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Generalizations of Some Hardy-Littlewood-Pólya Type Inequalities and Related Results

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Abstract. In this paper, we use an identity of Fink and present some interesting identities and inequalities for real valued functions and r-convex functions respectively. We also obtain generalizations of some Hardy-Littlewood-Pólya type inequalities. In addition, we use the Čebyšev functional and the Grüss type inequalities and find the bounds for the remainder in the obtained identities. Finally, we present an interesting result related to the Ostrowski type inequalities.

1. Introduction

A sequence $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is non-increasing ($\searrow$) in weighted mean (WM) (see [6]) if

$$\frac{1}{Q_m}\sum_{i=1}^{m} q_i a_i \geq \frac{1}{Q_{m+1}}\sum_{i=1}^{m+1} q_i a_i, \quad m \in \mathbb{N}, \tag{1}$$

where $a_i, q_i \in \mathbb{R}$ $(i \in \mathbb{N})$ such that $q_k > 0$ $(1 \leq k \leq i)$ and $Q_i := \sum_{k=1}^{i} q_k$ $(i \in \mathbb{N})$.
If (1) holds in the reversed direction, then the sequence $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ is called non-decreasing ($\nearrow$) in WM.

The following inequality is given in the renowned Hardy-Littlewood-Pólya book (see [4, Theorem 134]):

Theorem 1.1. *If f is a convex and continuous function defined on $[0,\infty)$ and a_i, $i \in \mathbb{N}$, are non-negative and $\searrow$, then*

$$f\left(\sum_{i=1}^{m} a_i\right) \geq f(0) + \sum_{i=1}^{m}\left[f(ia_i) - f((i-1)a_i)\right]. \tag{2}$$

If f' is a strictly increasing function, there is an equality only when a_i are equal up to a certain point and then zero.

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A weighted case of (2) was proved by Bennett [1] for power functions $f(x) = x^p$ in the following way: if $a_i \in [0,\infty)$ and a_i are $\searrow$ $(1 \le i \le m)$ and $q_i \in [0,\infty)$ for all $i \in \{1,\ldots,m\}$ such that $Q_i = \sum_{k=1}^{i} q_k$ $(1 \le i \le m)$, then for $p \in (1,\infty)$ the following inequality

$$\left(\sum_{i=1}^{m} q_i a_i\right)^p \ge \sum_{i=1}^{m} Q_i^p \left(a_i^p - a_{i+1}^p\right) = (q_1 a_1)^p + \sum_{i=2}^{m} a_i^p \left(Q_i^p - Q_{i-1}^p\right) \tag{3}$$

holds. If $p \in (0,1)$, then (3) holds in the reversed direction (see [1]).

A generalization of inequality (3) is presented in [6].

Theorem 1.2. *Let $a_i, q_i \in \mathbb{R}$ $(1 \le i \le m)$ such that $a_i \ge 0$ and $q_i > 0$. Let $q_1 a_1$, $\sum_{i=1}^{m} q_i a_i$, $Q_i a_i$, $Q_{i-1} a_i \in [s,t]$ for all $i \in \{2,\ldots,m\}$ and let $f : [s,t] \to \mathbb{R}$ be a convex function.*

(*i*) *If $\{a_i\}_{i=1}^m$ is $\searrow$ in WM, then*

$$f\left(\sum_{i=1}^{m} q_i a_i\right) \ge f(q_1 a_1) + \sum_{i=2}^{m} \left[f(Q_i a_i) - f(Q_{i-1} a_i)\right]. \tag{4}$$

(*ii*) *If $\{a_i\}_{i=1}^m$ is $\nearrow$ in WM, then*

$$f\left(\sum_{i=1}^{m} q_i a_i\right) \le f(q_1 a_1) + \sum_{i=2}^{m} \left[f(Q_i a_i) - f(Q_{i-1} a_i)\right]. \tag{5}$$

If f is concave, then (4) and (5) hold in the reversed direction.

Definition 1.3. *Let $\mathbf{q} = (q_1,\ldots,q_m)$ be a positive probability distribution. Then the Shannon entropy (see [7]) of $\mathbf{q}$ is defined by $S(\mathbf{q}) := \sum_{i=1}^{m} q_i \log\left(\frac{1}{q_i}\right)$.*

S. Khalid, Đ. Pečarić and J. Pečarić presented the following interesting result associated with the Shannon entropy in [5].

Theorem 1.4. *Let $q_i \in \mathbb{R}$ such that $q_i > 0$ $(1 \le i \le m)$ and let $f : [s,t] \to \mathbb{R}$ be a convex function.*

(*a*) *Let $0 < q_i < 1$ $(1 \le i \le m)$ and let $S(\mathbf{q})$, $q_1 \log\left(\frac{1}{q_1}\right)$, $Q_i \log\left(\frac{1}{q_i}\right)$, $Q_{i-1} \log\left(\frac{1}{q_i}\right) \in [s,t]$ for all $i \in \{2,\ldots,m\}$.*

(*i*) *If $\{q_i\}_{i=1}^m$ is $\searrow$, then*

$$f(S(\mathbf{q})) \le f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) + \sum_{i=2}^{m} \left[f\left(Q_i \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right)\right]. \tag{6}$$

(*ii*) *If $\{q_i\}_{i=1}^m$ is $\nearrow$, then*

$$f(S(\mathbf{q})) \ge f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) + \sum_{i=2}^{m} \left[f\left(Q_i \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right)\right]. \tag{7}$$

(*b*) *Let $q_i \ge 1$ $(1 \le i \le m)$ and let $-S(\mathbf{q})$, $q_1 \log q_1$, $Q_i \log q_i$, $Q_{i-1} \log q_i \in [s,t]$ for all $i \in \{2,\ldots,m\}$.*

(*i*) *If $\{q_i\}_{i=1}^m$ is $\searrow$, then*

$$f(-S(\mathbf{q})) \ge f(q_1 \log q_1) + \sum_{i=2}^{m} \left[f(Q_i \log q_i) - f(Q_{i-1} \log q_i)\right]. \tag{8}$$

(*ii*) *If* $\{q_i\}_{i=1}^m$ *is* $\nearrow$, *then*

$$f(-S(\mathbf{q})) \leq f(q_1 \log q_1) + \sum_{i=2}^{m} [f(Q_i \log q_i) - f(Q_{i-1} \log q_i)]. \tag{9}$$

If f is concave, then (6) - (9) *hold in the reversed direction.*

Remark 1.5. *Let* $f : [s,t] \to \mathbb{R}$ *be a convex function.*

(*i*) *If* $\{a_i\}_{i=1}^m$ *is* $\searrow$ *in WM, then Theorem* 1.2 (*i*) *implies that*

$$f\left(\sum_{i=1}^{m} q_i a_i\right) - f(q_1 a_1) - \sum_{i=2}^{m} [f(Q_i a_i) - f(Q_{i-1} a_i)] \geq 0. \tag{10}$$

(*ii*) *If* $\{q_i\}_{i=1}^m \subset \mathbb{R}$ *is* $\searrow$, *then Theorem* 1.4(*a*)(*i*) *and Theorem* 1.4(*b*)(*i*) *imply that*

$$f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) - f(S(\mathbf{q})) + \sum_{i=2}^{m} \left[f\left(Q_i \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right)\right] \geq 0 \tag{11}$$

and

$$f(-S(\mathbf{q})) - f(q_1 \log q_1) - \sum_{i=2}^{m} [f(Q_i \log q_i) - f(Q_{i-1} \log q_i)] \geq 0 \tag{12}$$

respectively.

In the first section, we will present inequalities of kind (10) - (12) for r-convex functions by using the following Fink's identity [3].

Theorem 1.6. *Let* $s, t \in \mathbb{R}$, $f : [s,t] \to \mathbb{R}$, $r \geq 1$ *and* $f^{(r-1)}$ *is absolutely continuous on* $[s,t]$. *Then*

$$\begin{aligned} f(x) = \quad & \frac{r}{t-s} \int_s^t f(u)\,du - \frac{1}{t-s} \sum_{n=1}^{r-1} \frac{r-n}{n!} \left(f^{(n-1)}(s)(x-s)^n - f^{(n-1)}(t)(x-t)^n\right) \\ & + \frac{1}{(r-1)!(t-s)} \int_s^t (x-u)^{r-1} k(u,x) f^{(r)}(u)\,du, \end{aligned} \tag{13}$$

where

$$k(u,x) = \begin{cases} u-s, & s \leq u \leq x \leq t, \\ u-t, & s \leq x < u \leq t. \end{cases} \tag{14}$$

To check the r-convexity of f (see [8, p. 16]), we will use the following criteria:

Lemma 1.7. *If* $f^{(r)}$ *exists, then f is r-convex if and only if* $f^{(r)} \geq 0$.

In the second section, we present some interesting results by using the Čebyšev functional and Grüss type inequalities (see [2]).

Let $L_p[s,t]$ $(1 \leq p < \infty)$ denotes the space of p-power integrable functions defined on $[s,t]$ equipped with the norm

$$\|\zeta\|_p = \left(\int_s^t |\zeta(u)|^p\,du\right)^{\frac{1}{p}}$$

and let $L_\infty[s,t]$ denotes the space of essentially bounded functions defined on $[s,t]$ together with the norm

$$\|\zeta\|_\infty = \operatorname*{ess\,sup}_{u\in[s,t]} |\zeta(u)|.$$

Suppose that $\zeta, \xi : [s,t] \to \mathbb{R}$ are two Lebesgue integrable functions. We consider the Čebyšev functional as follows:

$$\Gamma(\zeta,\xi) := \frac{1}{t-s}\int_s^t \zeta(u)\,\xi(u)\,du - \frac{1}{t-s}\int_s^t \zeta(u)\,du \cdot \frac{1}{t-s}\int_s^t \xi(u)\,du. \tag{15}$$

The next two results are related to the Grüss type inequalities.

Theorem 1.8. *Suppose that $\zeta, \xi : [s,t] \to \mathbb{R}$ are two absolutely continuous functions such that $(\cdot - s)(t-\cdot)(\xi')^2 \in L_1[s,t]$. Then*

$$|\Gamma(\zeta,\xi)| \leq \sqrt{\frac{\Gamma(\zeta,\zeta)}{2(t-s)}} \cdot \sqrt{\int_s^t (u-s)(t-u)(\xi'(u))^2\,du}. \tag{16}$$

Theorem 1.9. *Suppose that $\zeta : [s,t] \to \mathbb{R}$ is absolutely continuous such that $\zeta' \in L_\infty[s,t]$ and $\xi : [s,t] \to \mathbb{R}$ is monotonically $\nearrow$. Then*

$$|\Gamma(\zeta,\xi)| \leq \frac{1}{2(t-s)}\|\zeta'\|_\infty \int_s^t (u-s)(t-u)\,d\xi(u). \tag{17}$$

The constants $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are the best possible in (16) *and* (17) *respectively.*

In the second section, first we use the Fink's identity in the left hand sides of (10) - (12) and obtain some interesting identities and further we use the obtained identities and generalize inequalities of kind (10) - (12) for r-convex functions. In the third section, we use the Čebyšev functional and the Grüss type inequalities and present new bounds for the remainder in the obtained identities. In the fourth section, we present the Ostrowski type inequalities associated with the obtained identities.

2. Some interesting identities, inequalities and generalizations of some Hardy-Littlewood-Pólya type inequalities by the Fink's identity

The first main result of this section is related to the following identities:

Theorem 2.1. *Let $f : [s,t] \to \mathbb{R}$, $f^{(r-1)}$ be absolutely continuous for $r \geq 1$ and $k(u,x)$ be defined in* (14).

(*i*) *Let* q_1a_1, $\sum_{i=1}^{m} q_ia_i$, Q_ia_i, $Q_{i-1}a_i \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$. *Then*

$$
\begin{aligned}
& f\left(\sum_{i=1}^{m} q_ia_i\right) - f(q_1a_1) - \sum_{i=2}^{m}\left[f(Q_ia_i) - f(Q_{i-1}a_i)\right] \\
= \;& \frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\left[f^{(n-1)}(s)\left((q_1a_1-s)^n - \left(\sum_{i=1}^{m} q_ia_i - s\right)^n - \sum_{i=2}^{m}(Q_{i-1}a_i-s)^n\right.\right. \\
& \left. + \sum_{i=2}^{m}(Q_ia_i-s)^n\right) - f^{(n-1)}(t)\left((q_1a_1-t)^n - \left(\sum_{i=1}^{m} q_ia_i - t\right)^n - \sum_{i=2}^{m}(Q_{i-1}a_i-t)^n\right. \\
& \left.\left. + \sum_{i=2}^{m}(Q_ia_i-t)^n\right)\right] + \frac{1}{(r-1)!\,(t-s)}\int_s^t f^{(r)}(u)\left[\left(\sum_{i=1}^{m} q_ia_i - u\right)^{r-1} k\left(u, \sum_{i=1}^{m} q_ia_i\right)\right. \\
& - (q_1a_1-u)^{r-1}k(u,q_1a_1) - \sum_{i=2}^{m}(Q_ia_i-u)^{r-1}k(u,Q_ia_i) \\
& \left. + \sum_{i=2}^{m}(Q_{i-1}a_i-u)^{r-1}k(u,Q_{i-1}a_i)\right] du. \qquad (18)
\end{aligned}
$$

(*ii*) *Let* $S(\mathbf{q})$, $q_1\log\left(\frac{1}{q_1}\right)$, $Q_i\log\left(\frac{1}{q_i}\right)$, $Q_{i-1}\log\left(\frac{1}{q_i}\right) \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$, *where* $0 < q_i < 1$ $(1 \le i \le m)$. *Then*

$$
\begin{aligned}
& f\left(q_1\log\left(\frac{1}{q_1}\right)\right) - f(S(\mathbf{q})) + \sum_{i=2}^{m}\left[f\left(Q_i\log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)\right)\right] \\
= \;& \frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\left[f^{(n-1)}(s)\left((S(\mathbf{q})-s)^n - \left(q_1\log\left(\frac{1}{q_1}\right)-s\right)^n + \sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-s\right)^n\right.\right. \\
& \left. - \sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-s\right)^n\right) - f^{(n-1)}(t)\left((S(\mathbf{q})-t)^n - \left(q_1\log\left(\frac{1}{q_1}\right)-t\right)^n\right. \\
& \left.\left. + \sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-t\right)^n - \sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-t\right)^n\right)\right] + \frac{1}{(r-1)!\,(t-s)}\int_s^t f^{(r)}(u) \\
& \times\left[\left(q_1\log\left(\frac{1}{q_1}\right)-u\right)^{r-1} k\left(u, q_1\log\left(\frac{1}{q_1}\right)\right) - (S(\mathbf{q})-u)^{r-1}k(u,S(\mathbf{q}))\right. \\
& + \sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-u\right)^{r-1} k\left(u, Q_i\log\left(\frac{1}{q_i}\right)\right) \\
& \left. - \sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-u\right)^{r-1} k\left(u, Q_{i-1}\log\left(\frac{1}{q_i}\right)\right)\right] du. \qquad (19)
\end{aligned}
$$

(*iii*) *Let* $-S(\mathbf{q})$, $q_1 \log q_1$, $Q_i \log q_i$, $Q_{i-1} \log q_i \in [s, t]$ *for all* $i \in \{2, \ldots, m\}$, *where* $q_i \geq 1$ $(1 \leq i \leq m)$. *Then*

$$\begin{aligned} &f(-S(\mathbf{q})) - f(q_1 \log q_1) - \sum_{i=2}^{m} [f(Q_i \log q_i) - f(Q_{i-1} \log q_i)] \\ &= \frac{1}{t-s} \sum_{n=1}^{r-1} \frac{r-n}{n!} \Bigg[f^{(n-1)}(s) \Bigg((q_1 \log q_1 - s)^n - (-S(\mathbf{q}) - s)^n - \sum_{i=2}^{m} (Q_{i-1} \log q_i - s)^n \\ &\quad + \sum_{i=2}^{m} (Q_i \log q_i - s)^n \Bigg) - f^{(n-1)}(t) \Bigg((q_1 \log q_1 - t)^n - (-S(\mathbf{q}) - t)^n - \sum_{i=2}^{m} (Q_{i-1} \log q_i - t)^n \\ &\quad + \sum_{i=2}^{m} (Q_i \log q_i - t)^n \Bigg) \Bigg] + \frac{1}{(r-1)!(t-s)} \int_s^t f^{(r)}(u) \Big[(-S(\mathbf{q}) - u)^{r-1} k(u, -S(\mathbf{q})) \\ &\quad - (q_1 \log q_1 - u)^{r-1} k(u, q_1 \log q_1) - \sum_{i=2}^{m} (Q_i \log q_i - u)^{r-1} k(u, Q_i \log q_i) \\ &\quad + \sum_{i=2}^{m} (Q_{i-1} \log q_i - u)^{r-1} k(u, Q_{i-1} \log q_i) \Big] du. \end{aligned} \tag{20}$$

Proof. (*i*) Use the Fink's identity (13) in the L.H.S of (10), we have

$$\begin{aligned} &f\left(\sum_{i=1}^{m} q_i a_i\right) - f(q_1 a_1) - \sum_{i=2}^{m} [f(Q_i a_i) - f(Q_{i-1} a_i)] \\ &= \frac{1}{t-s} \sum_{n=1}^{r-1} \frac{r-n}{n!} \left[f^{(n-1)}(s) \left((q_1 a_1 - s)^n - \left(\sum_{i=1}^{m} q_i a_i - s \right)^n \right) \right. \\ &\quad \left. - f^{(n-1)}(t) \left((q_1 a_1 - t)^n - \left(\sum_{i=1}^{m} q_i a_i - t \right)^n \right) \right] - \frac{1}{(r-1)!(t-s)} \int_s^t f^{(r)}(u) \\ &\quad \times \left[(q_1 a_1 - u)^{r-1} k(u, q_1 a_1) - \left(\sum_{i=1}^{m} q_i a_i - u \right)^{r-1} k\left(u, \sum_{i=1}^{m} q_i a_i\right) \right] du - \frac{1}{t-s} \sum_{i=2}^{m} \sum_{n=1}^{r-1} \frac{r-n}{n!} \\ &\quad \times \Big[f^{(n-1)}(s) \left((Q_{i-1} a_i - s)^n - (Q_i a_i - s)^n \right) - f^{(n-1)}(t) \left((Q_{i-1} a_i - t)^n - (Q_i a_i - t)^n \right) \Big] \\ &\quad + \frac{1}{(r-1)!(t-s)} \sum_{i=2}^{m} \int_s^t f^{(r)}(u) \Big[(Q_{i-1} a_i - u)^{r-1} k(u, Q_{i-1} a_i) - (Q_i a_i - u)^{r-1} k(u, Q_i a_i) \Big] du. \end{aligned}$$

On simplifying the above expression and interchanging the summation and integral, (18) is immediate.

(*ii*) Use the Fink's identity in the LHS of (11), we have (19).

(*iii*) Use the Fink's identity in the LHS of (12), we obtain (20).
□

We present inequalities of kind (10) - (12) for r-convex functions as follows:

Theorem 2.2. *Let all the assumptions of Theorem 2.1 be satisfied and let for* $r \geq 1$, f *be* r*-convex.*

(*i*) *Let* q_1a_1, $\sum_{i=1}^{m} q_ia_i$, Q_ia_i, $Q_{i-1}a_i$, $u \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$. *If*

$$\left(\sum_{i=1}^{m} q_ia_i - u\right)^{r-1} k\left(u, \sum_{i=1}^{m} q_ia_i\right) - (q_1a_1 - u)^{r-1} k(u, q_1a_1) \geq \sum_{i=2}^{m} (Q_ia_i - u)^{r-1} k(u, Q_ia_i) - \sum_{i=2}^{m} (Q_{i-1}a_i - u)^{r-1} k(u, Q_{i-1}a_i) \tag{21}$$

holds, then

$$\begin{aligned} & f\left(\sum_{i=1}^{m} q_ia_i\right) - f(q_1a_1) - \sum_{i=2}^{m} \left[f(Q_ia_i) - f(Q_{i-1}a_i)\right] \\ \geq\ & \frac{1}{t-s}\sum_{n=1}^{r-1} \frac{r-n}{n!}\left[f^{(n-1)}(s)\left((q_1a_1-s)^n - \left(\sum_{i=1}^{m} q_ia_i - s\right)^n - \sum_{i=2}^{m}(Q_{i-1}a_i - s)^n \right.\right. \\ & \left. + \sum_{i=2}^{m}(Q_ia_i - s)^n\right) - f^{(n-1)}(t)\left((q_1a_1-t)^n - \left(\sum_{i=1}^{m} q_ia_i - t\right)^n - \sum_{i=2}^{m}(Q_{i-1}a_i - t)^n \right. \\ & \left.\left. + \sum_{i=2}^{m}(Q_ia_i - t)^n\right)\right]. \end{aligned} \tag{22}$$

(*ii*) *Let* $S(\mathbf{q})$, $q_1 \log\left(\frac{1}{q_1}\right)$, $Q_i \log\left(\frac{1}{q_i}\right)$, $Q_{i-1} \log\left(\frac{1}{q_i}\right)$, $u \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$, *where* $0 < q_i < 1$ $(1 \leq i \leq m)$. *If*

$$\begin{aligned} & \left(q_1 \log\left(\frac{1}{q_1}\right) - u\right)^{r-1} k\left(u, q_1 \log\left(\frac{1}{q_1}\right)\right) - (S(\mathbf{q}) - u)^{r-1} k(u, S(\mathbf{q})) \\ \geq\ & \sum_{i=2}^{m}\left(Q_{i-1} \log\left(\frac{1}{q_i}\right) - u\right)^{r-1} k\left(u, Q_{i-1} \log\left(\frac{1}{q_i}\right)\right) \\ & -\sum_{i=2}^{m}\left(Q_i \log\left(\frac{1}{q_i}\right) - u\right)^{r-1} k\left(u, Q_i \log\left(\frac{1}{q_i}\right)\right) \end{aligned} \tag{23}$$

holds, then

$$\begin{aligned} & f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) - f(S(\mathbf{q})) + \sum_{i=2}^{m}\left[f\left(Q_i \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right)\right] \\ \geq\ & \frac{1}{t-s}\sum_{n=1}^{r-1} \frac{r-n}{n!}\left[f^{(n-1)}(s)\left((S(\mathbf{q}) - s)^n - \left(q_1 \log\left(\frac{1}{q_1}\right) - s\right)^n + \sum_{i=2}^{m}\left(Q_{i-1} \log\left(\frac{1}{q_i}\right) - s\right)^n \right.\right. \\ & \left. -\sum_{i=2}^{m}\left(Q_i \log\left(\frac{1}{q_i}\right) - s\right)^n\right) - f^{(n-1)}(t)\left((S(\mathbf{q}) - t)^n - \left(q_1 \log\left(\frac{1}{q_1}\right) - t\right)^n \right. \\ & \left.\left. +\sum_{i=2}^{m}\left(Q_{i-1} \log\left(\frac{1}{q_i}\right) - t\right)^n - \sum_{i=2}^{m}\left(Q_i \log\left(\frac{1}{q_i}\right) - t\right)^n\right)\right]. \end{aligned} \tag{24}$$

(*iii*) *Let* $-S(\mathbf{q})$, $q_1 \log q_1$, $Q_i \log q_i$, $Q_{i-1} \log q_i$, $u \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$, *where* $q_i \geq 1$ $(1 \leq i \leq m)$. *If*

$$\begin{aligned} & (-S(\mathbf{q}) - u)^{r-1} k(u, -S(\mathbf{q})) - (q_1 \log q_1 - u)^{r-1} k(u, q_1 \log q_1) \\ \geq\ & \sum_{i=2}^{m} (Q_i \log q_i - u)^{r-1} k(u, Q_i \log q_i) - \sum_{i=2}^{m} (Q_{i-1} \log q_i - u)^{r-1} k(u, Q_{i-1} \log q_i) \end{aligned} \tag{25}$$

holds, then

$$\begin{aligned} & f\left(-S\left(\mathbf{q}\right)\right)-f\left(q_1\log q_1\right)-\sum_{i=2}^{m}\left[f\left(Q_i\log q_i\right)-f\left(Q_{i-1}\log q_i\right)\right] \\ \geq \; & \frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}\left(s\right)\Bigg(\left(q_1\log q_1-s\right)^n-\left(-S\left(\mathbf{q}\right)-s\right)^n-\sum_{i=2}^{m}\left(Q_{i-1}\log q_i-s\right)^n \\ & +\sum_{i=2}^{m}\left(Q_i\log q_i-s\right)^n\Bigg)-f^{(n-1)}\left(t\right)\Bigg(\left(q_1\log q_1-t\right)^n-\left(-S\left(\mathbf{q}\right)-t\right)^n-\sum_{i=2}^{m}\left(Q_{i-1}\log q_i-t\right)^n \\ & +\sum_{i=2}^{m}\left(Q_i\log q_i-t\right)^n\Bigg)\Bigg]. \end{aligned} \tag{26}$$

If the reversed inequalities hold in (21), (23) *and* (25), *then* (22),(24) *and* (26) *hold in the reversed direction.*

Proof. The absolute continuity of $f^{(r-1)}$ defined on $[s,t]$ implies the existence of $f^{(r)}$ almost everywhere. By using the r-convexity of f, from Lemma 1.7, we obtain $f^{(r)}(x)\geq 0$ for all $x\in[s,t]$.

(i) Use $f^{(r)}\geq 0$ and (21) in (18), we have (22).

(ii) By using the non-negativity of $f^{(r)}$ together with (23) in (19), (24) is immediate.

(iii) Follow the proof of (i) and (ii).

□

Now we present generalizations of the inequalities (10) - (12) as follows:

Theorem 2.3. *Let all the assumptions of Theorem 2.1 be satisfied. Let r be even such that $r\geq 2$ and let $\Lambda:[s,t]\to\mathbb{R}$ be a function defined by*

$$\Lambda\left(x\right)=\frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\left(\left(x-t\right)^n f^{(n-1)}\left(t\right)-\left(x-s\right)^n f^{(n-1)}\left(s\right)\right). \tag{27}$$

(a) *Let $a_i, q_i\in\mathbb{R}$ $(1\leq i\leq m)$ such that $a_i\geq 0$ and $q_i>0$. Let q_1a_1, $\sum_{i=1}^{m}q_ia_i$, Q_ia_i, $Q_{i-1}a_i\in[s,t]$ for all $i\in\{2,\dots,m\}$ and let the sequence $\{a_i\}_{i=1}^{m}$ be $\searrow$ in WM.*

- (i) *If $f:[s,t]\to\mathbb{R}$ is r-convex, then* (22) *holds.*
- (ii) *Let* (22) *be satisfied. If Λ is convex, then the RHS of* (22) *is non-negative and we have*

$$f\left(\sum_{i=1}^{m}q_ia_i\right)-f\left(q_1a_1\right)\geq\sum_{i=2}^{m}\left[f\left(Q_ia_i\right)-f\left(Q_{i-1}a_i\right)\right]. \tag{28}$$

(b) *Let $0<q_i<1$ $(1\leq i\leq m)$ and let $S(\mathbf{q})$, $q_1\log\left(\frac{1}{q_1}\right)$, $Q_i\log\left(\frac{1}{q_i}\right)$, $Q_{i-1}\log\left(\frac{1}{q_i}\right)\in[s,t]$ for all $i\in\{2,\dots,m\}$. Let the sequence $\{q_i\}_{i=1}^{m}\subset\mathbb{R}$ be $\searrow$.*

- (i) *If $f:[s,t]\to\mathbb{R}$ is r-convex, then* (24) *holds.*
- (ii) *Let* (24) *be satisfied. If Λ is convex, then the RHS of* (24) *is non-negative and we have*

$$f\left(q_1\log\left(\frac{1}{q_1}\right)\right)-f\left(S\left(\mathbf{q}\right)\right)\geq\sum_{i=2}^{m}\left[f\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)\right)-f\left(Q_i\log\left(\frac{1}{q_i}\right)\right)\right]. \tag{29}$$

(c) *Let $q_i\geq 1$ $(1\leq i\leq m)$ and let $-S(\mathbf{q})$, $q_1\log q_1$, $Q_i\log q_i$, $Q_{i-1}\log q_i\in[s,t]$ for all $i\in\{2,\dots,m\}$. Let the sequence $\{q_i\}_{i=1}^{m}\subset\mathbb{R}$ be $\searrow$.*

(*i*) *If* $f : [s, t] \to \mathbb{R}$ *is r-convex, then* (26) *holds.*

(*ii*) *Let* (26) *be satisfied. If* Λ *is convex, then the RHS of* (26) *is non-negative and we have*

$$f(-S(\mathbf{q})) - f(q_1 \log q_1) \geq \sum_{i=2}^{m} [f(Q_i \log q_i) - f(Q_{i-1} \log q_i)]. \tag{30}$$

Proof. By using Lemma 1.7, it is obvious that $\vartheta(x) := (x-u)^{r-1} k(u, x)$ is convex for even r, where $r \geq 2$.

(*a*) (*i*) As the sequence $\{a_i\}_{i=1}^m$ is $\searrow$ in WM, use the convex function $\vartheta(x)$ in (4), inequality (21) is immediate for even r, where $r \geq 2$. Now as f is r-convex for even r, apply Theorem 2.2 (i), we obtain (22).

(*ii*) It is obvious that the inequality (22) is equivalent to

$$f\left(\sum_{i=1}^{m} q_i a_i\right) - f(q_1 a_1) - \sum_{i=2}^{m} [f(Q_i a_i) - f(Q_{i-1} a_i)]$$
$$\geq \Lambda\left(\sum_{i=1}^{m} q_i a_i\right) - \Lambda(q_1 a_1) - \sum_{i=2}^{m} [\Lambda(Q_i a_i) - \Lambda(Q_{i-1} a_i)].$$

As the sequence $\{a_i\}_{i=1}^m$ is $\searrow$ in WM, replace f by Λ in Theorem 1.2 (*i*), the non-negativity of the RHS of (22) is immediate and we have (28).

(*b*) (*i*) As the sequence $\{q_i\}_{i=1}^m \subset \mathbb{R}$ is $\searrow$, use the convex function $\vartheta(x)$ in (6), we obtain (23) for even r, where $r \geq 2$. Use the r-convexity of f for even r and apply Theorem 2.2 (ii), we have (24).

(*ii*) It is clear that the inequality (24) is equivalent to

$$f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) - f(S(\mathbf{q})) - \sum_{i=2}^{m} \left[f\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_i \log\left(\frac{1}{q_i}\right)\right)\right]$$
$$\geq \Lambda\left(q_1 \log\left(\frac{1}{q_1}\right)\right) - \Lambda(S(\mathbf{q})) - \sum_{i=2}^{m} \left[\Lambda\left(Q_{i-1} \log\left(\frac{1}{q_i}\right)\right) - \Lambda\left(Q_i \log\left(\frac{1}{q_i}\right)\right)\right].$$

As the sequence $\{q_i\}_{i=1}^m$ is $\searrow$, replace f by Λ in Theorem 1.4 (a) (*i*), the RHS of (24) is non-negative and we have (29).

(*c*) (*i*) Use the convex function $\vartheta(x)$ in (8), we obtain (25) for even r, where $r \geq 2$. As f is r-convex for even r, apply Theorem 2.2 (iii), we have (26).

(*ii*) Clearly (26) is equivalent to

$$f(-S(\mathbf{q})) - f(q_1 \log q_1) - \sum_{i=2}^{m} [f(Q_i \log q_i) - f(Q_{i-1} \log q_i)]$$
$$\geq \Lambda(-S(\mathbf{q})) - \Lambda(q_1 \log q_1) - \sum_{i=2}^{m} [\Lambda(Q_i \log q_i) - \Lambda(Q_{i-1} \log q_i)].$$

Replace f by Λ in Theorem 1.4 (b) (*i*), the RHS of (26) is non-negative and (30) is immediate.

□

Remark 2.4. *For an arbitrary* 2*-convex, that is convex, function* f, Λ *from* (27) *takes the form*

$$\Lambda(x) = \frac{1}{t-s}\left((x-t)f(t) - (x-s)f(s)\right).$$

As this function is linear, it is both convex and concave. Clearly Theorem 2.3 *provide generalizations of the inequalities* (10) - (12).

3. Grüss type inequalities by the Fink's identity

In this section we present some new bounds for the remainder in the obtained identities. We denote

$$\alpha(u) = \left(\sum_{i=1}^{m} q_i a_i - u\right)^{r-1} k\left(u, \sum_{i=1}^{m} q_i a_i\right) - (q_1 a_1 - u)^{r-1} k(u, q_1 a_1) - \sum_{i=2}^{m} (Q_i a_i - u)^{r-1} k(u, Q_i a_i) + \sum_{i=2}^{m} (Q_{i-1} a_i - u)^{r-1} k(u, Q_{i-1} a_i), \tag{31}$$

where $q_1 a_1$, $\sum_{i=1}^{m} q_i a_i$, $Q_i a_i$, $Q_{i-1} a_i$, $u \in [s,t]$ for all $i \in \{2,\ldots,m\}$,

$$\beta(u) = \left(q_1 \log\left(\frac{1}{q_1}\right) - u\right)^{r-1} k\left(u, q_1 \log\left(\frac{1}{q_1}\right)\right) - (S(\mathbf{q}) - u)^{r-1} k(u, S(\mathbf{q})) - \sum_{i=2}^{m} \left(Q_{i-1} \log\left(\frac{1}{q_i}\right) - u\right)^{r-1} k\left(u, Q_{i-1} \log\left(\frac{1}{q_i}\right)\right) + \sum_{i=2}^{m} \left(Q_i \log\left(\frac{1}{q_i}\right) - u\right)^{r-1} k\left(u, Q_i \log\left(\frac{1}{q_i}\right)\right), \tag{32}$$

where $S(\mathbf{q})$, $q_1 \log\left(\frac{1}{q_1}\right)$, $Q_i \log\left(\frac{1}{q_i}\right)$, $Q_{i-1} \log\left(\frac{1}{q_i}\right)$, $u \in [s,t]$ for all $i \in \{2,\ldots,m\}$ such that $0 < q_i < 1$ $(1 \le i \le m)$ and

$$\gamma(u) = (-S(\mathbf{q}) - u)^{r-1} k(u, -S(\mathbf{q})) - (q_1 \log q_1 - u)^{r-1} k(u, q_1 \log q_1) - \sum_{i=2}^{m} (Q_i \log q_i - u)^{r-1} k(u, Q_i \log q_i) + \sum_{i=2}^{m} (Q_{i-1} \log q_i - u)^{r-1} k(u, Q_{i-1} \log q_i), \tag{33}$$

where $-S(\mathbf{q})$, $q_1 \log q_1$, $Q_i \log q_i$, $Q_{i-1} \log q_i$, $u \in [s,t]$ for all $i \in \{2,\ldots,m\}$ such that $q_i \ge 1$ $(1 \le i \le m)$ and in addition $k(u,.)$, appearing in (31) - (33), is defined in (14).

Theorem 3.1. *Let $f : [s,t] \to \mathbb{R}$, $f^{(r)}$ be absolutely continuous for $r \ge 1$ with $(\cdot - s)(t - \cdot)\left(f^{(r+1)}\right)^2 \in L_1[s,t]$ and let Γ be the same as defined in* (15).

(*i*) *Let $q_1 a_1$, $\sum_{i=1}^{m} q_i a_i$, $Q_i a_i$, $Q_{i-1} a_i \in [s,t]$ for all $i \in \{2,\ldots,m\}$. If α is defined in* (31), *then*

$$f\left(\sum_{i=1}^{m} q_i a_i\right) - f(q_1 a_1) - \sum_{i=2}^{m} \left[f(Q_i a_i) - f(Q_{i-1} a_i)\right] = \frac{1}{t-s} \sum_{n=1}^{r-1} \frac{r-n}{n!} \left[f^{(n-1)}(s) \left((q_1 a_1 - s)^n - \left(\sum_{i=1}^{m} q_i a_i - s\right)^n - \sum_{i=2}^{m} (Q_{i-1} a_i - s)^n + \sum_{i=2}^{m} (Q_i a_i - s)^n \right) - f^{(n-1)}(t) \left((q_1 a_1 - t)^n - \left(\sum_{i=1}^{m} q_i a_i - t\right)^n - \sum_{i=2}^{m} (Q_{i-1} a_i - t)^n + \sum_{i=2}^{m} (Q_i a_i - t)^n \right) \right] + \frac{f^{(r-1)}(t) - f^{(r-1)}(s)}{(t-s)^2 (r-1)!} \int_s^t \alpha(u)\, du + \Delta_r(s,t;f), \tag{34}$$

where the remainder $\Delta_r(s,t;f)$ satisfies the estimation

$$\left|\Delta_r(s,t;f)\right| \le \frac{1}{(r-1)!} \sqrt{\frac{\Gamma(\alpha(u), \alpha(u))}{2(t-s)}} \cdot \sqrt{\int_s^t (u-s)(t-u)\left(f^{(r+1)}(u)\right)^2 du}. \tag{35}$$

(*ii*) *Let* $S(\mathbf{q})$, $q_1 \log\left(\frac{1}{q_1}\right)$, $Q_i \log\left(\frac{1}{q_i}\right)$, $Q_{i-1} \log\left(\frac{1}{q_i}\right) \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$, *where* $0 < q_i < 1$ $(1 \le i \le m)$. *If* β *is defined in* (32), *then*

$$\begin{aligned}
&f\left(q_1 \log\left(\frac{1}{q_1}\right)\right) - f(S(\mathbf{q})) + \sum_{i=2}^{m}\left[f\left(Q_i \log\left(\frac{1}{q_i}\right)\right) - f\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)\right)\right]\\
&= \frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}(s)\Bigg((S(\mathbf{q})-s)^n - \left(q_1\log\left(\frac{1}{q_1}\right)-s\right)^n + \sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-s\right)^n\\
&\quad - \sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-s\right)^n\Bigg) - f^{(n-1)}(t)\Bigg((S(\mathbf{q})-t)^n - \left(q_1\log\left(\frac{1}{q_1}\right)-t\right)^n\\
&\quad + \sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-t\right)^n - \sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-t\right)^n\Bigg)\Bigg]\\
&\quad + \frac{f^{(r-1)}(t)-f^{(r-1)}(s)}{(t-s)^2(r-1)!}\int_s^t \beta(u)\,du + \Xi_r(s,t;f),
\end{aligned} \tag{36}$$

where

$$\left|\Xi_r(s,t;f)\right| \le \frac{1}{(r-1)!}\sqrt{\frac{\Gamma(\beta(u),\beta(u))}{2(t-s)}}\cdot\sqrt{\int_s^t (u-s)(t-u)\left(f^{(r+1)}(u)\right)^2du}.$$

(*iii*) *Let* $-S(\mathbf{q})$, $q_1\log q_1$, $Q_i\log q_i$, $Q_{i-1}\log q_i \in [s,t]$ *for all* $i \in \{2,\ldots,m\}$, *where* $q_i \ge 1$ $(1 \le i \le m)$. *If* γ *is defined in* (33), *then*

$$\begin{aligned}
&f(-S(\mathbf{q})) - f(q_1\log q_1) - \sum_{i=2}^{m}\left[f(Q_i\log q_i) - f(Q_{i-1}\log q_i)\right]\\
&= \frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}(s)\Bigg((q_1\log q_1 - s)^n - (-S(\mathbf{q})-s)^n - \sum_{i=2}^{m}(Q_{i-1}\log q_i - s)^n\\
&\quad + \sum_{i=2}^{m}(Q_i\log q_i - s)^n\Bigg) - f^{(n-1)}(t)\Bigg((q_1\log q_1 - t)^n - (-S(\mathbf{q})-t)^n - \sum_{i=2}^{m}(Q_{i-1}\log q_i - t)^n\\
&\quad + \sum_{i=2}^{m}(Q_i\log q_i - t)^n\Bigg)\Bigg] + \frac{f^{(r-1)}(t)-f^{(r-1)}(s)}{(t-s)^2(r-1)!}\int_s^t \gamma(u)\,du + \Upsilon_r(s,t;f),
\end{aligned} \tag{37}$$

where

$$\left|\Upsilon_r(s,t;f)\right| \le \frac{1}{(r-1)!}\sqrt{\frac{\Gamma(\gamma(u),\gamma(u))}{2(t-s)}}\cdot\sqrt{\int_s^t (u-s)(t-u)\left(f^{(r+1)}(u)\right)^2du}.$$

Proof. (*i*) Apply Theorem 1.8 for $\zeta \to \alpha$ and $\xi \to f^{(r)}$, we have

$$\begin{aligned}
&\left|\frac{1}{t-s}\int_s^t \alpha(u) f^{(r)}(u)\,du - \frac{1}{t-s}\int_s^t \alpha(u)\,du\cdot\frac{1}{t-s}\int_s^t f^{(r)}(u)\,du\right|\\
&\le \sqrt{\frac{\Gamma(\alpha(u),\alpha(u))}{2(t-s)}}\cdot\sqrt{\int_s^t (u-s)(t-u)\left(f^{(r+1)}(u)\right)^2du}.
\end{aligned} \tag{38}$$

Divide both sides of (38) by $(r-1)!$ and in the obtained expression denote

$$\Delta_r(s,t;f) = \frac{1}{(r-1)!\,(t-s)}\int_s^t \alpha(u)\, f^{(r)}(u)\,du - \frac{f^{(r-1)}(t)-f^{(r-1)}(s)}{(t-s)^2\,(r-1)!}\int_s^t \alpha(u)\,du, \tag{39}$$

(35) is immediate. Now take the value of $\frac{1}{(r-1)!(t-s)}\int_s^t \alpha(u)\, f^{(r)}(u)du$ from (39) and substitute in (18), we have (34).

(*ii*) Apply Theorem 1.8 for $\zeta \to \beta$ and $\xi \to f^{(r)}$, follow the proof of (*i*) and denote

$$\Xi_r(s,t;f) = \frac{1}{(r-1)!\,(t-s)}\int_s^t \beta(u)\, f^{(r)}(u)\,du - \frac{f^{(r-1)}(t)-f^{(r-1)}(s)}{(t-s)^2\,(r-1)!}\int_s^t \beta(u)\,du. \tag{40}$$

Here we use identity (19) instead of (18).

(*iii*) Apply Theorem 1.8 for $\zeta \to \gamma$ and $\xi \to f^{(r)}$, follow the proof of (*i*) and denote

$$\Upsilon_r(s,t;f) = \frac{1}{(r-1)!\,(t-s)}\int_s^t \gamma(u)\, f^{(r)}(u)\,du - \frac{f^{(r-1)}(t)-f^{(r-1)}(s)}{(t-s)^2\,(r-1)!}\int_s^t \gamma(u)\,du. \tag{41}$$

Here we use identity (20).
□

Theorem 3.2. *Let $f:[s,t]\to\mathbb{R}$, $f^{(r)}$ be absolutely continuous for $r\geq 1$ and let $f^{(r+1)}\geq 0$ on $[s,t]$. Let Γ be the same as defined in* (15).

(*i*) *If α is the same as defined in* (31), *then we obtain* (34) *and the remainder $\Delta_r(s,t;f)$ satisfies*

$$\left|\Delta_r(s,t;f)\right| \leq \frac{\|\alpha'(u)\|_\infty}{(r-1)!}\left(\frac{f^{(r-1)}(s)+f^{(r-1)}(t)}{2} - \frac{f^{(r-2)}(t)-f^{(r-2)}(s)}{t-s}\right). \tag{42}$$

(*ii*) *If β is the same as defined in* (32), *then we obtain* (36) *and $\Xi_r(s,t;f)$ satisfies*

$$\left|\Xi_r(s,t;f)\right| \leq \frac{\|\beta'(u)\|_\infty}{(r-1)!}\left(\frac{f^{(r-1)}(s)+f^{(r-1)}(t)}{2} - \frac{f^{(r-2)}(t)-f^{(r-2)}(s)}{t-s}\right).$$

(*iii*) *If γ is the same as defined in* (33), *then we obtain* (37) *and $\Upsilon_r(s,t;f)$ satisfies*

$$\left|\Upsilon_r(s,t;f)\right| \leq \frac{\|\gamma'(u)\|_\infty}{(r-1)!}\left(\frac{f^{(r-1)}(s)+f^{(r-1)}(t)}{2} - \frac{f^{(r-2)}(t)-f^{(r-2)}(s)}{t-s}\right).$$

Proof. (*i*) Apply Theorem 1.9 for $\zeta \to \alpha$ and $\xi \to f^{(r)}$, we have

$$\begin{aligned}&\left|\frac{1}{t-s}\int_s^t \alpha(u)\, f^{(r)}(u)\,du - \frac{1}{t-s}\int_s^t \alpha(u)\,du \cdot \frac{1}{t-s}\int_s^t f^{(r)}(u)\,du\right| \\ &\quad\leq \frac{1}{2(t-s)}\|\alpha'(u)\|_\infty\left(\int_s^t (u-s)(t-u)\, f^{(r+1)}(u)\,du\right).\end{aligned} \tag{43}$$

Now divide (43) by $(r-1)!$ and use

$$\begin{aligned}&\int_s^t (u-s)(t-u) f^{(r+1)}(u)\,du\\ &= \int_s^t (2u-(s+t)) f^{(r)}(u)\,du\\ &= (t-s)\left(f^{(r-1)}(s)+f^{(r-1)}(t)\right)-2\left(f^{(r-2)}(t)-f^{(r-2)}(s)\right)\end{aligned}$$

and in addition use the notation $\Delta_r(s,t;f)$ as defined in (39), we have

$$\left|\Delta_r(s,t;f)\right| \le \frac{1}{2(t-s)}\cdot\frac{\|\alpha'(u)\|_\infty}{(r-1)!}\left((t-s)\left(f^{(r-1)}(s)+f^{(r-1)}(t)\right)-2\left(f^{(r-2)}(t)-f^{(r-2)}(s)\right)\right). \tag{44}$$

After simplification, (44) reduces to (42) and by inserting the value of $\frac{1}{(r-1)!(t-s)}\int_s^t \alpha(u) f^{(r)}(u)\,du$ from (39) into (18), we have the representation (34).

(*ii*) Apply Theorem 1.9 for $\zeta\to\beta$ and $\xi\to f^{(r)}$, follow the proof of (*i*) and use the notation $\Xi_r(s,t;f)$ as defined in (40). Here we use identity (19) instead of (18).

(*iii*) Apply Theorem 1.9 for $\zeta\to\gamma$ and $\xi\to f^{(r)}$, follow the proof of (*i*) and use the notation $\Upsilon_r(s,t;f)$ as defined in (41). Here we use identity (20).
□

4. Ostrowski type inequalities by the Fink's identity

Theorem 4.1. *Let all the assumptions of Theorem 2.1 be satisfied and let α, β and γ be the same as defined in* (31) - (33). *Let $p,q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$ and let $|f^{(r)}|^p:[s,t]\to\mathbb{R}$ be an $\mathbb{R}$-integrable function for some $r\ge 2$.*

(*i*) *Let q_1a_1, $\sum_{i=1}^m q_ia_i$, Q_ia_i, $Q_{i-1}a_i\in[s,t]$ for all $i\in\{2,\dots,m\}$. Then*

$$\begin{aligned}&\Bigg|f\left(\sum_{i=1}^m q_ia_i\right)-f(q_1a_1)-\sum_{i=2}^m\left[f(Q_ia_i)-f(Q_{i-1}a_i)\right]\\ &-\frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}(s)\Bigg((q_1a_1-s)^n-\left(\sum_{i=1}^m q_ia_i-s\right)^n-\sum_{i=2}^m(Q_{i-1}a_i-s)^n\\ &+\sum_{i=2}^m(Q_ia_i-s)^n\Bigg)-f^{(n-1)}(t)\Bigg((q_1a_1-t)^n-\left(\sum_{i=1}^m q_ia_i-t\right)^n-\sum_{i=2}^m(Q_{i-1}a_i-t)^n\\ &+\sum_{i=2}^m(Q_ia_i-t)^n\Bigg)\Bigg]\Bigg|\le\left(\int_s^t\left|f^{(r)}(u)\right|^p du\right)^{\frac{1}{p}}\left(\int_s^t|\hat{\alpha}(u)|^q du\right)^{\frac{1}{q}},\end{aligned} \tag{45}$$

where

$$\hat{\alpha}(u) := \frac{\alpha(u)}{(r-1)!(t-s)}.$$

(*ii*) *Let* $S(\mathbf{q})$, $q_1\log\left(\frac{1}{q_1}\right)$, $Q_i\log\left(\frac{1}{q_i}\right)$, $Q_{i-1}\log\left(\frac{1}{q_i}\right)\in[s,t]$ *for all* $i\in\{2,\ldots,m\}$ *such that* $0<q_i<1$ $(1\le i\le m)$. *Then*

$$\Bigg|f\left(q_1\log\left(\frac{1}{q_1}\right)\right)-f(S(\mathbf{q}))+\sum_{i=2}^{m}\left[f\left(Q_i\log\left(\frac{1}{q_i}\right)\right)-f\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)\right)\right]$$
$$-\frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}(s)\Bigg((S(\mathbf{q})-s)^n-\left(q_1\log\left(\frac{1}{q_1}\right)-s\right)^n+\sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-s\right)^n$$
$$-\sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-s\right)^n\Bigg)-f^{(n-1)}(t)\Bigg((S(\mathbf{q})-t)^n-\left(q_1\log\left(\frac{1}{q_1}\right)-t\right)^n$$
$$+\sum_{i=2}^{m}\left(Q_{i-1}\log\left(\frac{1}{q_i}\right)-t\right)^n-\sum_{i=2}^{m}\left(Q_i\log\left(\frac{1}{q_i}\right)-t\right)^n\Bigg)\Bigg]\Bigg|$$
$$\le\left(\int_s^t\left|f^{(r)}(u)\right|^p du\right)^{\frac{1}{p}}\left(\int_s^t\left|\hat{\beta}(u)\right|^q du\right)^{\frac{1}{q}},\tag{46}$$

where

$$\hat{\beta}(u):=\qquad\frac{\beta(u)}{(r-1)!\,(t-s)}.$$

(*iii*) *Let* $-S(\mathbf{q})$, $q_1\log q_1$, $Q_i\log q_i$, $Q_{i-1}\log q_i\in[s,t]$ *for all* $i\in\{2,\ldots,m\}$ *such that* $q_i\ge 1$ $(1\le i\le m)$. *Then*

$$\Bigg|f(-S(\mathbf{q}))-f(q_1\log q_1)-\sum_{i=2}^{m}\left[f(Q_i\log q_i)-f(Q_{i-1}\log q_i)\right]$$
$$-\frac{1}{t-s}\sum_{n=1}^{r-1}\frac{r-n}{n!}\Bigg[f^{(n-1)}(s)\Bigg((q_1\log q_1-s)^n-(-S(\mathbf{q})-s)^n-\sum_{i=2}^{m}(Q_{i-1}\log q_i-s)^n$$
$$+\sum_{i=2}^{m}(Q_i\log q_i-s)^n\Bigg)-f^{(n-1)}(t)\Bigg((q_1\log q_1-t)^n-(-S(\mathbf{q})-t)^n-\sum_{i=2}^{m}(Q_{i-1}\log q_i-t)^n$$
$$+\sum_{i=2}^{m}(Q_i\log q_i-t)^n\Bigg)\Bigg]\Bigg|\le\left(\int_s^t\left|f^{(r)}(u)\right|^p du\right)^{\frac{1}{p}}\left(\int_s^t\left|\hat{\gamma}(u)\right|^q du\right)^{\frac{1}{q}},\tag{47}$$

where

$$\hat{\gamma}(u):=\qquad\frac{\gamma(u)}{(r-1)!\,(t-s)}.$$

The constants $\left(\int_s^t|\hat{\alpha}(u)|^q du\right)^{\frac{1}{q}}$, $\left(\int_s^t\left|\hat{\beta}(u)\right|^q du\right)^{\frac{1}{q}}$ *and* $\left(\int_s^t\left|\hat{\gamma}(u)\right|^q du\right)^{\frac{1}{q}}$ *in* (45), (46) *and* (47) *respectively are sharp for* $1<p\le\infty$ *and best possible for* $p=1$.

Proof. (i) From identity (18), we have

$$\left| f\left(\sum_{i=1}^{m} q_i a_i\right) - f(q_1 a_1) - \sum_{i=2}^{m} \left[f(Q_i a_i) - f(Q_{i-1} a_i) \right] - \frac{1}{t-s} \sum_{n=1}^{r-1} \frac{r-n}{n!} \left[f^{(n-1)}(s) \left((q_1 a_1 - s)^n - \left(\sum_{i=1}^{m} q_i a_i - s\right)^n - \sum_{i=2}^{m} (Q_{i-1} a_i - s)^n + \sum_{i=2}^{m} (Q_i a_i - s)^n \right) - f^{(n-1)}(t) \left((q_1 a_1 - t)^n - \left(\sum_{i=1}^{m} q_i a_i - t\right)^n - \sum_{i=2}^{m} (Q_{i-1} a_i - t)^n + \sum_{i=2}^{m} (Q_i a_i - t)^n \right) \right] \right| = \left| \int_s^t f^{(r)}(u)\, \hat{\alpha}(u)\, du \right|. \tag{48}$$

On the RHS of (48), we apply Hölder's inequality for integrals as follows

$$\left| \int_s^t f^{(r)}(u)\, \hat{\alpha}(u)\, du \right| \le \left(\int_s^t \left| f^{(r)}(u) \right|^p du \right)^{\frac{1}{p}} \left(\int_s^t |\hat{\alpha}(u)|^q du \right)^{\frac{1}{q}}. \tag{49}$$

Now inequality (49) together with (48) implies (45).

For the proof of the sharpness of the constant $\left(\int_s^t |\hat{\alpha}(u)|^q du\right)^{\frac{1}{q}}$, we define

$$f^{(r)}(u) = \begin{cases} \operatorname{sgn}\hat{\alpha}(u)\, |\hat{\alpha}(u)|^{\frac{1}{p-1}}, & 1 < p < \infty, \\ \operatorname{sgn}\hat{\alpha}(u), & p = \infty \end{cases}$$

such that the equality in (49) holds.

For $p = 1$, we will prove that the following inequality

$$\left| \int_s^t f^{(r)}(u)\, \hat{\alpha}(u)\, du \right| \le \max_{u \in [s,t]} |\hat{\alpha}(u)| \cdot \int_s^t \left| f^{(r)}(u) \right| du \tag{50}$$

is the best possible inequality.

Let $|\hat{\alpha}(u)|$ attains its maximum at $u_0 \in [s, t]$.

(Case 1) When $\hat{\alpha}(u_0) > 0$. For small enough ϵ, we define

$$f_\epsilon(u) = \begin{cases} 0, & s \le u \le u_0, \\ \frac{1}{\epsilon r!}(u-u_0)^r, & u_0 \le u \le u_0 + \epsilon, \\ \frac{1}{(r-1)!}(u-u_0)^{r-1}, & u_0 + \epsilon \le u \le t. \end{cases}$$

Clearly

$$\left| \int_s^t f_\epsilon^{(r)}(u)\, \hat{\alpha}(u)\, du \right| = \frac{1}{\epsilon} \int_{u_0}^{u_0+\epsilon} \hat{\alpha}(u)\, du \tag{51}$$

and

$$\int_s^t \left| f_\epsilon^{(r)}(u) \right| du = \frac{1}{\epsilon} \int_{u_0}^{u_0+\epsilon} du = 1. \tag{52}$$

Now use (51) and (52) in (50) and use the fact that $|\hat{\alpha}(u)|$ attains its maximum at $u_0 \in [s, t]$, we have

$$\frac{1}{\epsilon}\int_{u_0}^{u_0+\epsilon} \hat{\alpha}(u)\, du \leq \hat{\alpha}(u_0) \cdot 1 = \hat{\alpha}(u_0).$$

As $\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{u_0}^{u_0+\epsilon} \hat{\alpha}(u)\, du = \hat{\alpha}(u_0)$, the statement follows.

(Case 2) When $\hat{\alpha}(u_0) < 0$, we define

$$f_\epsilon(u) = \begin{cases} \frac{1}{(r-1)!}(u-u_0-\epsilon)^{r-1}, & s \leq u \leq u_0, \\ -\frac{1}{\epsilon r!}(u-u_0-\epsilon)^{r}, & u_0 \leq u \leq u_0+\epsilon, \\ 0, & u_0+\epsilon \leq u \leq t \end{cases}$$

and the remaining part is the same as above.

(*ii*) Use identity (19) and follow the proof of (*i*).

(*iii*) Use identity (20) and follow the proof of (*i*).

□

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