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On Certain Difference Operators and their Explicitly Inverses

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Abstract. In recent years, a number of extensive applications of difference operators through sequence spaces have been developed. The most crucial application is being used in the study of functional analysis, operator theory and matrix theory. In this context, the present article makes an attempt to provide a survey on various difference operators and unify them by introducing two $m+1$-th sequential band matrices. The purpose of this work is also to extend the determination of their inverses and derive an adaptive recursive free formula for matrix inversions. We provide two relevant formulas for inversion of $m+1$-th sequential lower and upper band matrices. Subsequently, the idea is being applied to develop a new explicitly formula for matrix inversion.

1. Introduction and preliminaries

Let w be the space of all real valued sequences. By $\mathbb{N}$, we denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $A = (a_{ij})$ $(i, j \in \mathbb{N})$ represent an infinite matrix. Then for any two sequence spaces X and Y, we define a matrix mapping $A : X \to Y$, as

$$(Ax)_n := \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}_0). \tag{1}$$

In fact, for a sequence $x = (x_k) \in X$, Ax is called as the $A-$transform of x provided the series in (1) converges for each $n \in \mathbb{N}_0$.

A matrix $A = (a_{nk})$ is called triangular if $a_{nk} = 0$ for $k > n$. It is called a triangle if it is triangular and $a_{nn} \neq 0$ for all n. It is well-known that a triangular matrix has an inverse if and only if it is a triangle(see [40]).

Now, using the idea of $A-$transform of the sequence $x = (x_k)$, the generalized difference operators $B_L(a[m])$ and $B_U(a[m])$ are defined by

$$(B_L(a[m])x)_k = \sum_{i=0}^{m} a_{k-i}(i)x_{k-i}$$

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and

$$(B_U(a[m])x)_k = \sum_{i=0}^{m} a_{k+i}(i)x_{k+i}$$

for all $k, m \in \mathbb{N}_0$, where $a[m] = \{a(0), a(1), \ldots a(m)\}$, the set of convergent sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0}$ $(0 \le i \le m)$ of real numbers (see [5, 6, 23]). In matrix notations, the difference operators $B_L(a[m])$ and $B_U(a[m])$ are being expressed as:

$$B_L(a[m]) := \begin{pmatrix} a_0(0) & 0 & 0 & \ldots & 0 & 0 & \ldots \\ a_0(1) & a_1(0) & 0 & \ldots & 0 & 0 & \ldots \\ a_0(2) & a_1(1) & a_2(0) & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & \ldots & a_m(0) & 0 & \ldots \\ 0 & a_1(m) & a_2(m-1) & \ldots & a_m(1) & a_{m+1}(0) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$B_U(a[m]) := \begin{pmatrix} a_0(0) & a_0(1) & a_0(2) & \ldots & a_0(m) & 0 & \ldots \\ 0 & a_1(0) & a_1(1) & \ldots & a_1(m-1) & a_1(m) & \ldots \\ 0 & 0 & a_2(0) & \ldots & a_2(m-2) & a_2(m-1) & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_m(0) & a_m(1) & \ldots \\ 0 & 0 & 0 & \ldots & 0 & a_{m+1}(0) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Examples:

- Let us take a sequence $x = (x_k)$, with $x_k = 1$ for all $k \in \mathbb{N}_0$, then we have $B_L(a[m])x \to l_{a[m]}$, where

$$l_{a[m]} = \lim_{k\to\infty} (a_k(0) + a_{k-1}(1) + \cdots + a_{k-m}(m)).$$

- Consider a sequence $x = (x_k)$, with $x_k = k$ for all $k \in \mathbb{N}_0$ and for a given positive integer m,

$$a_{k-i}(i) = \begin{cases} 1/(k-i), & (0 \le i \le m \text{ and } k \ne i) \\ 0, & (\text{ otherwise}). \end{cases}$$

  Then

$$\begin{aligned} (B_L(a[m])x)_k &= \sum_{i=0}^{m} a_{k-i}(i)(k-i) \\ &= k[a_k(0) + a_{k-1}(1) + a_{k-2}(2) + \cdots + a_{k-m}(m)] \\ &\qquad - [a_{k-1}(1) + 2a_{k-2}(2) + 3a_{k-3}(3) + \cdots + ma_{k-m}(m)] \\ &= k\left[\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + \frac{1}{k-m}\right] - \left[\frac{1}{k-1} + \frac{2}{k-2} + \frac{3}{k-3} + \cdots + \frac{m}{k-m}\right]. \end{aligned}$$

  Now, for a positive integer m, $(B_L(a[m])x)_k \to m+1$ as $k \to \infty$.

- Consider a sequence $x = (x_k)$, with $x_k = k$ for all $k \in \mathbb{N}_0$ and for a given integer $m > 1$,

$$a_{k-i}(i) = \begin{cases} (-1)^i\binom{m}{i}, & (0 \le i \le m) \\ 0, & (\text{ otherwise}) \end{cases}, \text{ for all } k \in \mathbb{N}_0.$$

Then it is clearly observed that

$$\begin{aligned} (B_L(a[m])x)_k &= \sum_{i=0}^{m}(-1)^i\binom{m}{i}(k-i) \\ &= k\left[\sum_{i=0}^{m}(-1)^i\binom{m}{i}\right] - m\left[\sum_{i=0}^{m}(-1)^i\binom{m-1}{i}\right] \\ &= 0 \end{aligned}$$

Therefore, for any positive integer $m > 1$, $(B_L(a[m])x)_k \to 0$ as $k \to \infty$.

It is remarked that the proposed difference operators induce several difference operators under suitable choice of the sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0}$ $(0 \le i \le m)$. We state some of the special cases of these operators in the following table:

Table 1

Sl.No.	$a(i); (0 \leq i \leq m)$	Special cases
1.	$a_{k\pm i}(i) = \begin{cases} (-1)^i, & (i = 0 \text{ and } i = 1) \\ 0, & (2 \leq i \leq m) \end{cases}; (k \in \mathbb{N}_0)$	$\Delta^{(1)}, \Delta$ [2, 32]
2.	$a_{k\pm i}(i) = \begin{cases} (-1)^i \binom{m}{i}, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$\Delta^{(m)}, \Delta^m$[1, 26]
3.	$a_{k-i}(i) = \begin{cases} r, & (i = 0) \\ s, & (i = 1) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$B(r, s)$[4, 27]
4.	$a_{k-i}(i) = \begin{cases} r, & (i = 0) \\ s, & (i = 1) \\ t, & (i = 2) \\ 0, & (i > 2) \end{cases}; (k \in \mathbb{N}_0)$	$B(r, s, t)$[28]
5.	$a_{k-i}(i) = \begin{cases} r_k, & (i = 0) \\ s_{k-1}, & (i = 1) \\ 0, & (i > 1) \end{cases}; (k \in \mathbb{N}_0)$	$B(\widetilde{r}, \widetilde{s})$[38]
6.	$a_{k-i}(i) = \begin{cases} r_k, & (i = 0) \\ s_{k-1}, & (i = 1) \\ t_{k-2}, & (i = 2) \\ 0, & (i > 2) \end{cases}; (k \in \mathbb{N}_0)$	$B(\widetilde{r}, \widetilde{s}, \widetilde{t})$[7]
7.	$a_{k-i}(i) = \begin{cases} r_k, & (i = 0) \\ s_{k-1}, & (i = 1) \\ t_{k-3}, & (i = 2) \\ u_{k-3}, & (i = 3) \\ 0, & (i > 3) \end{cases}; (k \in \mathbb{N}_0)$	$B(\widetilde{r}, \widetilde{s}, \widetilde{t}, \widetilde{u})$[7]
8.	$a_{k-i}(i) = \begin{cases} (-1)^i \binom{m}{i} v_{k-i}, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	Δ_v^m[8, 24, 25]
9.	$a_{k-i}(i) = \begin{cases} u_k v_{k-i}, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$G(u, v)$[3]
10.	$a_{k-i}(i) = \begin{cases} u_k v_k, & (i = 0) \\ u_k(v_{k-i} - v_{k-i+1}) & (1 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$G(u, v; \Delta)$[39]
11.	$a_{k-i}(i) = \begin{cases} \frac{1}{k+1}, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$(C, 1)$[20]
12.	$a_{k-i}(i) = \begin{cases} \frac{s_i t_{k-i}}{r_k}, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (k \in \mathbb{N}_0)$	$A(r, s, t)$[34]
13.	$a_{k-i}(i) = \begin{cases} \binom{k}{i}(1-p)^{k-i} p^i, & (0 \leq i \leq m) \\ 0, & (i > m) \end{cases}; (0 < p < 1, k \in \mathbb{N}_0)$	E^p[31]
14.	$a_{k-i}(i) = \begin{cases} \frac{f_k}{f_{k+1}}, & (i = 0) \\ -\frac{f_{k+1}}{f_k}, & (i = 1) \\ 0, & (i > 2) \end{cases}; (k \in \mathbb{N}_0)$	$\widehat{F}$[19]
15.	$a_{k-i}(i) = \begin{cases} \frac{\binom{k}{i}}{2^k(i+1)}, & (0 \leq i \leq k) \\ 0, & (i > k) \end{cases}; (k \in \mathbb{N}_0)$	$\tilde{B}$[15]
16.	$a_{k-i}(i) = \begin{cases} \frac{t_{k-i}}{T_k}, & (0 \leq i \leq k) \\ 0, & (i > k) \end{cases}; (k \in \mathbb{N}_0)$	N^t[16, 17]

Nowadays, one of the most interesting areas of research in mathematics is the study of difference operators and related sequence spaces which has been attracted in different areas of mathematical sciences especially in applied and computational mathematics involving calculus, matrix and approximation theory. Nevertheless, the applications are also found in the theory of function spaces, modular spaces, fractional difference operators, operational matrices and differential equations. The idea of difference sequence spaces plays a significant role in most of the applied and scientific problems involving the spectral properties of bounded linear operators (see [2, 4, 7, 8, 18, 22, 24, 25, 27, 28]), topological structures including matrix transformations (see [1, 3, 13, 19–21, 26, 29, 30, 33–36, 38, 39]), approximation theory and fractional calculus (see [9–12, 14, 37]), etc. In fact, the study of all the ideas discussed earlier is only feasible and convenient upon the determination of related inverse operators. The primary objective of this work is to find the inverse of the most of the difference operators and apply this idea in matrix inversions. For matrix inversion, several methods have been employed and most of the popular methods such as Gaussian Elimination, Gauss Jordan, Cholesky decomposition etc. are involved with a series of recursive calculations. Without taking the evaluations of previous elements it is observed that any arbitrary element of the inverse matrix can not be computed explicitly. Recently, using difference operators, Baliarsingh et al.[6] proposed an algorithm to find the inverse of a nonsingular matrix in non pivot case, but the proposed algorithm does not work in transition cases (pivot cases). In this work, using some existing results, we find the explicit formula for matrix inversion in both pivot and non pivot cases.

Let $M_n(\mathbb{R})$ be the set of all $n \times n$ (square) matrices over $\mathbb{R}$ and $A \in M_n(\mathbb{R})$ be a non singular matrix, given by

$$A := (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

Then A can be factorized, called LU factorization and expressed as products LU with L, a lower-unitriangular matrix and U, an upper-triangular matrix provided such product exists. In general, a triangular matrix is called unitriangular if all of its diagonal entries are equal to one. Now, we may write $A := LU$, where

$$L := (l_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{pmatrix} \quad \text{and} \quad U := (u_{ij}) = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{pmatrix}.$$

and

$$\det(L) = 1 \text{ and } \det(U) = \prod_{i=1}^{n} u_{ii}.$$

An $n \times n$ permutation is a matrix with precisely one entry whose value is '1' in each column and row, and all of whose others are '0'. The rows of a permutation matrix are k-permutation of the rows of identity matrix I_n. In componentwise, the permutation matrix $P = (p_{ij})$ can be expressed as

$$p_{ij} = \begin{cases} 1, & (j = k_i) \\ 0, & \text{(otherwise)}, \end{cases}$$

where k_i signifies the k-permutation of i−th row of identity matrix I_n. It is remarked that P is invertible, $\det(P) = \pm 1$, and more precisely, $P^{-1} = P^T$, transpose matrix of P.

However, for any non singular arbitrary matrix A, sometimes the corresponding LU factorization may not be possible directly, but it is noted that appropriate permutations of the rows will convert any invertible matrix A to a matrix of the form LU. In fact, there should be a permutation matrix $P \in M_n(\mathbb{R})$ such that PA has an LU factorization, where P is a permutation matrix. Now, we have the following important results which are being used for proving main theorems:

Remark 1.1. *For any invertible matrix $A \in M_n(\mathbb{R})$, there exists a permutation matrix $P \in M_n(\mathbb{R})$, a lower-unitriangular matrix $L \in M_n(\mathbb{R})$, and an upper-triangular matrix $U \in M_n(\mathbb{R})$ with $PA = LU$, or equivalently $A = P^T LU$.*

Remark 1.2. *It is noted that if the matrix A has directly LU-factorization, then the permutation matrix given in Remark 1.1 can be chosen as an identity matrix of same size.*

Remark 1.3. *For any invertible matrix $A \in M_n(\mathbb{R})$, let the LU-factorization of the matrix A via permutation matrix P be $A = P^T LU$, then inverse of A is given by*

$$A^{-1} = U^{-1}L^{-1}P.$$

2. Main results

In this section, we compute the individual inverses of the lower-unitriangular matrix L and the upper-triangular matrix U in simplified form. Finally, applying these results, we construct a new algorithm for matrix inversion.

Theorem 2.1. *The explicit formula for inverse of the matrix L is given by*

$$l^{-1}_{nk} = \begin{cases} 1, & (k = n) \\ \displaystyle\sum_{i=1}^{n-k} (-1)^{n-k+i-1} l_{k+i,k} D^{(k)}_{n-k-i}(L), & (0 \le k \le n-1) \\ 0, & (k > n) \end{cases}, \ (n, k \in \mathbb{N}_0).$$

where

$$D^{(k)}_{n-k}(L) = \begin{vmatrix} l_{k+1,k} & 1 & 0 & \dots & 0 \\ l_{k+2,k} & l_{k+2,k+1} & 1 & \dots & 0 \\ l_{k+3,k} & l_{k+3,k+1} & l_{k+3,k+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n,k} & l_{n,k+1} & l_{n,k+2} & \dots & l_{n,n-1} \end{vmatrix}, (n, k > 1) \text{ with } D^{(k)}_0(L) = 1.$$

Proof. From [6], it is observed that the inverse elements of the matrix L are given by

$$l^{-1}_{nk} = \begin{cases} 1, & (k = n) \\ (-1)^{n-k} D^{(k)}_{n-k}(L), & (0 \le k \le n-1) \\ 0, & (k > n) \end{cases}, \ (n, k \in \mathbb{N}_0).$$

Now, on expansion of the determinant $D^{(k)}_{n-k}(L)$, we get

$$\begin{aligned}
&D^{(k)}_{n-k}(L)\\
&= l_{k+1,k}\begin{vmatrix} l_{k+2,k+1} & 1 & 0 & \dots & 0\\ l_{k+3,k+1} & l_{k+3,k+2} & 1 & \dots & 0\\ l_{k+4,k+1} & l_{k+4,k+2} & l_{k+4,k+3} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ l_{n,k+1} & l_{n,k+2} & l_{n,k+3} & \dots & l_{n,n-1}\end{vmatrix} - \begin{vmatrix} l_{k+2,k} & 1 & 0 & \dots & 0\\ l_{k+3,k} & l_{k+3,k+2} & 1 & \dots & 0\\ l_{k+4,k} & l_{k+4,k+2} & l_{k+4,k+3} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ l_{n,k} & l_{n,k+2} & l_{n,k+3} & \dots & l_{n,n-1}\end{vmatrix}\\
&= l_{k+1,k}D^{(k+1)}_{n-k-1}(L) - l_{k+2,k}D^{(k+2)}_{n-k-2}(L) + l_{k+3,k}D^{(k+3)}_{n-k-3}(L) + \cdots + (-1)^{n-k}l_{n-1,k}D^{(n-1)}_{1}(L) + (-1)^{n-k-1}l_{n,k}\\
&= \sum_{i=1}^{n-k}(-1)^{i-1}l_{k+i,k}D^{(k+i)}_{n-k-i}(L).
\end{aligned}$$

This concludes the proof. □

Theorem 2.2. *The explicit formula for inverse of the matrix U is given by*

$$u^{-1}_{nk} = \begin{cases} \dfrac{1}{u_{nn}}, & (k = n)\\ \displaystyle\sum_{i=1}^{k-n} \frac{(-1)^{k-n+i-1}u_{n,n+i}D^{(n+i)}_{k-n-i}(U)}{u_{nn}\prod_{j=i+n}^{k} u_{jj}}, & (k > n)\\ 0, & (k < n)\end{cases}, \ (n,k \in \mathbb{N}_0).$$

where

$$D^{(n)}_{k-n}(U) = \begin{vmatrix} u_{n,n+1} & u_{n,n+2} & u_{n,n+3} & \dots & u_{n,k}\\ u_{n+1,n+1} & u_{n+1,n+2} & u_{n+1,n+3} & \dots & u_{n+1,k}\\ 0 & u_{n+2,n+2} & u_{n+2,n+3} & \dots & u_{n+2,k}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & u_{k-1,k}\end{vmatrix}, (n,k > 1) \text{ with } D^{(n)}_{0}(U) = 1.$$

Proof. Proof follows from [6] and subsequently, we represent the inverse of the matrix U as

$$u^{-1}_{nk} = \begin{cases} \dfrac{1}{u_{nn}}, & (k = n)\\ \dfrac{(-1)^{k-n}}{\prod_{j=n}^{k} u_{jj}}D^{(n)}_{k-n}(U), & (k > n)\\ 0, & (k < n)\end{cases}, \ (n,k \in \mathbb{N}_0).$$

However, the value of determinants $D_{k-n}^{(n)}(U)$ for each $n,k\in\mathbb{N}_0$ is being calculated as

$$\begin{aligned}
D_{k-n}^{(n)}(U) &= u_{n,n+1}\begin{vmatrix} u_{n+1,n+2} & u_{n+1,n+3} & u_{n+1,n+4} & \dots & u_{n+1,k}\\ u_{n+2,n+2} & u_{n+2,n+3} & u_{n+2,n+4} & \dots & u_{n+2,k}\\ 0 & u_{n+3,n+3} & u_{n+3,n+4} & \dots & u_{n+3,k}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & u_{k-1,k}\end{vmatrix}\\
&\quad - u_{n+1,n+1}\begin{vmatrix} u_{n,n+2} & u_{n,n+3} & u_{n,n+4} & \dots & u_{n,k}\\ u_{n+2,n+2} & u_{n+2,n+3} & u_{n+2,n+4} & \dots & u_{n+2,k}\\ 0 & u_{n+3,n+3} & u_{n+3,n+4} & \dots & u_{n+3,k}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & u_{k-1,k}\end{vmatrix}\\
&= u_{n,n+1}D_{k-n-1}^{(n+1)}(U) - u_{n+1,n+1}u_{n,n+2}D_{k-n-2}^{(n+2)}(U) + u_{n+1,n+1}u_{n+2,n+2}u_{n,n+3}D_{k-n-3}^{(n+3)}(U)+\\
&\quad \dots + (-1)^{k-n}u_{n+1,n+1}\dots u_{k-2,k-2}u_{n,k-1}D_1^{(k-1)}(U) + (-1)^{k-n-1}u_{n+1,n+1}\dots u_{k-1,k-1}u_{n,k}\\
&= u_{n,n+1}D_{k-n-1}^{(n+1)}(U) + \sum_{i=2}^{k-n}(-1)^{i-1}u_{n,n+i}\prod_{j=1}^{i-1}u_{n+j,n+j}D_{k-n-i}^{(n+i)}(U)\\
&= \sum_{i=1}^{k-n}(-1)^{i-1}u_{n,n+i}\prod_{j=1}^{i-1}u_{n+j,n+j}D_{k-n-i}^{(n+i)}(U).
\end{aligned}$$

Now, for $k>n$ it is calculated that

$$\begin{aligned}
u_{nk}^{-1} &= \frac{(-1)^{k-n}}{\prod_{j=n}^{k}u_{jj}}\sum_{i=1}^{k-n}(-1)^{i-1}u_{n,n+i}\prod_{j=1}^{i-1}u_{n+j,n+j}D_{k-n-i}^{(n+i)}(U)\\
&= \sum_{i=1}^{k-n}\frac{(-1)^{k-n+i-1}u_{n,n+i}D_{k-n-i}^{(n+i)}(U)}{u_{nn}\prod_{j=i+n}^{k}u_{jj}}.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *The explicit formula for inverse of the matrix A without pivoting via matrices $L=(l_{ij})$ and $U=(u_{ij})$ is given by*

$$a_{ij}^{-1} = \begin{cases} \sum_{k=j+1}^{n}\sum_{m=1}^{k-i}\sum_{p=1}^{k-j}(-1)^{m+p-i-j}u_{i,i+m}l_{j+p,j}\frac{D_{k-i-m}^{(i+m)}(U)D_{k-j-p}^{(j)}(L)}{u_{ii}\prod_{r=i+m}^{k}u_{rr}} \\ + \sum_{m=1}^{j-i}(-1)^{j-i+m-1}u_{i,i+m}\frac{D_{j-i-m}^{(i+m)}(U)}{u_{ii}\prod_{r=i+m}^{j}u_{rr}}, & (i\le j)\\ \sum_{k=i+1}^{n}\sum_{m=1}^{k-i}\sum_{p=1}^{k-j}(-1)^{m+p-i-j}u_{i,i+m}l_{j+p,j}\frac{D_{k-i-m}^{(i+m)}(U)D_{k-j-p}^{(j)}(L)}{u_{ii}\prod_{r=i+m}^{k}u_{rr}} \\ + \frac{1}{u_{ii}}\left[\sum_{m=1}^{i-j}(-1)^{i-j+m-1}l_{j+m,j}D_{i-j-m}^{(j)}(L)\right], & (i>j)\end{cases},$$

Proof. For non pivoting case, consider a matrix A which has LU factorizations directly, then its inverse can be found out immediately as

$$A^{-1}=U^{-1}L^{-1}.$$

More precisely, using Theorems 2.1 and 2.2, the elements of the inverse matrix $A^{-1}=(a_{ij}^{-1})$ $(i,j\in\mathbb{N})$ are being calculated explicitly as follows:

For all $i \leq j$,

$$
\begin{aligned}
a_{ij}^{-1} &= \sum_{k=j}^{n} u_{ik}^{-1} l_{kj}^{-1} \\
&= u_{ij}^{-1} l_{jj}^{-1} + \sum_{k=j+1}^{n} u_{ik}^{-1} l_{kj}^{-1} \\
&= \sum_{m=1}^{j-i} (-1)^{j-i+m-1} u_{i,i+m} \frac{D_{j-i-m}^{(i+m)}(U)}{u_{ii} \prod_{r=i+m}^{j} u_{rr}} + \sum_{k=j+1}^{n} \sum_{m=1}^{k-i} \sum_{p=1}^{k-j} (-1)^{m+p-i-j} u_{i,i+m} l_{j+p,j} \frac{D_{k-i-m}^{(i+m)}(U) D_{k-j-p}^{(j)}(L)}{u_{ii} \prod_{r=i+m}^{k} u_{rr}}.
\end{aligned}
$$

For all $i > j$,

$$
\begin{aligned}
a_{ij}^{-1} &= \sum_{k=i}^{n} u_{ik}^{-1} l_{kj}^{-1} \\
&= u_{ii}^{-1} l_{ij}^{-1} + \sum_{k=i+1}^{n} u_{ik}^{-1} l_{kj}^{-1} \\
&= \frac{1}{u_{ii}} \left[\sum_{m=1}^{i-j} (-1)^{i-j+m-1} l_{j+m,j} D_{i-j-m}^{(j)}(L) \right] + \sum_{k=i+1}^{n} \sum_{m=1}^{k-i} \sum_{p=1}^{k-j} (-1)^{m+p-i-j} u_{i,i+m} l_{j+p,j} \frac{D_{k-i-m}^{(i+m)}(U) D_{k-j-p}^{(j)}(L)}{u_{ii} \prod_{r=i+m}^{k} u_{rr}}
\end{aligned}
$$

□

Theorem 2.4. *The explicit formula for A^{-1} with pivoting via matrices $L = (l_{ij})$, $U = (u_{ij})$ and P is given by*

$$A^{-1} = (r_{ij}) = (a_{is}^{-1}),$$

where s-th row of I_n is being k-permutated to get permutation matrix P and

$$
a_{is}^{-1} = \begin{cases}
\sum_{k=s+1}^{n} \sum_{m=1}^{k-i} \sum_{p=1}^{k-s} (-1)^{m+p-i-s} u_{i,i+m} l_{s+p,s} \frac{D_{k-i-m}^{(i+m)}(U) D_{k-s-p}^{(s)}(L)}{u_{ii} \prod_{r=i+m}^{k} u_{rr}} \\
+ \sum_{m=1}^{s-i} (-1)^{s-i+m-1} u_{i,i+m} \frac{D_{s-i-m}^{(i+m)}(U)}{u_{ii} \prod_{r=i+m}^{s} u_{rr}}, & (i \leq s) \\
\sum_{k=i+1}^{n} \sum_{m=1}^{k-i} \sum_{p=1}^{k-s} (-1)^{m+p-i-s} u_{i,i+m} l_{s+p,s} \frac{D_{k-i-m}^{(i+m)}(U) D_{k-s-p}^{(s)}(L)}{u_{ii} \prod_{r=i+m}^{k} u_{rr}} \\
+ \frac{1}{u_{ii}} \left[\sum_{m=1}^{i-s} (-1)^{i-s+m-1} l_{s+m,s} D_{i-s-m}^{(s)}(L) \right], & (i > s)
\end{cases},
$$

Proof. The proof follows from Remark 1.1. □

Remark 2.5. *In Theorem 2.4, it is seen that the elements of $A^{-1} = (r_{ij})$ can be determined by taking m-permutation of j-th column of (a_{ij}^{-1}), defined in Theorem 2.3.*

Remark 2.6. *Theorem 2.4 is valid for both pivoting and non pivoting cases. In particular, for non pivoting case, the permutation matrix P is reduced to the identity matrix I_n, therefore the result of Theorem 3 is immediately obtained from Theorem 2.4.*

As an application of Theorems 2.1 and 2.2, we may list the inverses of matrices provided in Table 1 given in previous section as follow:

Table 2

Operators	Corresponding Matrix($a = (a_{nk})$)	Inverse($B = (b_{nk})$)
$\Delta^{(1)}$	$\begin{cases} 1, & (k = n) \\ -1, & (k = n+1) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} 1, & (k \geq n) \\ 0, & (\text{otherwise}) \end{cases}$
Δ	$\begin{cases} 1, & (k = n) \\ -1, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} 1, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$
Δ^m	$\begin{cases} (-1)^{k-n}\binom{m}{k-n}, & (k \geq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} (-1)^{k-n}\binom{m+k-n-1}{k-n}, & (k \geq n) \\ 0, & (\text{otherwise}) \end{cases}$
$\Delta^{(m)}$	$\begin{cases} (-1)^{n-k}\binom{m}{n-k}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} (-1)^{n-k}\binom{m+n-k-1}{n-k}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$
$B(r,s)$	$\begin{cases} r, & (k = n) \\ s, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{r}, & (k = n) \\ \frac{(-1)^{n-k}s^{n-k}}{r^{n-k+1}}, & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$
$B(r,s,t)$	$\begin{cases} r, & (k = n) \\ s, & (k = n-1) \\ t, & (k = n-2) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{r}, & (k = n) \\ \frac{1}{r}\sum_{j=0}^{n-k} z_1^{n-k-j} z_2^j (\text{c.f.}(4)), & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$
$G(u,v)$	$\begin{cases} u_n v_k, & (k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{u_n v_n}, & (k = n) \\ -\frac{1}{u_{n-k-1} v_{n-k}}, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$
$G(u,v;\Delta)$	$\begin{cases} u_n v_n, & (k = n) \\ u_n(v_k - v_{k+1}), & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{u_n v_n}, & (k = n) \\ -\frac{(v_k - v_{k+1})}{u_k v_k v_{k+1}}, & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$
$(C,1)$	$\begin{cases} \frac{1}{n+1}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} n, & (k = n) \\ -(n-1), & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$
$B(\tilde{r},\tilde{s})$	$\begin{cases} r_k, & (k = n) \\ s_k, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{r_n}, & (k = n) \\ \frac{(-1)^{n-k}\prod_{i=k}^{n-1} s_i}{\prod_{j=k}^{n} r_j}, & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$
$B(\tilde{r},\tilde{s},\tilde{t})$	$\begin{cases} r_k, & (k = n) \\ s_k, & (k = n-1) \\ t_k, & (k = n-2) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{1}{r_n}, & (k = n) \\ \frac{(-1)^{n-k} D_{n-k}^{(k)}(\tilde{r},\tilde{s},\tilde{t})}{\prod_{j=k}^{n} r_j} (\text{c.f.}(2)), & (0 \leq k < n) \\ 0, & (\text{otherwise}) \end{cases}$
$\widehat{F}$	$\begin{cases} \frac{f_n}{f_{n+1}}, & (k = n) \\ -\frac{f_{n+1}}{f_n}, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$
$\widetilde{B}$	$\begin{cases} \frac{\binom{n}{k}}{2^k(k+1)}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} \sum_{j=k}^{n} \binom{j}{k} 2^k (2j - k + 1) & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$
N^t	$\begin{cases} \frac{t_{n-k}}{T_n}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} (-1)^{n-k} D_{n-k} T_k (\text{c.f.}(3)), & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$
E^p	$\begin{cases} \binom{n}{k}(1-p)^{n-k}p^k, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$	$\begin{cases} (-1)^{n-k}\binom{n}{k}(1-p)^{n-k}p^{-n}, & (0 \leq k \leq n) \\ 0, & (\text{otherwise}) \end{cases}$

where

$$D_{n-k}^{(k)}(\tilde{r},\tilde{s},\tilde{t}) = \begin{vmatrix} s_k & r_{k+1} & 0 & \dots & 0 \\ t_k & s_{k+1} & r_{k+2} & \dots & 0 \\ 0 & t_{k+1} & s_{k+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s_{n-1} \end{vmatrix}, \tag{2}$$

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & \dots & 0 \\ t_2 & t_1 & 1 & \dots & 0 \\ t_3 & t_2 & t_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_n & t_{n-1} & t_{n-3} & \dots & t_1 \end{vmatrix}, D_0 = 1, \tag{3}$$

$$z_1 = \frac{-s + \sqrt{s^2 - 4rt}}{2r}, z_2 = \frac{-s - \sqrt{s^2 - 4rt}}{2r} \text{ and } T_n = \sum_{k=0}^{n} t_k. \tag{4}$$

Now, using Theorems 2.3 and 2.4, we construct the following algorithm for matrix inversion:
Algorithm (Matrix inversion with and without pivoting):

Step 1 : LU factorization

- Input n, size of matrix A and elements of A.
- Decompose A as $A = PLU$, and compute P, L and U.

Step 2 : Inverse of L

- Set $L = (l_{ij})_{n\times n} = (\mathbf{r}^1; \mathbf{r}^2; \dots ; \mathbf{r}^n)^T$, where $\mathbf{r}^k = (r_1^k, r_2^k \dots r_n^k)$, kth row vector of L with $r_j^k = l_{kj}$, $(1 \le j < k), l_{jj} = 1$ and 0 otherwise.
- Compute the matrix $[D_{n-k}^{(k)}(L)] = (d_{ij}(L))_{(n-k)\times(n-k)} = (\mathbf{r}^{k+1}; \mathbf{r}^{k+2}; \dots ; \mathbf{r}^n)^T$.
- Compute $D_{n-k}^{(k)}(L) = \sum_{i=1}^{n-k} (-1)^{i-1} l_{k+i,k} D_{n-k-i}^{(k+i)}(L)$ for $n > k$, 1 for $n = k$, and 0 for $n < k$.
- Compute inverse of L as $L^{-1} = (l_{nk}^{-1}) = (-1)^{n-k} D_{n-k}^{(k)}(L)$ for $n > k$ and set $l_{nk}^{-1} = 1$ for $k = n$, and 0 for $k > n$.

Step 3 : Inverse of U

- Set $U = (u_{ij})_{n\times n} = (\mathbf{c}^1; \mathbf{c}^2; \dots ; \mathbf{c}^n)$, where $\mathbf{c}^k = (c_1^k, c_2^k \dots c_n^k)^T$, kth column vector of U with $c_i^k = u_{ik}$, $(i < k \le n), u_{ii} \ne 0$ and 0 otherwise.
- Compute the matrix $[D_{k-n}^{(n)}(U)] = (d_{ij}(U))_{(k-n)\times(k-n)} = (\mathbf{c}^{n+1}; \mathbf{c}^{n+2}; \dots ; \mathbf{c}^k)$.
- Compute $D_{k-n}^{(n)}(U) = \sum_{i=1}^{k-n-1} (-1)^{i-1} u_{n,n+i} \prod_{j=1}^{i-1} u_{n+j,n+j} D_{k-n-i}^{(n+i)}(U)$ for $k > n$, 1 for $k = n$, and 0 for $k < n$
- Compute inverse of U as $U^{-1} = (u_{nk}^{-1}) = \sum_{i=1}^{k-n} \frac{(-1)^{k-n+i-1} u_{n,n+i} D_{k-n-i}^{(n+i)}(U)}{u_{nn} \prod_{j=i}^{k}}$ for $k > n$ and set $u_{nk}^{-1} = \frac{1}{u_{nn}}$ for $k = n$ and 0 for $k < n$.

Step 4 : Inverse of A

- Compute inverse of A as $A^{-1} = U^{-1} L^{-1} P$.

3. Example, discussion and verification

In this section, we provide a numerical example illustrating which the detail of the proposed algorithm is described. The time complexity and other features of the algorithm are discussed. We also design the relevant MATLAB codes for the algorithm and implement it against different type and size of matrices.
Example 1: Consider a non singular matrix A of order 4, where

$$A = \begin{pmatrix} 0 & 2 & 3 & 5 \\ 4 & 0 & 6 & 5 \\ 8 & 1 & 10 & 0 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Note that factorization of A is only possible by pivoting rows. On LU factorization of A (Step 1 of the algorithm), we have

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1/2 & 1 & 0 \\ 1/2 & 3/2 & 1 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 4 & 0 & 6 & 5 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & -7/2 & -25/2 \\ 0 & 0 & 0 & 7/2 \end{pmatrix}$$

From Step 2 of the algorithm, we can calculate the values of $D_{n-k}^{(k)}(L) = D_{n-k}^{(k)}$ as follows:

$$D_1^{(1)} = 0; \qquad D_2^{(1)} = \begin{vmatrix} 0 & 1 \\ 2 & 1/2 \end{vmatrix} = -2; \qquad D_3^{(1)} = \begin{vmatrix} 0 & 1 & 0 \\ 2 & 1/2 & 1 \\ 1/2 & 3/2 & 1 \end{vmatrix} = -3/2;$$

$$D_1^{(2)} = 1/2; \qquad D_2^{(2)} = \begin{vmatrix} 1/2 & 1 \\ 3/2 & 1 \end{vmatrix} = -1; \qquad D_1^{(3)} = 1.$$

The inverse of L can be calculated as

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (-1)^{2-1}D_1^{(1)} & 1 & 0 & 0 \\ (-1)^{3-1}D_2^{(1)} & (-1)^{3-2}D_1^{(2)} & 1 & 0 \\ (-1)^{4-1}D_3^{(1)} & (-1)^{4-2}D_2^{(2)} & (-1)^{4-3}D_1^{(3)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -1/2 & 1 & 0 \\ 3/2 & -1 & -1 & 1 \end{pmatrix}$$

Similarly, from Step 3 of the algorithm, one may calculate the values of $D_{k-n}^{(n)}(U) = D_{k-n}^{(n)}$ as follows:

$$D_1^{(1)} = 0; \qquad D_2^{(1)} = \begin{vmatrix} 0 & 6 \\ 2 & 3 \end{vmatrix} = -12; \qquad D_3^{(1)} = \begin{vmatrix} 0 & 6 & 5 \\ 2 & 3 & 5 \\ 0 & -7/2 & -25/2 \end{vmatrix} = 115;$$

$$D_1^{(2)} = 3; \qquad D_2^{(2)} = \begin{vmatrix} 3 & 5 \\ -7/2 & -25/2 \end{vmatrix} = -20; \qquad D_1^{(3)} = -25/2.$$

Using Step 3 of the algorithm, the inverse of U can be calculated as

$$U^{-1} = \begin{pmatrix} 1/4 & \frac{(-1)^{2-1}D_1^{(1)}}{8} & \frac{(-1)^{3-1}D_2^{(1)}}{-28} & \frac{(-1)^{4-1}D_3^{(1)}}{-98} \\ 0 & 1/2 & \frac{(-1)^{3-2}D_1^{(2)}}{-7} & \frac{(-1)^{4-2}D_2^{(2)}}{-49/2} \\ 0 & 0 & -2/7 & \frac{(-1)^{4-3}D_1^{(3)}}{-49/4} \\ 0 & 0 & 0 & 2/7 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 3/7 & 115/98 \\ 0 & 1/2 & 3/7 & 40/49 \\ 0 & 0 & -2/7 & -50/49 \\ 0 & 0 & 0 & 2/7 \end{pmatrix}$$

Finally, using Step 4 of the algorithm, we obtain

$$A^{-1} = U^{-1}L^{-1}P = \begin{pmatrix} -1.3878 & 1.1531 & -0.7449 & 1.1735 \\ -0.5306 & 0.3673 & -0.3878 & 0.8163 \\ 1.1633 & -0.9592 & 0.7347 & -1.0204 \\ -0.2857 & 0.4286 & -0.2857 & 0.2857 \end{pmatrix}$$

As the central theme of the present algorithm is the evaluations of determinants $D_n^{(k)}(L)$ and $D_n^{(k)}(U)$ by breaking down them to the simpler form, we need to find the computational costs for both of the evaluation of the determinants. Theorems 2.1 and 2.2 suggest that the computational cost for both of the algorithms for complete evaluations of these determinant is $\frac{n(n+1)}{2}$, where n denotes the size of the determinant. As we know the computational costs for LU decomposition and matrix multiplication are, respectively $n^3/3$ and n^2, all together the entire algorithm takes $\frac{2n^3+15n^2+3n}{6}$ and $\frac{2n^3+9n^2+3n}{6}$ computations for pivot and non pivot cases. However, this time complexity will be remarkably reduced in the case of sparse and Cesàro type matrices, where the evaluations of the determinants of Hessenberg matrices $[D_n^{(k)}(L)]$ and $[D_n^{(k)}(U)]$ takes less number of computations comparatively. In fact, for a Cesàro matrix the prosed algorithm needs only $\frac{n^3+3n^2+3}{6}$ computations for non pivot case. The main advantage of the proposed algorithm is that even for transient stage of LU factorization, it does not become unstable and it works normally by adding few numbers of additional computations. As it is known that the algorithm involves the evaluation of determinants Hessenberg matrices $[D_n^{(k)}(L)]$ and $[D_n^{(k)}(U)]$, we conclude that it seems to be stable if $\|[D_n^{(k)}(L)]\|$ and $\|[D_n^{(k)}(U)]\|$ are finite. An other important feature of this algorithm is that one can find the explicit expression of any elements of the inverse matrix without considering its previous or neighboring elements.
Verification: Above algorithms are being programmed using MATLAB codes and also verified by taking different size of matrices. For instances, we take an example of a matrices of size 10 and by compiling the respective program in MATLAB we get following output.

Example 2 :

```
enter the matrix: [0 2 3 4 5 6 7 8 9 1; 2 0 3 5 6 3 2 8 2 4; 1 8 0
5 1 7 2 9 3 6; 1 8 3 0 3 6 1 5 0 3; 1 2 5 6 0 3 2 7 8 2; 1 3 1 4 5
0 0 3 4 5; 6 1 3 9 3 4 0 4 1 7; 0 4 1 8 3 4 6 0 1 0; 1 4 8 3 5 9 2
7 0 3; 9 8 3 7 9 1 4 8 2 0]

inv_matrix =
 0.0801  0.0280  0.0904 0.1539  0.0098  0.1202  0.1561  0.0590  0.1201 0.0340
 0.0556  0.0076  0.0158 0.1331  0.0554  0.0225  0.0377 0.0440  0.0594  0.0117
 0.0896  0.3511  0.3753 0.6288  0.2541  0.1864 0.0704 0.1182  0.3394  0.1229
 0.0632  0.2192  0.2424  0.5095  0.0571  0.1591  0.0855 0.0068  0.3104 0.0896
 0.0582  0.2799 0.2049  0.5163  0.2134 0.2722  0.0998  0.0892  0.3818 0.1144
 0.1370  0.5656 0.4056  0.7961  0.2854 0.2247  0.0362  0.1659  0.5937 0.1340
 0.0028 0.8088  0.6466 1.3313  0.3050 0.5125  0.2040 0.2820  0.9407  0.2642
 0.0547 0.1598 0.0529 0.0034  0.0502 0.0806  0.0709  0.0205  0.0378 0.0131
 0.1118  0.3180 0.1415  0.3400  0.0669 0.1835  0.0154  0.1093  0.2185 0.0658
 0.0289 0.5536  0.4411 0.9574 0.2003  0.2501  0.2004  0.1453  0.6546  0.2357
P =

 0 1 0 0 0 0 0 0 0 0
 1 0 0 0 0 0 0 0 0 0
 0 0 1 0 0 0 0 0 0 0
 0 0 0 1 0 0 0 0 0 0
 0 0 0 0 1 0 0 0 0 0
 0 0 0 0 0 1 0 0 0 0
```

0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1

4. Conclusion

In this study, using difference operators, two generalized sequential banded matrices have been introduced and adaptive formulas for their inversion has been formulated. As a result, a new algorithm for matrix inversion has been proposed which is based on LU factorizations and individual inverse of sequential banded matrices L and U. In the present algorithm, we have evaluated explicitly the elements of the inverse matrix without taking any recursive calculations. We have basically designed the algorithm involving three steps such as LU factorizations with or without pivoting, computation of permutation matrix and new inversion formulas for triangular matrices including evaluation of determinant of certain matrices called Hessenberg matrices.

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