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Approximation by Sampling-Type Nonlinear Discrete Operators in φ-Variation

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Abstract. In the present paper, our purpose is to obtain a nonlinear approximation by using convergence in φ-variation. Angeloni and Vinti prove some approximation results considering linear sampling-type discrete operators. These types of operators have close relationships with generalized sampling series. By improving Angeloni and Vinti's one, we aim to get a nonlinear approximation which is also generalized by means of summability process. We also evaluate the rate of approximation under appropriate Lipschitz classes of φ-absolutely continuous functions. Finally, we give some examples of kernels, which fulfill our kernel assumptions.

1. Introduction

Sampling-type operators have numerous applications in speech processing, geophysics, medicine and etc (see [4, 9, 20–28, 42]). These operators are dealing with the generalized sampling series. In this study, we concentrate on the paper [2], where Angeloni and Vinti have some convergence results concerning sampling-type discrete operators. Our goal is to obtain more general approximations than their studies. To this end, we construct a nonlinear form of the operators

$$T_w(f;x) = \sum_{k\in\mathbb{Z}} f\left(x - \frac{k}{w}\right) l_{k,w} \quad (x \in \mathbb{R} \text{ and } w \in \mathbb{N}) \tag{1}$$

given in [7, 8] and we improve them via Bell-type summability method [18, 19]. Note that, Bell's method is considerably general and beside the classical convergence, it includes Cesàro convergence, almost convergence and so on (see [30, 32, 33, 36]). Although there are many works about usages of Bell's methods on positive linear operators [10, 29, 34, 35, 40, 44, 46], there are only a few works on nonlinear cases [11–14] in approximation theory.

Assume that $\mathcal{A} = \{A^v\} = \{(a^v_{nw})\}$ $(w, n, v \in \mathbb{N})$ is a family of infinite matrices of real or complex numbers. Then for a given sequence $(x_w)_{w\in\mathbb{N}}$, the double sequence $t^v_n := \sum_{w=1}^{\infty} a^v_{nw} x_w$ is called $\mathcal{A}$–transform of (x_w) provided that it is convergent for all $n, v \in \mathbb{N}$. In addition, it is called "(x_w) is $\mathcal{A}$–summable to L" if

$$\lim_{n\to\infty} \sum_{w=1}^{\infty} a^v_{nw} x_w = L \ \text{ (uniformly in } v) \text{ [18].}$$

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This approximation is denoted by $\mathcal{A}-\lim x = L$. $\mathcal{A}$ is called regular if $\lim_w x_w = L$ implies $\mathcal{A}-\lim x = L$. A characterization of regularity of $\mathcal{A}$ is also given by Bell in [19]: $\mathcal{A}$ is regular if and only if

(a_1) for every $w \in \mathbb{N}$, $\lim_n a_{nw}^{v} = 0$ (uniformly in v)
(a_2) $\lim_n \sum_{w=1}^{\infty} a_{nw}^{v} = 1$ (uniformly in v)
(a_3) for each $n, v \in \mathbb{N}$, $\sum_{w=1}^{\infty} \left|a_{nw}^{v}\right| =: a_{n,v}$ is finite and there exist positive integers N, M satisfying that $\sup_{n \geq N, v \in \mathbb{N}} \sum_{w=1}^{\infty} \left|a_{nw}^{v}\right| \leq M$.

The variation of a function was first given by Jordan in [31] and then it was developed, e.g., in [37, 45, 47, 48]. Afterwards, taking these generalizations into account, Musielak and Orlicz introduced φ-variation [41], which is known as the Musielak Orlicz φ-variation. This concept is a strict generalization of classical Jordan variation and retains many properties of it. For other applications about φ-variation, see [1, 4, 6–8, 17, 39]. We also refer to [5, 15], which are related to the topic of this paper.

Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a φ-function, that is, φ is continuous, nondecreasing such that $\varphi(0) = 0$, $\varphi(x) > 0$ for all $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

Throughout the paper, we assume that $\mathcal{A}$ is regular with nonnegative real entries and φ is a convex φ-function together with the following limit condition

$$\lim_{x \to 0^+} \frac{\varphi(x)}{x} = 0. \tag{+}$$

Note that, this limit condition is needed to have the following inclusion $BV(\mathbb{R}) \subset BV_\varphi(\mathbb{R})$, i.e., the inclusion is strict in general (for further information, see Remark 4.5. in [1]).

Suppose that $\mathcal{P} = \{x_i\}_{i=0}^{m}$ is an increasing sequence in $\mathbb{R}$. Then φ-variation of a given measurable function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$V_\varphi[f] = \sup_{\mathcal{P}} \sum_{i=1}^{m} \varphi\left(\left|f(x_i) - f(x_{i-1})\right|\right) \quad [41].$$

In addition, f is called bounded φ-variation, if there exists a $\lambda > 0$ such that $V_\varphi[\lambda f] < \infty$. By $BV_\varphi(\mathbb{R})$, we denote the space of all functions of bounded φ-variation.

One significant property of φ-variation is that,

$$V_\varphi[\sum_{i=1}^{n} f_i] \leq \frac{1}{n} \sum_{i=1}^{n} V_\varphi[n f_i] \tag{2}$$

holds for every measurable function $f_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, \ldots, n$) (see [41]).

By $AC_\varphi(\mathbb{R})$, we denote the space of all φ-absolutely continuous functions on $\mathbb{R}$, namely, the space of all functions of bounded φ-variation such that there exists a $\lambda > 0$ for which for all $\varepsilon > 0$ and for all bounded interval $I = [a, b] \subset \mathbb{R}$, there exists a $\delta > 0$ satisfying that

$$\sum_{i=1}^{m} \varphi\left(\lambda \left|f(\beta_i) - f(\alpha_i)\right|\right) < \varepsilon$$

holds for any collections of non-overlapping intervals $[\alpha_i, \beta_i] \subset I$, whenever

$$\sum_{i=1}^{m} \varphi(\beta_i - \alpha_i) < \delta.$$

Now that we have given some basic concepts, we can define our operator as follows.

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Then consider the following operator

$$\mathcal{T}_{n,v}(f; x) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k \in \mathbb{Z}} H_w\left(f\left(x - \tfrac{k}{w}\right)\right) l_{k,w} \quad (x \in \mathbb{R} \text{ and } n, v \in \mathbb{N}), \tag{3}$$

where $H_w : \mathbb{R} \to \mathbb{R}$, $H_w(0) = 0$ and H_w is a ψ-Lipschitz kernel ($\left|H_w(x) - H_w(y)\right| \leq K\psi\left(\left|x - y\right|\right)$ for all $x, y \in \mathbb{R}$). Here, ψ is a φ-function and $l_{k,w} \in l^1(\mathbb{Z})$ is a family of discrete kernels for every $w \in \mathbb{N}$. Then, it is not hard to see that (3) is well-defined for all real-valued bounded functions f.

In this work, by using φ-absolutely continuous functions, we investigate the existence of $\mu > 0$ such that the following limit holds

$$\lim_{n\to\infty} V_\varphi\left[\mu\left(\mathcal{T}_{n,v}(f) - f\right)\right] = 0 \text{ (uniformly in } v \in \mathbb{N}),$$

where $\mathcal{T}_{n,v}(f)$ is defined above.

Then, we will check the rate of approximation under some Lipschitz classes of φ-absolutely continuous functions. By using the relation between them, we also get the following result

$$\lim_{n\to\infty} V_\varphi\left[\mu\left(\mathcal{S}_{n,v}(f) - f\right)\right] = 0 \text{ (uniformly in } v \in \mathbb{N}),$$

where

$$\mathcal{S}_{n,v}(f;x) = \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} H_w\left(f\left(\frac{k}{w}\right)\right)\chi(wx-k), \tag{4}$$

namely, $\mathcal{S}_{n,v}(f)$ is $\mathcal{A}$−transform of nonlinear generalized sampling series. Furthermore, we give an application of Theorem 2.4 and Theorem 3.1 at the end of the paper.

2. Convergence in φ-Variation

In this section, we prove our main approximation theorem using convergence in φ-variation.

We require the following conditions:

(l_1) $\sup_{w\in\mathbb{N}} \left\|l_{k,w}\right\|_{l^1} = A < \infty$ for some constant $A > 0$,

(l_2) $\mathcal{A} - \lim\left(\sum_{k\in\mathbb{Z}} l_{k,w}\right) = 1$,

(l_3) $\exists r > 0$ such that $\mathcal{A} - \lim\left(\sum_{|k|\geq r} \left|l_{k,w}\right|\right) = 0$,

(h) For every $\gamma > 0$, there exists a $\lambda > 0$ such that, for every (proper) bounded interval $J \subset \mathbb{R}$, $\mathcal{A}$− $\lim \dfrac{V_\varphi[\lambda G_w, J]}{\varphi(\gamma m(J))} = 0$ uniformly in $J \subset \mathbb{R}$, where $G_w(u) = H_w(u) - u$ and $V_\varphi[\lambda G_w, J]$ denotes the φ-variation of λG_w on the interval J.

It can be easily seen that taking $\mathcal{A} = \{I\}$, the identity matrix, then (l_1) − (l_3) turn into (**A1**)-(**A2**) given in [2]. Here, condition (h) is a natural condition due to the nonlinearity of the kernel. For the examples of H_w in case of $\mathcal{A} = \{I\}$, see [1, 8]. At the end of the paper, we give a specific kernel satisfying (l_1) − (l_3) and (h).

The following growth condition on ψ corresponding to ψ−Lipschitz condition of H_w is also needed.

Definition 2.1. *Let φ, η, ψ be a φ-function. If for all $\gamma \in (0, 1)$, there exists a constant C_γ such that*

$$\varphi\left(C_\gamma \psi\left(\left|g\right|\right)\right) \leq \eta\left(\gamma\left|g\right|\right) \tag{5}$$

for every measurable function $g : \mathbb{R} \to \mathbb{R}$, then (φ, η, ψ) is called properly directed.

Throughout the paper, we will assume that (φ, η, ψ) is properly directed. In the nonlinear setting, this condition is common (see [1, 7, 16, 17, 38, 43]) and some examples of the triple (φ, η, ψ) can be found in [1].

Lemma 2.2. *Let $f \in BV_{\eta}(\mathbb{R})$. If (l_1) is satisfied, then $\mathcal{T}_{n,v}$ maps from $BV_{\eta}(\mathbb{R})$ into $BV_{\varphi}(\mathbb{R})$, namely, there exists a $\mu > 0$ such that*

$$V_{\varphi}\left[\mu \mathcal{T}_{n,v} f\right] \leq V_{\eta}\left[\lambda f\right]$$

holds, where $\lambda > 0$ is sufficiently small for which $V_{\eta}[\lambda f] < \infty$.

Proof. Let $\{x_i\}_{i\in\{1,\ldots,m\}}$ be an increasing sequence in $\mathbb{R}$. For all $\mu > 0$, it is not hard to see from Jensen's inequality that

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;x_i) - \mathcal{T}_{n,v}(f;x_{i-1})\right|\right) \\
&\leq \sum_{i=1}^{m} \varphi\left(\mu \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \left|H_w\left(f\left(x_i - \tfrac{k}{w}\right)\right) - H_w\left(f\left(x_{i-1} - \tfrac{k}{w}\right)\right)\right|\right) \\
&\leq \frac{1}{a_{n,v}} \sum_{i=1}^{m} \sum_{w=1}^{\infty} a_{nw}^{v} \varphi\left(\mu a_{n,v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \left|H_w\left(f\left(x_i - \tfrac{k}{w}\right)\right) - H_w\left(f\left(x_{i-1} - \tfrac{k}{w}\right)\right)\right|\right)
\end{aligned}$$

where $a_{n,v} = \sum_{w=1}^{\infty} a_{nw}^{v} < \infty$ by (a_3). Then, using Jensen's inequality one more time and taking supremum, we get the following inequality,

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;x_i) - \mathcal{T}_{n,v}(f;x_{i-1})\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \varphi\left(\mu a_{n,v} A \left|H_w\left(f\left(x_i - \tfrac{k}{w}\right)\right) - H_w\left(f\left(x_{i-1} - \tfrac{k}{w}\right)\right)\right|\right).
\end{aligned}$$

Since H_w is ψ-Lipschitz, then there holds

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;x_i) - \mathcal{T}_{n,v}(f;x_{i-1})\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \varphi\left(\mu a_{n,v} A K \psi\left(\left|f\left(x_i - \tfrac{k}{w}\right) - f\left(x_{i-1} - \tfrac{k}{w}\right)\right|\right)\right)
\end{aligned}$$

where K is ψ-Lipschitz constant of H_w. Now, from (5) for every $\lambda \in (0,1)$ for which $V_{\eta}[\lambda f] < \infty$, there exists a constant $C_{\lambda} \in (0,1)$ such that

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;x_i) - \mathcal{T}_{n,v}(f;x_{i-1})\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \eta\left(\lambda \left|f\left(x_i - \tfrac{k}{w}\right) - f\left(x_{i-1} - \tfrac{k}{w}\right)\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| V_{\eta}\left[\lambda f\left(\cdot - \tfrac{k}{w}\right)\right]
\end{aligned}$$

holds for all $0 < \mu \leq C_{\lambda}/(a_{n,v}AK)$. Since

$$V_{\eta}\left[\lambda f\left(\cdot - \tfrac{k}{w}\right)\right] = V_{\eta}\left[\lambda f\right],$$

we derive from (l_1) that

$$\begin{aligned}
\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;x_i) - \mathcal{T}_{n,v}(f;x_{i-1})\right|\right) &\leq \frac{V_{\eta}[\lambda f]}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \\
&\leq V_{\eta}[\lambda f].
\end{aligned}$$

Consequently, if we take supremum over $\{x_i\}_{i\in\{1,\ldots,m\}}$, the proof is done. □

Lemma 2.3. *Let $f \in AC_\eta(\mathbb{R})$. If (l_1) is satisfied, then $\mathcal{T}_{n,v}(f) \in AC_\varphi(\mathbb{R})$ for all $n, v \in \mathbb{N}$.*

Proof. Assume that $\varepsilon > 0$ be given and let $\delta := \delta(\varepsilon) > 0$ corresponds to η-absolute continuity of f where $\{[\alpha_i, \beta_i]\}_{i=1}^m$ be a finite nonoverlapping intervals of $I = [a, b] \subset \mathbb{R}$ such that $\sum_{i=1}^m \varphi(\beta_i - \alpha_i) < \delta$. Then, applying Jensen's inequality we may clearly see that

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;\beta_i) - \mathcal{T}_{n,v}(f;\alpha_i)\right|\right) \\
&\leq \sum_{i=1}^{m} \varphi\left(\mu \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \left|H_w\left(f\left(\beta_i - \tfrac{k}{w}\right)\right) - H_w\left(f\left(\alpha_i - \tfrac{k}{w}\right)\right)\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \varphi\left(\mu a_{n,v} A \left|H_w\left(f\left(\beta_i - \tfrac{k}{w}\right)\right) - H_w\left(f\left(\alpha_i - \tfrac{k}{w}\right)\right)\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \varphi\left(\mu a_{n,v} A K \psi \left|f\left(\beta_i - \tfrac{k}{w}\right) - f\left(\alpha_i - \tfrac{k}{w}\right)\right|\right).
\end{aligned}$$

Since (φ, η, ψ) is properly directed, then for every $\lambda \in (0, 1)$ there exists a $C_\lambda > 0$ such that

$$\begin{aligned}
&\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;\beta_i) - \mathcal{T}_{n,v}(f;\alpha_i)\right|\right) \\
&\leq \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \sum_{i=1}^{m} \eta\left(\lambda \left|f\left(\beta_i - \tfrac{k}{w}\right) - f\left(\alpha_i - \tfrac{k}{w}\right)\right|\right)
\end{aligned}$$

holds for all $0 < \mu \leq C_\lambda/(a_{n,v}AK)$. Moreover, seeing that f is η-absolutely continuous, then there exists a $\gamma > 0$ such that

$$\sum_{i=1}^{m} \eta\left(\gamma \left|f\left(\beta_i - \tfrac{k}{w}\right) - f\left(\alpha_i - \tfrac{k}{w}\right)\right|\right) < \varepsilon$$

whenever

$$\sum_{i=1}^{m} \eta\left(\left(\beta_i - \tfrac{k}{w}\right) - \left(\alpha_i - \tfrac{k}{w}\right)\right) = \sum_{i=1}^{m} \eta(\beta_i - \alpha_i) < \delta.$$

Using the previous expression together with (l_1) and (a_3), we finally get

$$\begin{aligned}
\sum_{i=1}^{m} \varphi\left(\mu \left|\mathcal{T}_{n,v}(f;\beta_i) - \mathcal{T}_{n,v}(f;\alpha_i)\right|\right) &< \frac{1}{a_{n,v}A} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \varepsilon \\
&\leq \varepsilon
\end{aligned}$$

for all $0 < \lambda \leq \gamma$. □

Now, we state our main approximation theorem.

Theorem 2.4. *Assume that $(l_1) - (l_3)$ and (h) hold. Then, there exists a $\mu > 0$ such that for a given $f \in AC_\varphi(\mathbb{R}) \cap BV_\eta(\mathbb{R})$, we have*

$$\lim_{n\to\infty} V_\varphi\left[\mu\left(\mathcal{T}_{n,v}(f) - f\right)\right] = 0 \text{ (uniformly in } v \in \mathbb{N}). \tag{6}$$

Proof. Let $\{x_i\}_{i\in\{1,\dots,m\}}$ be an increasing sequence in $\mathbb{R}$. Then, for all $\mu > 0$

$$\begin{aligned}
I &= \sum_{i=1}^{m}\varphi\left(\mu\left|\mathcal{T}_{n,\upsilon}\left(f;x_i\right)-f\left(x_i\right)-\mathcal{T}_{n,\upsilon}\left(f;x_{i-1}\right)+f\left(x_{i-1}\right)\right|\right)\\
&= \sum_{i=1}^{m}\varphi\left(\mu\left|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}l_{k,w}\left\{H_w\left(f\left(x_i-\tfrac{k}{w}\right)\right)-f\left(x_i-\tfrac{k}{w}\right)\right.\right.\right.\\
&\left.-H_w\left(f\left(x_{i-1}-\tfrac{k}{w}\right)\right)+f\left(x_{i-1}-\tfrac{k}{w}\right)\right\}\\
&+\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}l_{k,w}\left\{f\left(x_i-\tfrac{k}{w}\right)-f\left(x_i\right)-f\left(x_{i-1}-\tfrac{k}{w}\right)+f\left(x_{i-1}\right)\right\}\\
&\left.\left.+\left\{f\left(x_i\right)-f\left(x_{i-1}\right)\right\}\left(\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}l_{k,w}-1\right)\right|\right)
\end{aligned}$$

holds. Now, using the convexity of φ, one can observe the following,

$$\begin{aligned}
I &\le \frac{1}{3}\sum_{i=1}^{m}\varphi\left(3\mu\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|\left|H_w\left(f\left(x_i-\tfrac{k}{w}\right)\right)-f\left(x_i-\tfrac{k}{w}\right)\right.\right.\\
&\left.\left.-H_w\left(f\left(x_{i-1}-\tfrac{k}{w}\right)\right)+f\left(x_{i-1}-\tfrac{k}{w}\right)\right|\right)\\
&+\frac{1}{3}\sum_{i=1}^{m}\varphi\left(3\mu\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|\left|f\left(x_i-\tfrac{k}{w}\right)-f\left(x_i\right)-f\left(x_{i-1}-\tfrac{k}{w}\right)+f\left(x_{i-1}\right)\right|\right)\\
&+\frac{1}{3}\sum_{i=1}^{m}\varphi\left(3\mu\left|f\left(x_i\right)-f\left(x_{i-1}\right)\right|\left|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}l_{k,w}-1\right|\right)\\
&=: I_1+I_2+I_3.
\end{aligned}$$

In I_1, using two times Jensen's inequality we immediately get

$$\begin{aligned}
I_1 &\le \frac{1}{3a_{n,\upsilon}A}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|\sum_{i=1}^{m}\varphi\left(3\mu a_{n,\upsilon}A\left|H_w\left(f\left(x_i-\tfrac{k}{w}\right)\right)-f\left(x_i-\tfrac{k}{w}\right)\right.\right.\\
&\left.\left.-H_w\left(f\left(x_{i-1}-\tfrac{k}{w}\right)\right)+f\left(x_{i-1}-\tfrac{k}{w}\right)\right|\right).
\end{aligned}$$

It is known from (a_3) that $a_{n,\upsilon} := \sum_{w=1}^{\infty}a_{nw}^{\upsilon} \le M$ for sufficiently large $n \in \mathbb{N}$. Then, from the convexity of φ

$$\begin{aligned}
I_1 &\le \frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|\sum_{i=1}^{m}\varphi\left(3\mu MA\left|H_w\left(f\left(x_i-\tfrac{k}{w}\right)\right)-f\left(x_i-\tfrac{k}{w}\right)\right.\right.\\
&\left.\left.-H_w\left(f\left(x_{i-1}-\tfrac{k}{w}\right)\right)+f\left(x_{i-1}-\tfrac{k}{w}\right)\right|\right)\\
&\le \frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu MA\left(H_w\left(f\left(\cdot-\tfrac{k}{w}\right)\right)-f\left(\cdot-\tfrac{k}{w}\right)\right)\right]
\end{aligned}$$

yields. Now, using the fact that

$$V_{\varphi}\left[3\mu MA\left(H_w\left(f\left(\cdot-\tfrac{k}{w}\right)\right)-f\left(\cdot-\tfrac{k}{w}\right)\right)\right]=V_{\varphi}\left[3\mu MA\left(H_w\left(f\right)-f\right)\right],$$

then there holds

$$I_1 \le \frac{1}{3M}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}V_{\varphi}\left[3\mu MA\left(H_w\left(f\right)-f\right)\right].$$

Considering (h) together with Lemma 1 in [8], we observe that for all $\gamma > 0$, there exists a $\lambda > 0$ such that $\forall\varepsilon > 0$, there exists a number n_0 satisfying that

$$I_1 < \frac{V_{\varphi}\left[\gamma f\right]}{3M}\varepsilon$$

for all $n > n_0$ and $0 < \mu \leq \frac{\lambda}{3MA}$.

About I_2, using the convexity of φ, Jensen's inequality and (a_3), there holds

$$\begin{aligned} & I_2 \\ & \leq \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| \varphi\left(3\mu MA \left| f\left(x_i - \tfrac{k}{w}\right) - f(x_i) - f\left(x_{i-1} - \tfrac{k}{w}\right) + f(x_{i-1})\right|\right) \\ & \leq \frac{1}{3MA} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| V_{\varphi}\left[3\mu MA\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right] \end{aligned} \tag{7}$$

for sufficiently large $n \in \mathbb{N}$. Here, one can observe the φ-modulus of smoothness of $f \in AC_{\varphi}(\mathbb{R})$ by Subsection 2.4. in [41], that is, if φ satisfies (+), then $\lim_{\delta\to 0^+} \sup_{|t|<\delta} V_{\varphi}\left[\lambda\left(f(\cdot - t) - f(\cdot)\right)\right] = 0$ for some $\lambda > 0$ if and only if $f \in AC_{\varphi}(\mathbb{R})$. So, one can find a $\lambda_1 > 0$ such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$V_{\varphi}\left[\lambda_1\left(f(\cdot - t) - f(\cdot)\right)\right] < \varepsilon \tag{8}$$

whenever $|t| < \delta$. Now, from (2) we can divide the sum in (7) as follows

$$\begin{aligned} I_2 & \leq \frac{1}{3MA} \sum_{w=1}^{w_1} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| V_{\varphi}\left[3\mu MA\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right] \\ & + \frac{1}{3MA} \sum_{w=w_1+1}^{\infty} a_{nw}^{v} \sum_{|k|<r} \left|l_{k,w}\right| V_{\varphi}\left[3\mu MA\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right] \\ & + \frac{1}{3MA} \sum_{w=w_1+1}^{\infty} a_{nw}^{v} \sum_{|k|\geq r} \left|l_{k,w}\right| V_{\varphi}\left[3\mu MA\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right] \\ & := I_2^1 + I_2^2 + I_2^3 \end{aligned}$$

where $r > 0$ is given in (l_3) and w_1 is such that

$$\frac{k}{w} < \frac{r}{w} < \delta$$

for all $w > w_1$.

In I_2^1, since $V_{\varphi}\left[6\mu MAf\left(\cdot - \frac{k}{w}\right)\right] = V_{\varphi}\left[6\mu MAf\right]$, it can easily be observed from (l_1) that

$$\begin{aligned} I_2^1 & \leq \frac{1}{3MA} \sum_{w=1}^{w_1} a_{nw}^{v} \sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| V_{\varphi}\left[6\mu MAf\right] \\ & \leq \frac{1}{3M} \sum_{w=1}^{w_1} a_{nw}^{v} V_{\varphi}\left[6\mu MAf\right]. \end{aligned}$$

Then, there holds

$$I_2^1 < \frac{w_1 V_{\varphi}\left[6\mu MAf\right]}{3M} \varepsilon$$

for all $0 < \mu \leq \tilde{\mu}/(6MA)$ and for sufficiently large $n \in \mathbb{N}$.

From (8), (l_1) and (a_3) we obtain

$$I_2^2 < \frac{\varepsilon}{3}$$

for all $0 < \mu \leq \lambda_1/(6MA)$.

From (l_3), we get

$$I_2^3 \leq \frac{V_{\varphi}\left[6\mu MAf\right]}{3MA} \varepsilon$$

for sufficiently large $n \in \mathbb{N}$.

On the other hand, since $\left| \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} l_{k,w} - 1 \right| < 1$ for sufficiently large $n \in \mathbb{N}$, by the convexity of φ

$$I_3 \leq \frac{1}{3} \sum_{i=1}^{m} \varphi\left(3\mu \left| f(x_i) - f(x_{i-1}) \right|\right) \left| \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} l_{k,w} - 1 \right|$$
$$\leq \frac{1}{3} V_{\varphi}[3\mu f] \left| \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} l_{k,w} - 1 \right|$$

holds. Then from (l_2), we get

$$I_3 < \frac{V_{\varphi}[3\mu f]}{3} \varepsilon$$

for sufficiently large $n \in \mathbb{N}$. Finally, taking supremum over $\{x_i\}_{i\in\{1,\dots,m\}}$ in the first inequality, we complete the proof. □

3. Order of Approximation

In this section, we examine the order of approximation. For this reason, we first consider the following Lipschitz class

$$V_{\varphi}Lip(\alpha) = \left\{ f \in AC_{\varphi}(\mathbb{R}) : \exists \rho > 0 \text{ s.t. } V_{\varphi}\left[\rho \left| f(\cdot - t) - f(\cdot) \right|\right] = O(|t|^{\alpha}) \text{ as } t \to 0 \right\}$$

for any $\alpha > 0$ (see also [3]).

For a given nonnegative regular method $\mathcal{A} = \{(a_{nw}^{v})\}_{v\in\mathbb{N}}$ and $\alpha > 0$, we take into account the following orders of approximations:

$$\left(\sum_{w=1}^{\infty} a_{nw}^{v} \sum_{k\in\mathbb{Z}} l_{k,w} - 1 \right) = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v), \tag{9}$$

there exists a number $\bar{r} > 0$ such that

$$\sum_{w=1}^{\infty} a_{nw}^{v} \sum_{|k|<\bar{r}} \frac{1}{w^{\alpha}} = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v), \tag{10}$$

$$\sum_{w=1}^{\infty} a_{nw}^{v} \sum_{|k|\geq\bar{r}} \left| l_{k,w} \right| = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v) \tag{11}$$

and for each $w \in \mathbb{N}$,

$$a_{nw}^{v} = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v). \tag{12}$$

Theorem 3.1. *Assume that (9)-(12) and (l_1) hold. Assume further that for every $\gamma > 0$, there exists a $\lambda > 0$ such that*

$$\sum_{w=1}^{\infty} a_{nw}^{v} \frac{V_{\varphi}[\lambda G_w, J]}{\varphi(\gamma m(J))} = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v \text{ and} \tag{13}$$
uniformly in every proper bounded interval $J \subset \mathbb{R}$).

Then, there exists a $\mu > 0$ such that

$$V_{\varphi}[\mu(\mathcal{T}_{n,v}(f) - f)] = O(n^{-\alpha}) \text{ as } n \to \infty \text{ (uniformly in } v)$$

for all $f \in V_{\varphi}Lip(\alpha) \cap BV_{\eta}(\mathbb{R})$.

Proof. By the proof of Theorem 2.4, we may easily obtain the following inequality

$$\begin{aligned} V_{\varphi}\left[\mu\left(\mathcal{T}_{n,\upsilon}(f)-f\right)\right] &\le \frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu AM\left(H_w(f)-f\right)\right] \\ &+\frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu AM(f(\cdot-\tfrac{k}{w})-f(\cdot))\right] \\ &+\frac{V_{\varphi}\left[3\mu f\right]}{3}\left|\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{k\in\mathbb{Z}}l_{k,w}-1\right| \\ &=: J_1+J_2+J_3 \end{aligned}$$

for sufficiently large $n\in\mathbb{N}$. Considering (13) in [8], there exists a constant $L>0$ such that

$$\begin{aligned} J_1 &= \frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}V_{\varphi}\left[3\mu AM\left(H_w(f)-f\right)\right]\sum_{k\in\mathbb{Z}}\left|l_{k,w}\right| \\ &\le \frac{L}{3M}V_{\varphi}\left[\gamma f\right]n^{-\alpha} \\ &= O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon) \end{aligned}$$

for sufficiently small $\mu>0$.

In J_2, since $f\in V_{\varphi}Lip(\alpha)$, there exist $\rho,N,\delta>0$ s.t. $V_{\varphi}\left[\rho\left|f(\cdot-t)-f(\cdot)\right|\right]\le N|t|^{\alpha}$ if $|t|<\delta$. Moreover, for a given $\bar{r}>0$, we can find a number w' such that

$$\frac{k}{w}<\frac{\bar{r}}{w}<\delta$$

for every $w>w'$. Taking these arguments into account, we divide J_2 as follows,

$$\begin{aligned} J_2 &= \frac{1}{3MA}\sum_{w=1}^{w'}a_{nw}^{\upsilon}\sum_{|k|<\bar{r}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu AM\left(f\left(\cdot-\tfrac{k}{w}\right)-f(\cdot)\right)\right] \\ &+\frac{1}{3MA}\sum_{w=w'+1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|<\bar{r}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu AM\left(f\left(\cdot-\tfrac{k}{w}\right)-f(\cdot)\right)\right] \\ &+\frac{1}{3MA}\sum_{w=1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|\ge\bar{r}}\left|l_{k,w}\right|V_{\varphi}\left[3\mu AM\left(f\left(\cdot-\tfrac{k}{w}\right)-f(\cdot)\right)\right] \\ &=: J_2^1+J_2^2+J_2^3. \end{aligned}$$

Then, it follows from (10) that

$$\begin{aligned} J_2^2 &\le \frac{N}{3MA}\sum_{w=w'+1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|<\bar{r}}\left|l_{k,w}\right|\left|\tfrac{k}{w}\right|^{\alpha} \\ &\le \frac{N\bar{r}^{\alpha}}{3M}\sum_{w=w'+1}^{\infty}a_{nw}^{\upsilon}\sum_{|k|<\bar{r}}\frac{1}{w^{\alpha}} \\ &= O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon) \end{aligned}$$

for all $0<\mu\le\frac{\rho}{3MA}$. On the other hand, for J_2^1 it is not hard to see from (2) that

$$J_2^1\le\frac{1}{3M}\sum_{w=1}^{w'}a_{nw}^{\upsilon}V_{\varphi}\left[6\mu AMf\right]$$

and therefore, from (12)

$$J_2^1=O\left(n^{-\alpha}\right) \text{ as } n\to\infty \text{ (uniformly in } \upsilon)$$

holds. About J_2^3, from (2) and (11), we observe the following

$$\begin{aligned} J_2^3 &\leq \frac{V_\varphi [6\mu AMf]}{3MA} \sum_{w=1}^{\infty} a_{nw}^v \sum_{|k|\geq \tilde{r}} \left| l_{k,w} \right| \\ &= O\left(n^{-\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } v\text{)}. \end{aligned}$$

Finally, directly from (9) we get

$$J_3 = O\left(n^{-\alpha}\right) \text{ as } n \to \infty \text{ (uniformly in } v\text{)}.$$

□

Now, we investigate a special case of the operator (3), where $l_{k,w} \equiv \chi(k)$ and $\chi : \mathbb{R} \to \mathbb{R}$, namely, $l_{k,w}$ is not depending on w. Then, (3) reduces to

$$\bar{\mathcal{T}}_{n,v}(f;x) = \sum_{w=1}^{\infty} a_{nw}^v \sum_{k\in\mathbb{Z}} H_w \left(f\left(x - \tfrac{k}{w} \right) \right) \chi(k),$$

which is (in some cases) equivalent to $\mathcal{A}$-transform of nonlinear generalized sampling series given in (4)

Under these considerations, (l_1) and (l_2) turn into the following assumptions

(l_1') $\chi \in l^1(\mathbb{Z})$

(l_2') $\sum_{k\in\mathbb{Z}} \chi(k) = 1$

where on the other hand (l_3) is clearly not satisfied. But these two conditions are still enough to verify the following theorem.

Theorem 3.2. *Let $f \in AC_\varphi(\mathbb{R}) \cap BV_\eta(\mathbb{R})$. If (l_1'), (l_2') and (h) hold, then there exists a $\mu > 0$ such that*

$$\lim_{n\to\infty} V_\varphi \left[\mu \left(\bar{\mathcal{T}}_{n,v}(f) - f \right) \right] = 0 \text{ (uniformly in } v \in \mathbb{N}\text{)}.$$

Proof. Considering (l_2') in the proof of Theorem 2.4, then for every $\mu > 0$

$$\begin{aligned} V_\varphi \left[\mu \left(\bar{\mathcal{T}}_{n,v}(f) - f \right) \right] &\leq \frac{1}{3M\bar{A}} \sum_{w=1}^{\infty} a_{nw}^v \sum_{k\in\mathbb{Z}} |\chi(k)| V_\varphi [3\mu M\bar{A}(H_w \circ f - f)] \\ &+ \frac{1}{3M\bar{A}} \sum_{w=1}^{\infty} a_{nw}^v \sum_{k\in\mathbb{Z}} |\chi(k)| V_\varphi \left[3\mu M\bar{A} \left(f\left(\cdot - \tfrac{k}{w} \right) - f(\cdot) \right) \right] \\ &+ \frac{1}{3} V_\varphi [3\mu f] \left| \sum_{w=1}^{\infty} a_{nw}^v - 1 \right| \\ &=: L_1 + L_2 + L_3 \end{aligned}$$

holds, where $\bar{A} = \|\chi\|_{l^1}$. From (h), (l_1'), and Lemma 1 in [8], one can clearly see that

$$L_1 < \frac{V_\varphi [\gamma f]}{3M} \varepsilon$$

for sufficiently large $n \in \mathbb{N}$ and for all $0 < \mu \leq \lambda/(3M\bar{A})$ where λ and γ correspond to Lemma 1 in [8]. On the other hand, since $\chi \in l^1(\mathbb{Z})$, for all $\varepsilon > 0$ there exists a $\tilde{r} > 0$ such that

$$\sum_{|k|\geq \tilde{r}} |\chi(k)| < \varepsilon.$$

Hence, if we divide L_2 into two parts as follows,

$$L_2 = \frac{1}{3M\bar{A}} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{|k| \geq \tilde{r}} |\chi(k)| V_\varphi \left[3\mu M\bar{A}\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right]$$
$$+ \frac{1}{3M\bar{A}} \sum_{w=1}^{\infty} a_{nw}^{v} \sum_{|k| < \tilde{r}} |\chi(k)| V_\varphi \left[3\mu M\bar{A}\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right]$$
$$=: L_2^1 + L_2^2$$

then, there holds

$$L_2^1 < \frac{V_\varphi \left[6\mu M\bar{A} f\right]}{3\bar{A}} \varepsilon$$

For L_2^2, using φ-modulus of smoothness of the function $f \in AC_\varphi(\mathbb{R})$, we obviously see that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\tfrac{k}{w} < \tfrac{\tilde{r}}{w} < \delta$$

for all $w > \tilde{w}$, which implies

$$V_\varphi \left[3\mu M A (f(\cdot - t) - f(\cdot))\right] < \varepsilon.$$

Then, dividing L_2^2 as follows,

$$L_2^2 = \frac{1}{3M\bar{A}} \sum_{w=1}^{\tilde{w}} a_{nw}^{v} \sum_{|k| < \tilde{r}} |\chi(k)| V_\varphi \left[3\mu M\bar{A}\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right]$$
$$+ \frac{1}{3M\bar{A}} \sum_{w=\tilde{w}+1}^{\infty} a_{nw}^{v} \sum_{|k| < \tilde{r}} |\chi(k)| V_\varphi \left[3\mu M\bar{A}\left(f\left(\cdot - \tfrac{k}{w}\right) - f(\cdot)\right)\right]$$

we may easily obtain

$$L_2^2 < \left(\frac{\tilde{w} V_\varphi \left[6\mu M\bar{A} f\right]}{3M} + \frac{1}{3}\right) \varepsilon.$$

Finally, using (a_2) we conclude

$$L_3 < \frac{V_\varphi \left[3\mu f\right]}{3} \varepsilon$$

for sufficiently large $n \in \mathbb{N}$, which completes the proof. □

Remark 3.3. *Note that, the operators $\bar{\mathcal{T}}$ and $\mathcal{S}$ are different in general but, in some cases, they coincide.*

Corollary 3.4. *Assume that $f \in B_{\pi w}^1(\mathbb{R}) \cap BV_\eta(\mathbb{R})$ and $\psi\left(\left|f\right|\right) \in B_{\pi w}^1(\mathbb{R})$ (the Paley-Wiener Space $B_{\pi w}^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}) : f$ has an extension to whole $\mathbb{C}$ s.t. $\left|f(z)\right| \leq \exp(\pi w |z|) \left\|f\right\|$ for every $z \in \mathbb{C}\}$) for some $w > 0$, where $\|\cdot\|$ denotes supremum norm. If $\chi \in B_\pi^\infty(\mathbb{R})$ and $\left(l_1'\right)$, $\left(l_2'\right)$, (h) are satisfied, then there exists a $\mu > 0$ such that*

$$\lim_{n\to\infty} V_\varphi \left[\mu \left(\mathcal{S}_{n,v}(f) - f\right)\right] = 0 \text{ (uniformly in } v \in \mathbb{N}).$$

Proof. First of all, we should say that since $\left|H_w(f)\right| \leq K\psi\left(\left|f\right|\right)$ and $\psi\left(\left|f\right|\right) \in B_{\pi w}^1(\mathbb{R})$, then $H_w(f) \in B_{\pi w}^1(\mathbb{R})$. From Proposition 4.3. in [2] and (+), we may easily see that $B_{\pi w}^1(\mathbb{R}) \subset AC_\varphi(\mathbb{R})$. Therefore, using the similar arguments on Lemma 4.2. in [2], we deduce that

$$\mathcal{S}_{n,v}(f) = \bar{\mathcal{T}}_{n,v}(f)$$

for all $n, v \in \mathbb{N}$. Consequently, by the Theorem 3.2 the proof completes. □

An example of $\chi \in B_\pi^\infty(\mathbb{R})$ satisfying $\left(l_1'\right)$ and $\left(l_2'\right)$ can be found in Example 4.5. in [2].

4. Conclusions and Applications

We remark that operator (3) can be written as

$$\mathcal{T}_{n,v}(f;x)=\sum_{w=1}^{\infty}a_{nw}^{v}T_{w}(f;x)$$

where $T_w(f;x)$ is introduced by

$$T_w(f;x)=\sum_{k\in\mathbb{Z}}H_w\left(f\left(x-\frac{k}{w}\right)\right)l_{k,w}.$$

Using certain methods, some significant results of Theorem 2.4 are given below:

- If we take $\mathcal{A}=\{C_1\}$, Cesàro matrix [30], where $C_1=[c_{nw}]$ is such that

$$c_{nw}=\begin{cases}\frac{1}{n}; & \text{if } 1\leq w\leq n\\ 0; & \text{otherwise,}\end{cases}$$

  then we get

$$\lim_{n\to\infty}V_{\varphi}\left[\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right]=0$$

  for all $f\in AC_{\varphi}(\mathbb{R})$.

- Putting $\mathcal{A}=\mathcal{F}$, the almost convergence matrix [36], where $\mathcal{F}=\{[c_{nw}^{v}]\}$ is such that

$$c_{nw}^{v}=\begin{cases}\frac{1}{n}; & \text{if } v\leq w\leq n+v-1\\ 0; & \text{otherwise,}\end{cases}$$

  then we get

$$\lim_{n\to\infty}V_{\varphi}\left[\frac{T_v(f)+T_{v+1}(f)+\cdots+T_{n+v-1}(f)}{n}-f\right]=0 \text{ uniformly in } v$$

  for all $f\in AC_{\varphi}(\mathbb{R})$.

- If $\mathcal{A}=\{I\}$, the identity matrix, then we get

$$\lim_{n\to\infty}V_{\varphi}\left[T_n(f)-f\right]=0,$$

  where T_n is nonlinear form of (1).

- If one take $H_w(u)=u$, then T_n reduces to linear case given in (1) and the previous estimations hold for the operator (1).

- On the other hand, all the previous results are still valid for the generalized sampling series $\mathcal{S}_{n,v}(f)$ given in (4).

Now, we will investigate the existence of kernels which satisfy $(l_1)-(l_3)$, (h) and conditions (9)–(13). Let $\mathcal{A}=\mathcal{F}=\{F^v\}$, $\alpha=1/2$ and $l_{k,w}$, H_w and ψ are defined by

$$l_{k,w}:=\begin{cases}\dfrac{1}{2^{w|k|-1}}; & w=m^2\ (m\in\mathbb{N})\\ \dfrac{2^w-1}{2^{w|k|}\left(2^w+1\right)}; & w\neq m^2\ (m\in\mathbb{N}),\end{cases}$$

$H_w(u) := u + \tanh\left(\frac{u}{w}\right)$ and $\psi(|u|) := |u|$. Then, if $w = m^2$ $(m \in \mathbb{N})$, we have

$$\sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| = 2\left(\frac{2^w+1}{2^w-1}\right) \le 6$$

and if $w \neq m^2$, we have

$$\sum_{k\in\mathbb{Z}} \left|l_{k,w}\right| = 1,$$

which implies (l_1) for $A = 6$.

For (l_2) and (9), consider the following inequality

$$\begin{aligned}\left|\sum_{w=v}^{n+v-1}\frac{1}{n}\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| &\le \sum_{w=v}^{n+v-1}\frac{1}{n}\left|\sum_{k\in\mathbb{Z}} l_{k,w} - 1\right| \\ &= \sum_{w=v,w=m^2}^{n+v-1}\frac{1}{n}\left|2\left(\frac{2^w+1}{2^w-1}\right)-1\right| \\ &\le \sum_{w=v,w=m^2}^{n+v-1}\frac{5}{n} \\ &\le \frac{5\left(\sqrt{n+v-1}-\sqrt{v}+1\right)}{n} \\ &= \frac{5(n-1)}{n\left(\sqrt{n+v-1}+\sqrt{v}\right)}+\frac{5}{n} \\ &\le \frac{5}{\sqrt{n+v-1}+\sqrt{v}}+\frac{5}{n} \\ &\le \frac{10}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } v\text{)},\end{aligned}$$

which proves (l_2) and (9).

For (l_3) and (11), if $w = m^2$, then

$$\sum_{|k|\ge r} \left|l_{k,w}\right| = 4\left(\frac{2^w}{2^w-1}\right)\frac{1}{2^{wr}}$$

and if $w \neq m^2$,

$$\sum_{|k|\ge r} \left|l_{k,w}\right| = 2\left(\frac{2^w-1}{2^w+1}\right)\left(\frac{2^w}{2^w-1}\right)\frac{1}{2^{wr}}$$

hold. Therefore, we get the following expression

$$\begin{aligned}\sum_{w=v}^{n+v-1}\frac{1}{n}\sum_{|k|\ge r}\left|l_{k,w}\right| &\le \frac{4}{n}\sum_{w=v}^{n+v-1}\left(\frac{2^w}{2^w-1}\right)\frac{1}{2^{wr}} \\ &\le \frac{8}{n}\sum_{w=v}^{n+v-1}\frac{1}{2^{wr}} \\ &\le \frac{8}{n}\sum_{w=0}^{\infty}\frac{1}{2^{wr}} \\ &= \frac{8}{n}\left(\frac{2^r}{2^r-1}\right),\end{aligned}$$

which shows (l_3) is satisfied for $r = 1$. Furthermore, by the fact that for all $r \geq 1$

$$\left(\frac{2^r}{2^r - 1}\right) \leq 2$$

and so, we conclude

$$\begin{aligned}\sum_{w=v}^{n+v-1} \frac{1}{n} \sum_{|k|\geq r} \left|l_{k,w}\right| &\leq \frac{16}{n} \leq \frac{16}{\sqrt{n}} \\ &= O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } v).\end{aligned}$$

For the condition (10), we may clearly get

$$\begin{aligned}\frac{1}{n}\sum_{w=v}^{n+v-1} \frac{1}{\sqrt{w}} &\leq \frac{2\left(\sqrt{n+v-1} - \sqrt{v}\right)}{n} \\ &\leq \frac{2(n-1)}{n\left(\sqrt{n+v-1} + \sqrt{v}\right)} \\ &\leq \frac{2}{\left(\sqrt{n}\right)} \\ &= O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } v).\end{aligned} \tag{14}$$

Moreover, by the definition of $\mathcal{F}$, we obtain the following

$$\begin{aligned}c_{nw}^{v} &\leq \frac{1}{n} \leq \frac{1}{\sqrt{n}} \\ &= O\left(\frac{1}{\sqrt{n}}\right) \text{ (uniformly in } v).\end{aligned}$$

On the other hand, by the definition of H_w, it is clear that $H_w(0) = 0$ and H_w is 1-Lipschitz (see also Figure 1). In addition, $G_w(u) = H_w(u) - u = \tanh\left(\frac{u}{w}\right)$ is an increasing function and hence choosing $\lambda = \gamma$ and $J = [a, b]$ we have the following equality

$$\frac{V_\varphi\left[\gamma G_w, J\right]}{\varphi\left(\gamma m(J)\right)} = \frac{\varphi\left(\gamma\left(G_w(b) - G_w(a)\right)\right)}{\varphi\left(\gamma m(J)\right)}.$$

Furthermore, by the convexity of φ

$$\begin{aligned}\frac{V_\varphi\left[\gamma G_w, J\right]}{\varphi\left(\gamma m(J)\right)} &\leq \frac{\varphi\left(\gamma\left(\frac{b}{w} - \frac{a}{w}\right)\right)}{\varphi\left(\gamma m(J)\right)} \\ &\leq \frac{1}{w}\frac{\varphi\left(\gamma(b-a)\right)}{\varphi\left(\gamma m(J)\right)} \\ &= \frac{1}{w}\end{aligned}$$

holds, where $1/w \to 0$ as $w \to \infty$. Then we obtain from (14) that

$$\begin{aligned}\frac{1}{n}\sum_{w=v}^{n+v-1} \frac{1}{w} &\leq \frac{1}{n}\sum_{w=v}^{n+v-1} \frac{1}{\sqrt{w}} \\ &= O\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \to \infty \text{ (uniformly in } v)\end{aligned}$$

which verifies (13) and (h).

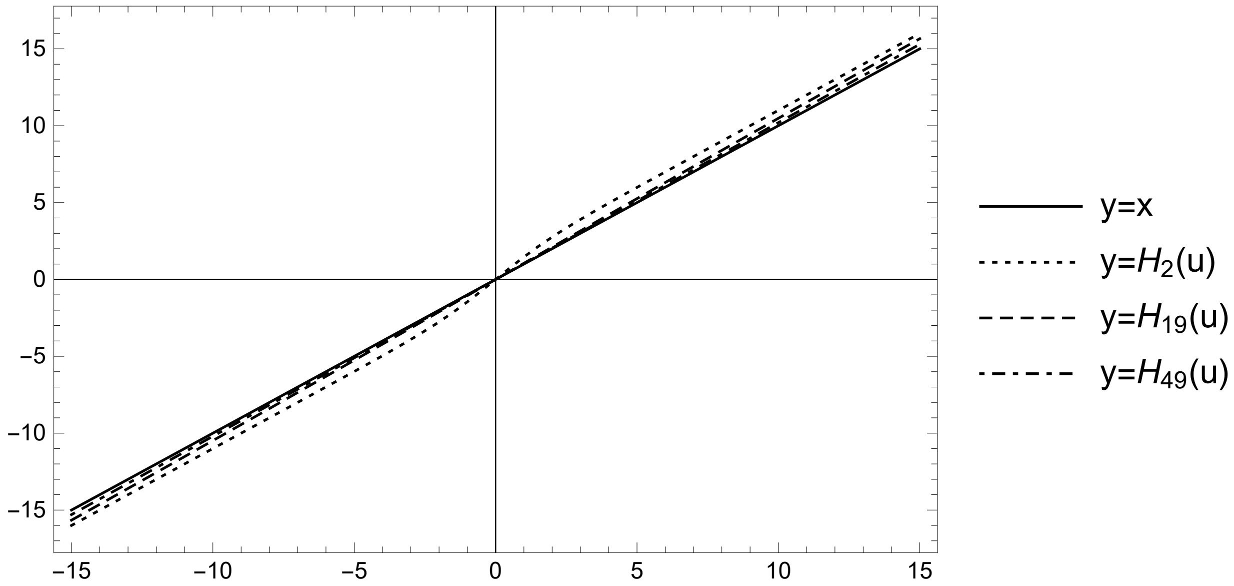

Figure 1: The kernel function H_w

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