}} \right) \text{ (uniformly in } v \text{).} \]

Moreover, by the convexity of φ
\[ \frac{V_\phi [\gamma G_{2w}, J]}{\phi (\gamma m (J))} \leq \frac{\phi (\gamma (G_{2w} (b) - G_{2w} (a)))}{\phi (\gamma m (J))} \]
Furthermore, by the convexity of φ
\[ \frac{V_\phi [\gamma G_{2w}, J]}{\phi (\gamma m (J))} \leq \frac{\phi (\gamma (\frac{b}{w} - \frac{a}{w}))}{\phi (\gamma m (J))} \]
\[ \leq \frac{1}{w} \frac{\phi (\gamma (b - a))}{\phi (\gamma m (J))} \]
\[ = \frac{1}{w} \]
holds, where \( 1/w \to 0 \) as \( w \to \infty \). Then we obtain from (14) that
\[ \frac{1}{n} \sum_{n=0}^{n+1-1} \frac{1}{w} \leq \frac{1}{n} \sum_{n=0}^{n+1-1} \frac{1}{\sqrt{w}} \]
\[ = O\left( \frac{1}{\sqrt{n}} \right) \text{ as } n \to \infty \text{ (uniformly in } v \text{)} \]
which verifies (13) and (h).
Figure 1: The kernel function $H_w$.

References


[18] H. T. Bell, $\mathcal{A}$-summability, Dissertation, (Lehigh University, Bethlehem, Pa., 1971).