



Integral Modification of Apostol-Genocchi Operators

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Abstract. In this article, we consider Jain-Durrmeyer operators associated with the Apostol-Genocchi polynomials and study the approximation properties of these Durrmeyer operators. Furthermore, we examine the approximation behaviour of these operators including K -functional. We estimate the rate of convergence of the proposed operators for function in Lipschitz-type space and local approximation results by using modulus of continuity. Employing Mathematica software, to show the approximation and the absolute error graphically by varying the values of given parameters.

1. Introduction and motivation

The classical Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, and Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B^{(\alpha)}(x)$, $E^{(\alpha)}(x)$ and $G^{(\alpha)}(x)$ of (real or complex) order α , are usually by means of the following generating functions (see, [3, 31, 32, 36, 39, 43] for details):

$$\left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < 2\pi; 1^\alpha := 1), \quad (1)$$

$$\left(\frac{2}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi; 1^\alpha := 1), \quad (2)$$

and

$$\left(\frac{2z}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi; 1^\alpha := 1, \alpha \in \mathbb{N}_0). \quad (3)$$

so that, obviously

$$B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x), \quad \text{and} \quad G_n(x) := G_n^{(1)}(x), \quad (n \in \mathbb{N}_0), \quad (4)$$

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where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ($\mathbb{N} := \{1, 2, 3, \dots\}$).

The classical Bernoulli numbers $B_n(x)$, Euler numbers $E_n(x)$ and Genocchi numbers $G_n(x)$ can be readily found from Equations (1), (2) and (3) that

$$B_n := B_n(0) = B_n^{(1)}(0), \quad E_n := E_n(0) = E_n^{(1)}(0), \quad \text{and} \quad G_n := G_n(0) = G_n^{(1)}(0),$$

respectively.

Analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [9] and later on Srivastav [37]. An analogous extension of the generalized Euler polynomials as the Apostol-Euler polynomials studied by Luo.

Moreover, Luo [25–28] introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order α , which are defined as follows:

Definition 1.1. [26] The Apostol-Genocchi polynomials $G_n^{(\alpha)}(x; \lambda)$ ($\lambda \in \mathbb{C}$) of (real or complex) order α in variable x are defined by means of the generating function:

$$\left(\frac{2z}{\lambda e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|; 1^\alpha := 1). \tag{5}$$

with, of course,

$$G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1) \quad \text{and} \quad G_n^{(\alpha)}(\lambda) := G_n^{(\alpha)}(0; \lambda)$$

and

$$G_n(x; \lambda) := G_n^{(1)}(x; \lambda) \quad \text{and} \quad G_n(\lambda) := G_n^{(1)}(\lambda),$$

where $G_n(\lambda)$, $G_n^{(\alpha)}(\lambda)$ and $G_n(x; \lambda)$ denote the so-called Apostol-Genocchi numbers, Apostol-Genocchi numbers of order α and Apostol-Genocchi polynomials, respectively.

Remark 1.2. When $\lambda \neq -1$ in (5), the order α of the generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x; \lambda)$ should tacitly be restricted to non negative integer values.

For our convenience, we consider the operators in the following form:

For $f \in C[0, \infty)$, the operator is defined as:

$$\mathcal{A}_n^{\alpha, \lambda}(f; x) = \sum_{k=0}^{\infty} b_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) = e^{-nx} \left(\frac{1 + e\lambda}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{G_k^{(\alpha)}(nx; \lambda)}{k!} f\left(\frac{k}{n}\right) \tag{6}$$

where $G_k^{(\alpha)}(x; \lambda)$ is generalized Apostol-Genocchi polynomials, which have the generating function of the form

$$\left(\frac{2t}{1 + \lambda e^t}\right)^\alpha e^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!}, \quad (|t| < \pi). \tag{7}$$

The Apostol-Genocchi polynomials and their properties are studied by many researchers for the detail here we refer (cf. [8, 14, 19, 20, 23, 27, 28, 30, 33, 34, 40–42]).

In [29], the following explicit formula for the Apostol-Genocchi polynomials $G_k^{(\alpha)}(x; \lambda)$ is given:

$$G_k^{(\alpha)}(x; \lambda) = 2^\alpha \alpha! \binom{k}{\alpha} \sum_{i=0}^{k-\alpha} \frac{\lambda^i}{(1 + \lambda)^{\alpha+i}} \binom{k-\alpha}{i} \binom{\alpha+i-1}{i} \times \sum_{j=0}^i (-1)^j \binom{i}{j} j^i (x+j)^{k-i-\alpha} {}_2F_1[\alpha+i-k, i; i+1; j/(x+j)], \tag{8}$$

where $k, \alpha \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}$, $\lambda \in \mathbb{R} \setminus \{-1\}$ and ${}_2F_1[a, b; c; z]$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c} \frac{z^2}{2!} + \dots,$$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$, $(n \geq 1)$ and $0 \leq \alpha < 1$ (see [2], pp. 37).

Over the most recent two decades, an amazing number of papers showed up contemplating Genocchi numbers, their combinatorial relations, Genocchi polynomials, and their speculations alongside their different extensions and integral representations, which provides a new direction in the field of positive linear operators. To the readers, we suggest the following articles (cf. [4, 10, 17, 21]).

Jain [22] introduced a new class of linear operators as:

$$J_n^{(\beta)}(f; x) = \sum_{k=0}^{\infty} p_{n,k}^{(\beta)}(x) f\left(\frac{k}{n}\right), \quad x \geq 0, \tag{9}$$

where $0 \leq \beta < 1$ and

$$p_{n,k}^{(\beta)}(x) = \frac{nx(nx + k\beta)^{k-1} e^{-(nx+k\beta)}}{k!}.$$

For $\beta = 0$, these operators reduce to Szász-Mirakyan operators. Several researchers studied Jain operators and their integral variant (See [5, 6, 11, 16, 35, 44]).

Durrmeyer variants of various operators are studied by several researchers (see [1, 11–13, 45]) but in the year 2015, Gupta and Greubel [18] introduced the Durrmeyer variant of Jain operators (9). Motivated from [18], we now consider a Durrmeyer type modification of Apostol-Genocchi operators based on Jain operators. For $f \in C[0, \infty)$ the operators are defined as:

$$\begin{aligned} \mathcal{H}_n(f; x) &= \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) f(\xi) d\xi \\ &= \sum_{k=0}^{\infty} \frac{\langle p_{n,k}^{(\beta)}(\xi), f(\xi) \rangle}{\langle p_{n,k}^{(\beta)}(\xi), 1 \rangle} b_{n,k}^{(\alpha)}(x), \end{aligned} \tag{10}$$

where $\langle f, g \rangle = \int_0^{\infty} f(\xi)g(\xi) d\xi$.

Some interesting results are studied by several mathematicians which have given a new direction in the field of positive linear operators (cf. [4, 10, 17, 21]).

The main goal of this article is to construct Durrmeyer type operators of Apostol-Genocchi operators (6) based on the Jain operators (9) with real parameters α, β , and λ . We obtain moments and estimate the rate of convergence of Jain-Durrmeyer operators associated with the Apostol-Genocchi operator. We establish approximation estimates such as a global approximation theorem along with some convergence estimates in terms of usual modulus of continuity and examine the approximation behaviour of these operators (10) including K -functional. We also estimate the rate of convergence of the proposed operators for function in Lipschitz-type space. Moreover, the graphical interpretation to find the absolute error for some particular values of parameters by using Mathematica software.

2. Basic Properties

Lemma 2.1. [18] For $0 \leq \beta < 1$, we have

$$\frac{\langle p_{n,k}^{(\beta)}(\xi), \xi^r \rangle}{\langle p_{n,k}^{(\beta)}(\xi), 1 \rangle} = \mathcal{P}_r(k; \beta), \quad r = 0, 1, 2, \dots$$

where $\mathcal{P}_r(k; \beta)$ is a polynomial of order r in variable k and $\langle f, g \rangle = \int_0^\infty f(\xi)g(\xi)d\xi$. In particular

$$\begin{aligned} \mathcal{P}_0(k; \beta) &= 1; \\ \mathcal{P}_1(k; \beta) &= \frac{1}{n} \left[(1 - \beta)k + \frac{1}{1 - \beta} \right]; \\ \mathcal{P}_2(k; \beta) &= \frac{1}{n^2} \left[(1 - \beta)^2 k^2 + 3k + \frac{2!}{1 - \beta} \right]; \\ \mathcal{P}_3(k; \beta) &= \frac{1}{n^3} \left[(1 - \beta)^3 k^3 + 6(1 - \beta)k^2 + \frac{(11 - 8\beta)k}{1 - \beta} + \frac{3!}{1 - \beta} \right]; \\ \mathcal{P}_4(k; \beta) &= \frac{1}{n^4} \left[(1 - \beta)^4 k^4 + 10(1 - \beta)^2 k^3 + 5(7 - 4\beta)k^2 \right. \\ &\quad \left. + \frac{10(5 - 3\beta)k}{1 - \beta} + \frac{4!}{1 - \beta} \right]; \\ \mathcal{P}_5(k; \beta) &= \frac{1}{n^5} \left[(1 - \beta)^5 k^5 + 15(1 - \beta)^3 k^4 + 5(1 - \beta)(17 - 8\beta)k^3 \right. \\ &\quad \left. + \frac{15(15 - 20\beta + 6\beta^2)k^2}{1 - \beta} + \frac{(274 - 144\beta)k}{1 - \beta} + \frac{5!}{1 - \beta} \right]. \end{aligned}$$

Lemma 2.2. [34] For $\mathcal{A}_n^{\alpha, \lambda}(t^m; x)$, $m = 0, 1, 2, 3$ and 4 , we have

$$\begin{aligned} \mathcal{A}_n^{\alpha, \lambda}(1; x) &= 1; \\ \mathcal{A}_n^{\alpha, \lambda}(\xi; x) &= x + \frac{\alpha}{n(1 + e\lambda)}; \\ \mathcal{A}_n^{\alpha, \lambda}(\xi^2; x) &= x^2 + \frac{(1 + 2\alpha + e\lambda)}{n(1 + e\lambda)}x + \frac{\alpha^2 - 2\alpha e\lambda - \alpha e^2 \lambda^2}{n^2(1 + e\lambda)^2}; \\ \mathcal{A}_n^{\alpha, \lambda}(\xi^3; x) &= x^3 + \frac{(3 + 3\alpha + 3e\lambda)}{n(1 + e\lambda)}x^2 + \frac{(3\alpha^2 + 3\alpha + e^2 \lambda^2 - 3\alpha e^2 \lambda^2 - 3\alpha e\lambda + 2e\lambda + 1)}{n^2(1 + e\lambda)^2}x \\ &\quad + \frac{(\alpha^3 - 6\alpha^2 e\lambda - 3\alpha^2 e^2 \lambda^2 - 5\alpha e\lambda - 4\alpha e^2 \lambda^2 - \alpha e^3 \lambda^3)}{n^3(1 + e\lambda)^3}; \\ \mathcal{A}_n^{\alpha, \lambda}(\xi^4; x) &= x^4 + \frac{(3 + 2\alpha + 3e\lambda)}{n(1 + e\lambda)}x^3 + \frac{(-6\alpha^2 - 25e^2 \lambda^2 - 50e\lambda + 6\alpha e^2 \lambda^2 - 12\alpha - 25)}{n^2(1 + e\lambda)^2}x^2 \\ &\quad + \frac{x}{n^3(1 + e\lambda)^3} \left[2\alpha^3 + 7e^3 \lambda^3 - 5\alpha e^3 \lambda^3 + 21e^2 \lambda^2 + 3\alpha^2 - 6\alpha^2 e^2 \lambda^2 + 3\alpha e^2 \lambda^2 \right. \\ &\quad \left. - 9\alpha^2 e\lambda + 21e\lambda + 20\alpha + 24\alpha e\lambda \right]. \end{aligned}$$

Lemma 2.3. For the operators \mathcal{H}_n given by (10), the moments up to second order are given by

$$\mathcal{H}_n(1; x) = 1, \quad \mathcal{H}_n(\xi; x) = (1 - \beta)x + \frac{1}{n} \left\{ \frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta} \right\},$$

and

$$\begin{aligned} \mathcal{H}_n(\xi^2; x) &= (1 - \beta)^2 x^2 + \left\{ \frac{(1 + 2\alpha + e\lambda)(1 - \beta)^2}{1 + e\lambda} + 3 \right\} \frac{x}{n} \\ &\quad + \frac{1}{n^2} \left\{ \frac{(1 - \beta)^2(\alpha^2 - 2\alpha e\lambda - \alpha e^2 \lambda^2)}{(1 + e\lambda)^2} + \frac{3\alpha}{1 + e\lambda} + \frac{2!}{1 - \beta} \right\}. \end{aligned}$$

Proof. By using Lemma 2.1 and Lemma 2.2, we get

$$\mathcal{H}_n(1; x) = \sum_{k=0}^{\infty} \mathcal{P}_0(k, \beta) b_{n,k}^{(\alpha)}(x) = 1;$$

$$\begin{aligned} \mathcal{H}_n(\xi; x) &= \sum_{k=0}^{\infty} \mathcal{P}_1(k, \alpha) b_{n,k}^{(\alpha)}(x) \\ &= \sum_{k=0}^{\infty} \frac{1}{n} \left[(1 - \beta)k + \frac{1}{1 - \beta} \right] b_{n,k}^{(\alpha)}(x) \\ &= (1 - \beta) \mathcal{A}_n^{(\alpha)}(\xi; x) + \frac{1}{n(1 - \beta)} \mathcal{A}_n^{(\alpha)}(1; x) \\ &= (1 - \beta)x + \frac{1}{n} \left\{ \frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta} \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_n(\xi^2; x) &= \sum_{k=0}^{\infty} \mathcal{P}_2(k, \alpha) b_{n,k}^{(\alpha)}(x) \\ &= \sum_{k=0}^{\infty} \frac{1}{n^2} \left[(1 - \beta)^2 k^2 + 3k + \frac{2!}{1 - \beta} \right] b_{n,k}^{(\alpha)}(x) \\ &= (1 - \beta)^2 \mathcal{A}_n^{(\alpha)}(\xi^2; x) + \frac{3}{n} \mathcal{A}_n^{(\alpha)}(\xi; x) + \frac{2!}{n^2(1 - \beta)} \mathcal{A}_n^{(\alpha)}(1; x) \\ &= (1 - \beta)^2 \left\{ x^2 + \frac{(1 + 2\alpha + e\lambda)}{n(1 + e\lambda)} x + \frac{\alpha^2 - 2\alpha e\lambda - \alpha e^2 \lambda^2}{n^2(1 + e\lambda)^2} \right\} \\ &\quad + \frac{3}{n} \left\{ x + \frac{\alpha}{n(1 + e\lambda)} \right\} + \frac{2!}{n^2(1 - \beta)} \\ &= (1 - \beta)^2 x^2 + \left\{ \frac{(1 + 2\alpha + e\lambda)(1 - \beta)^2}{1 + e\lambda} + 3 \right\} \frac{x}{n} \\ &\quad + \frac{1}{n^2} \left\{ \frac{(1 - \beta)^2(\alpha^2 - 2\alpha e\lambda - \alpha e^2 \lambda^2)}{(1 + e\lambda)^2} + \frac{3\alpha}{1 + e\lambda} + \frac{2!}{1 - \beta} \right\}. \end{aligned}$$

□

Lemma 2.4. *By direct computation, we have*

$$\mathcal{H}_n((\xi - x); x) = \frac{1}{n} \left\{ \frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta} \right\} - \beta x,$$

and

$$\begin{aligned} \mathcal{H}_n((\xi - x)^2; x) &= \beta^2 x^2 + \left\{ \frac{(1 + 2\alpha + e\lambda)(1 - \beta)^2}{1 + e\lambda} - \frac{2\alpha(1 - \beta)}{1 + e\lambda} - \frac{2}{1 - \beta} + 3 \right\} \frac{x}{n} \\ &+ \left\{ \frac{(\alpha^2 - 2\alpha e\lambda - \alpha e^2 \lambda^2)(1 - \beta)^2}{(1 + e\lambda)^2} + \frac{3\alpha}{1 + e\lambda} + \frac{2}{1 - \beta} \right\} \frac{1}{n^2}. \end{aligned}$$

Lemma 2.5. *For the operators \mathcal{H}_n , we have*

$$|\mathcal{H}_n(f; x)| \leq \|f\|,$$

where $f \in C[0, \infty)$ and $\|f\| = \sup_{x \in [0, \infty)} |f(x)|$.

Proof. From operators (10) and using Lemma 2.3, we get

$$\begin{aligned} |\mathcal{H}_n(f; x)| &\leq \left| \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) f(\xi) d\xi \right| \\ &\leq \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) |f(\xi)| d\xi \leq \|f\|. \end{aligned}$$

□

3. Main Results

Let $f \in C_B^2[0, \infty)$ be denoted the space of all functions $f \in C_B[0, \infty)$ such that f', f'' define in $C[0, \infty)$. Let $\|f\|$ be denoted the usual supremum norm of a bounded function f . Then the Peetre’s K -functional

$$K(f, \delta) = \inf_{g \in C_B^2[0, \infty)} \{ \|f - g\| + \delta \|g''\| \}, \tag{11}$$

and for $\delta > 0$ the modulus of continuity of second order

$$\omega_2(f, \delta) = \sup_{0 < h \leq \delta, x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|. \tag{12}$$

Also from ([15], p. 177, Theorem 2.4), there exists a constant $C > 0$ such that

$$K(f, \delta) \leq C\omega_2(f, \sqrt{\delta}). \tag{13}$$

Now we get the following approximation results.

With the help of Bohman-Korovkin-Popoviciu theorem (see [24]) we prove the uniform convergence of the Apostol-Genocchi-Jain-Durremyer operators (10).

Theorem 3.1. *Let us $f \in C[0, \infty) \cap \mathcal{O}$ and this function also belongs to the class*

$$\mathcal{O} := \left\{ f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \rightarrow \infty \right\}.$$

Then, the uniformly on each compact subset of $[0, \infty)$, where $C[0, \infty)$ is the space of all real-valued continuous functions on $[0, \infty)$, i.e.,

$$\lim_{n \rightarrow \infty} \mathcal{H}_n(f; x) = f(x),$$

where $\alpha(n)$ be such that $\alpha \rightarrow 0$ as $n \rightarrow \infty$.

Proof. As $\alpha \rightarrow 0$ as $n \rightarrow \infty$, from Lemma 2.3, we have

$$\lim_{n \rightarrow \infty} \mathcal{H}_n(\xi^i; x) = x^i, \quad i = 0, 1, 2.$$

uniformly on each compact subset of the non-negative half line real axis. Hence, we get the desired result by applying the well-known Korovkin-type theorem [7] regarding the convergence of a sequence of positive linear operators. \square

Theorem 3.2. If $f \in C_B[0, \infty)$ then for $x \in [0, \infty)$, we have

$$|\mathcal{H}_n(f; x) - f(x)| \leq 2\omega\left(f; \sqrt{\mathcal{H}_n((\xi - x)^2; x)}\right),$$

where ω is the modulus of continuity of f [15] defined as:

$$\omega(f; x) := \sup_{\substack{x, y \in [0, \infty) \\ |x - y| \leq \delta}} |f(x) - f(y)|.$$

Proof. Applying the well-known property of $\omega(f; x)$, Lemma 2.3, and from operators (10), we have

$$\begin{aligned} |\mathcal{H}_n(f; x) - f(x)| &= \left| \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) (f(\xi) - f(x)) d\xi \right| \\ &\leq \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) |f(\xi) - f(x)| d\xi \\ &\leq \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} \omega(f; \delta) p_{n,k}^{(\beta)}(\xi) \left(1 + \frac{1}{\delta} |\xi - x|\right) d\xi \\ &= \left[1 + \frac{1}{\delta} \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) |\xi - x| d\xi \right] \omega(f; \delta). \end{aligned}$$

For the integration, the following result holds by using Cauchy-Schwarz inequality

$$\begin{aligned} |\mathcal{H}_n(f; x) - f(x)| &\leq \left[1 + \frac{1}{\delta} \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{1/2} \right. \\ &\quad \left. \times \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) (\xi - x)^2 d\xi \right)^{1/2} d\xi \right] \omega(f; \delta). \end{aligned}$$

Now using the last inequality for infinite sum and we have

$$\begin{aligned} &|\mathcal{H}_n(f; x) - f(x)| \\ &\leq \left[1 + \frac{1}{\delta} \left\{ \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right\}^{1/2} \right. \\ &\quad \left. \times \left\{ \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) (\xi - x)^2 d\xi \right\}^{1/2} \right] \omega(f; \delta) \\ &= \left[1 + \frac{1}{\delta} \{\mathcal{H}_n(1; x)\}^{1/2} \{\mathcal{H}_n((\xi - x)^2; x)\}^{1/2} \right] \omega(f; \delta). \end{aligned}$$

By taking

$$\delta = \left\{ \mathcal{H}_n \left((\xi - x)^2 ; x \right) \right\}^{1/2}.$$

We get the required result. \square

Now, for $0 < \varrho \leq 1$ and let us present approximation in terms of Lipschitz constant defined as:

$$Lip_K^\varrho = \left\{ f \in C_B[0, \infty) : |f(\eta_1) - f(\eta_2)| \leq K \frac{|\eta_1 - \eta_2|^\varrho}{(\eta_1 + \eta_2)^{\varrho/2}} \right\}, \quad \eta_1, \eta_2 \in [0, \infty),$$

where $K > 0$ is a constant.

Theorem 3.3. Suppose that $f \in Lip_K^\varrho$, then

$$|\mathcal{H}_n(f; x) - f(x)| \leq K \left\{ \frac{1}{x} \mathcal{H}_n \left((\xi - x)^2 ; x \right) \right\}^{\varrho/2}.$$

Proof. Since $f \in Lip_K^\varrho$ and $0 < \varrho \leq 1$, we have

$$\begin{aligned} |\mathcal{H}_n(f; x) - f(x)| &= |\mathcal{H}_n(f(\xi) - f(x); x)| \\ &\leq \mathcal{H}_n(|f(\xi) - f(x)|; x) \\ &\leq K \mathcal{H}_n \left(\frac{|\xi - x|^\varrho}{(\xi + x)^{\varrho/2}} ; x \right). \end{aligned} \tag{14}$$

From (14), it becomes

$$\begin{aligned} \mathcal{H}_n \left(\frac{|\xi - x|^\varrho}{(\xi + x)^{\varrho/2}} ; x \right) &= \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) \frac{|\xi - x|^\varrho}{(\xi + x)^{\varrho/2}} d\xi \\ &\leq \left\{ \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right\}^{(2-\varrho)/2} \\ &\quad \times \left\{ \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi) d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) \frac{(\xi - x)^2}{(\xi + x)} d\xi \right\}^{\varrho/2}, \end{aligned}$$

where we use the Hölder inequality by taking $p = \frac{2}{\varrho}$ and $q = \frac{2}{2-\varrho}$. Now using the fact that $\frac{1}{\xi+x} \leq \frac{1}{x}$, we have

$$\mathcal{H}_n \left(\frac{|\xi - x|^\varrho}{(\xi + x)^{\varrho/2}} ; x \right) = \{\mathcal{H}_n(1; x)\}^{(2-\varrho)/2} \left\{ \frac{1}{x} \mathcal{H}_n \left((\xi - x)^2 ; x \right) \right\}^{\varrho/2}.$$

This proves the required results. \square

Theorem 3.4. For every $f \in C_B[0, \infty)$ we have

$$|\mathcal{H}_n(f; x) - f(x)| \leq M \omega_2 \left(f, \frac{\delta_n(x, \alpha)}{\sqrt{2}} \right) + \omega(f, \tau_\omega),$$

where $M > 0$ is a constant.

Proof. We consider the auxiliary operators

$$\tilde{\mathcal{H}}_n(f; x) := \mathcal{H}_n(f; x) - f \left((1 - \beta)x + \frac{1}{n} \left\{ \frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta} \right\} \right) + f(x). \tag{15}$$

Using Lemma 2.3, we obtain

$$\tilde{\mathcal{H}}_n(e_0; x) = 1, \quad \text{and} \quad \tilde{\mathcal{H}}_n(e_1; x) = x,$$

i.e., $\tilde{\mathcal{H}}_n$ preserve constants and linear functions. Therefore

$$\tilde{\mathcal{H}}_n((e_1 - x); x) = 0. \tag{16}$$

Let $f \in C_B^2[0, \infty)$ and using Taylor's expansion

$$g(\xi) = g(x) + (\xi - x)g'(x) + \int_x^\xi (\xi - u)g''(u)du, \quad \xi, x \in [0, \infty). \tag{17}$$

Applying $\tilde{\mathcal{H}}_n$ to above expansion and using (17), we have

$$\tilde{\mathcal{H}}_n(g; x) - g(\xi) = g'(x)\tilde{\mathcal{H}}_n(\xi - x; x) + \tilde{\mathcal{H}}_n\left(\int_x^\xi (\xi - u)g''(u)du; x\right),$$

and from (16), we have

$$\begin{aligned} & \left| \tilde{\mathcal{H}}_n(g; x) - g(\xi) \right| \\ & \leq \tilde{\mathcal{H}}_n\left(\left|\int_x^\xi (\xi - u)g''(u)du\right|; x\right) \\ & \leq \mathcal{H}_n\left(\left|\int_x^\xi (\xi - u)g''(u)du\right|; x\right) \\ & - \left| \int_x^{(1-\beta)x + \frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda} + \frac{1}{1-\beta}\right\}} \left((1-\beta)x + \frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda} + \frac{1}{1-\beta}\right\} - u \right) g''(u) du \right|. \end{aligned}$$

Now we consider

$$\left| \int_\xi^x (\xi - u)g''(u)du \right| \leq (\xi - x)^2 \|g''\|,$$

then we get

$$\left| \tilde{\mathcal{H}}_n(g; x) - g(x) \right| \leq \left(\mathcal{H}_n((\xi - x)^2; x) + \left(\mathcal{H}_n(\xi - x; x)^2 \right) \right) \|g''\|.$$

From Lemma 2.3 and using (15), for the operators $\tilde{\mathcal{H}}_n^{(\alpha)}$, we have

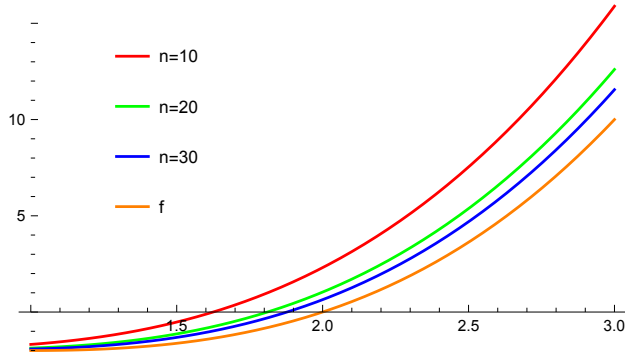
$$\begin{aligned} \left| \tilde{\mathcal{H}}_n(f; x) - f(x) \right| & \leq \left| \tilde{\mathcal{H}}_n(f - g; x) \right| + \left| \tilde{\mathcal{H}}_n(g; x) - g(x) \right| + |g(x) - f(x)| \\ & + \left| f\left((1-\beta)x + \frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda} + \frac{1}{1-\beta}\right\} \right) - f(x) \right| \\ & \leq 4\|f - g\| + \left(\mathcal{H}_n((\xi - x)^2; x) + \left(\mathcal{H}_n(\xi - x; x)^2 \right) \right) \|g''\| \\ & + \omega(f, \mathcal{H}_n(\xi - x; x)). \end{aligned}$$

Taking infimum on the right hand side over $g \in W_\infty^2$ and from (13), we have

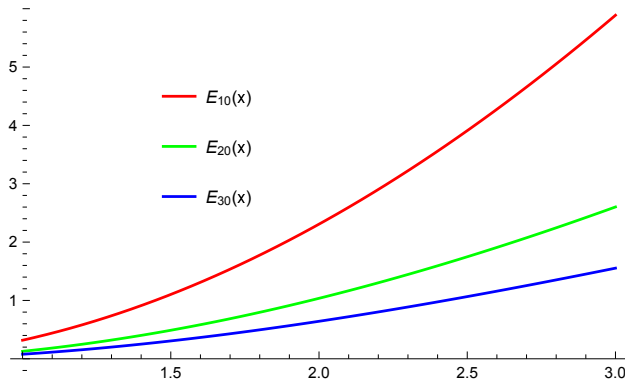
$$\begin{aligned} \left| \tilde{\mathcal{H}}_n(f; x) - f(x) \right| & \leq 4K_2 \left(f, \frac{\tau_n^4(x, \alpha)}{4} \right) + \omega(f, \tau_\omega) \\ & \leq M\omega_2 \left(f, \frac{\delta_n(x, \alpha)}{2} \right) + \omega(f, \tau_\omega), \end{aligned}$$

where $\tau_n^2(x, \alpha) = \mathcal{H}_n((\xi - x)^2; x) + \left(\mathcal{H}_n(\xi - x; x)^2 \right)$ and $\tau_\omega = \mathcal{H}_n(\xi - x; x)$. \square

Example 3.5. Let $f(x) = x^3 - 2x^2 + x - 2$, $\alpha = 2$, $\beta = .01$, $\lambda = 4$ and $n \in \{10, 20, 30\}$. The convergence of the defined operators \tilde{H}_n towards the function $f(x)$ and the absolute error $E_n(x) = |\tilde{H}_n(f; x) - f(x)|$ of the operators are shown in Fig(a) and Fig(b) respectively. The absolute error of the operators are also computed in Table 1 for some values in [1, 3].



Fig(a)



Fig(b)

Table 1: Absolute error of the considered operators with function $f(x) = x^3 - 2x^2 + x - 2$ for $n = \{10, 20, 30\}$.

x	n = 10	n = 20	n = 30
1	0.31827	0.12793	0.07812
1.2	0.58023	0.24848	0.15482
1.4	0.91155	0.40104	0.25106
1.6	1.31081	0.58416	0.36541
1.8	1.77657	0.79643	0.49645
2.0	2.30742	1.03641	0.64276
2.2	2.90192	1.30269	0.80290
2.4	3.55866	1.59383	0.97545
2.6	4.27620	1.90842	1.15890
2.8	5.05312	2.24502	1.35209
3.0	5.88800	2.60221	1.55333

4. Conclusion:

Genocchi numbers and polynomials have been extensively studied in many different contexts in branches of mathematics. Prakash et al. [34] constructed a sequence of operators by using these polynomials and studied some approximation properties. Our prime objective in this paper is to consider the Jain-Durrmeyer operators associated with the Apostol-Genocchi polynomials depending on some parameters α, β , and λ . We analyze the approximation behaviour of these operators including K -functional and find local approximation results by using modulus of continuity. We have also calculated the rate of convergence of operators by means of Lipschitz-type space. Lastly, we discussed the graphical interpretation to find the absolute error for some particular values of parameters using Mathematica software.

Motivated by a recently-published survey-cum-expository review article [38], the interested reader's attention is drawn toward the possibility of investigating the basic (or q -) extensions of the results which are presented in this paper. However, as already pointed out on Page 340 in [38], their further extensions using the so-called (p, q) -calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical q -calculus, the additional parameter p being redundant or superfluous.

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