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Matrix LSQR Algorithms for Solving Constrained Quadratic Inverse Eigenvalue Problem

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Abstract. The inverse eigenvalue problem appears in many applications such as control design, seismic tomography, exploration and remote sensing, molecular spectroscopy, particle physics, structural analysis, and mechanical system simulation. This paper investigates the matrix form of LSQR methods for solving the quadratic inverse eigenvalue problem with partially bisymmetric matrices under a prescribed submatrix constraint. In order to illustrate the effectiveness and feasibility of our results, one numerical example is presented.

1. Introduction

Notation: Throughout this paper, we assume that $\mathbf{R}^{m\times n}$, $I_n = (e_1, e_2, ..., e_n)$, $S_n = (e_n, e_{n-1}, ..., e_1)$, $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$ and $D_{p,n} = \{d = (d_1, d_2, ..., d_p) : 1 \leq d_1 < d_2 < ... < d_p \leq n\}$ respectively represent the $m\times n$ real matrix set, the $n\times n$ unit matrix, the $n\times n$ reverse unit matrix, the Frobenius norm of the matrix A and the strictly increasing sequences of p elements from $1, 2, ..., n$. For $s = (s_1, s_2, ..., s_p) \in D_{p,n}$, $t = (t_1, t_2, ..., t_q) \in D_{q,n}$ and $u = (u_1, u_2, ..., u_r) \in D_{r,n}$, we assume that $E_s = (e_{s_1}, e_{s_2}, ..., e_{s_p}) \in \mathbf{R}^{n\times p}$, $E_t = (e_{t_1}, e_{t_2}, ..., e_{t_q}) \in \mathbf{R}^{n\times q}$ and $E_u = (e_{u_1}, e_{u_2}, ..., e_{u_r}) \in \mathbf{R}^{n\times r}$. The symbol $A[s|t]$ exhibits the $p\times q$ submatrix of A determined by rows indexed by s and columns indexed by t. The notation $A[\bar{s}|\bar{t}]$ represents the $(m-p)\times(n-q)$ submatrix of A determined by deleting rows indexed by s and columns indexed by t. A real $n\times n$ matrix $A = (a_{i,j})$ is named a bisymmetric matrix if its elements satisfy the properties $a_{i,j} = a_{j,i}$ and $a_{i,j} = a_{n-j+1,n-i+1}$ for $1 \leq i, j \leq n$. Let $\mathbf{BSR}^{n\times n}$ denote the set of $n\times n$ bisymmetric matrices. It can be verified that a matrix $X \in \mathbf{BSR}^{n\times n}$ if and only if $X = X^T = S_nXS_n$. The bisymmetric matrices including symmetric Toeplitz matrices and persymmetric Hankel matrices as special cases have wide applications in applied sciences [1, 4].
Various types of inverse eigenvalue problem such as

$$\begin{cases} \text{Given } X \in \mathbf{R}^{n\times m}, \text{ and } \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbf{R}^{m\times m} \\ \text{Find } \; C \in \mathbf{R}^{n\times n} \text{ such that } CX = X\Lambda, \end{cases} \qquad \textit{Inverse eigenvalue problem (IEP)},$$

$$\begin{cases} \text{Given } X \in \mathbf{R}^{n\times m}, \text{ and } \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbf{R}^{m\times m} \\ \text{Find } \; C, B \in \mathbf{R}^{n\times n} \text{ such that } CX = BX\Lambda, \end{cases} \qquad \textit{Generalized IEP (GIEP)},$$

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$$\begin{cases} \textit{Given } X \in \mathbf{R}^{n\times m}, \textit{ and } \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbf{R}^{m\times m} \\ \textit{Find } C, B, A \in \mathbf{R}^{n\times n} \textit{ such that } AX\Lambda^2 + BX\Lambda + CX = 0, \end{cases} \quad \textit{Quadratic IEP (QIEP)},$$

play a fundamental role in a variety of fields of wireless communications, pole assignment problem, inverse Sturm Liouville problem and quantum mechanics, signal and data processing [5]. The inverse eigenvalue problem attracted a lot of research attention over the last few years due to the growing importance of inverse problems [3, 6–8, 13]. In practice we usually require that the resulting matrix from a specific inverse eigenvalue problem is physically realizable and thus additional structural constraints are imposed [14]. Therefore so far several constrained and specific inverse eigenvalue problems have been studied [15–18]. Wei and Dai proposed two numerical algorithms to solve the inverse eigenvalue problem of Jacobi matrix [12]. In [9], the inverse eigenvalue problem and the associated optimal approximation problem for Hermitian reflexive matrices with respect to a normal $k+1$-potent matrix were studied considered.
The present article deals with the quadratic inverse eigenvalue problem with partially bisymmetric matrices under a prescribed submatrix constraint. We will develop the LSQR methods for a constrained quadratic inverse eigenvalue problem as follows:

Problem 1. *Given* $X \in \mathbf{R}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbf{R}^{m\times m}$, $s = (s_1, s_2, ..., s_p, n+1-s_p, ..., n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, ..., t_q, n+1-t_q, ..., n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, ..., u_r, n+1-u_r, ..., n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbf{R}^{2p\times 2p}$, $B_q \in \mathbf{R}^{2q\times 2q}$ *and* $C_r \in \mathbf{R}^{2r\times 2r}$, *let* $S_1 = \{X | X[s,s] = A_p,\ X[\bar{s},\bar{s}] \in \mathbf{BSR}^{(n-2p)\times(n-2p)}\}$, $S_2 = \{X | X[t,t] = B_q,\ X[\bar{t},\bar{t}] \in \mathbf{BSR}^{(n-2q)\times(n-2q)}\}$ *and* $S_3 = \{X | X[u,u] = C_r,\ X[\bar{u},\bar{u}] \in \mathbf{BSR}^{(n-2r)\times(n-2r)}\}$, *find* $A \in S_1$, $B \in S_2$ *and* $C \in S_3$ *such that* $AX\Lambda^2 + BX\Lambda + CX = 0$.

2. Main Results

In this section, first we propose a simplified form of Problem 1. By introducing the sets

$$S_1 = \underbrace{\{X | X[s,s] = A_p,\ X[\bar{p},\bar{p}] = 0\}}_{=\widetilde{S}_1} \oplus \underbrace{\{X | X \in \mathbf{BSR}^{n\times n},\ X[s,s] = 0\}}_{=\widehat{S}_1},$$

$$S_2 = \underbrace{\{X | X[t,t] = B_q,\ X[\bar{q},\bar{q}] = 0\}}_{=\widetilde{S}_2} \oplus \underbrace{\{X | X \in \mathbf{BSR}^{n\times n},\ X[t,t] = 0\}}_{=\widehat{S}_2},$$

$$S_3 = \underbrace{\{X | X[u,u] = C_r,\ X[\bar{u},\bar{u}] = 0\}}_{=\widetilde{S}_3} \oplus \underbrace{\{X | X \in \mathbf{BSR}^{n\times n},\ X[u,u] = 0\}}_{=\widehat{S}_3},$$

Problem 1 can be transformed into the following equivalent problem.

Problem 2. *Given* $X \in \mathbf{R}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbf{R}^{m\times m}$, $s = (s_1, s_2, ..., s_p, n+1-s_p, ..., n+1-s_2, n+1-s_1) \in D_{2p,n}$, $t = (t_1, t_2, ..., t_q, n+1-t_q, ..., n+1-t_2, n+1-t_1) \in D_{2q,n}$, $u = (u_1, u_2, ..., u_r, n+1-u_r, ..., n+1-u_2, n+1-u_1) \in D_{2r,n}$, $A_p \in \mathbf{R}^{2p\times 2p}$, $B_q \in \mathbf{R}^{2q\times 2q}$ *and* $C_r \in \mathbf{R}^{2r\times 2r}$, *find* $\widehat{A} \in \widehat{S}_1$, $\widehat{B} \in \widehat{S}_2$ *and* $\widehat{C} \in \widehat{S}_3$ *such that*

$$\widehat{A}X\Lambda^2 + \widehat{B}X\Lambda + \widehat{C}X = \widehat{Z}, \tag{1}$$

where $\widehat{Z} = -\widehat{A}_pX\Lambda^2 - \widehat{B}_qX\Lambda - \widehat{C}_rX$, *in which* $\widehat{A}_p$, $\widehat{B}_q$ *and* $\widehat{C}_r$ *denote the matrices satisfying* $\widehat{A}_p[s,s] = A_p$, $\widehat{B}_q[t,t] = B_q$, $\widehat{C}_r[u,u] = C_r$ *and zeros elsewhere.*

It is obvious that $\widehat{A}^*$, $\widehat{B}^*$ and $\widehat{C}^*$ are the solutions of Problem 2 iff $\widehat{A}^* + \widehat{A}_p$, $\widehat{B}^* + \widehat{B}_p$ and $\widehat{C}^* + \widehat{C}_p$ are the solutions of Problem 1. Now we can solve Problem 2 more easier than Problem 1.
By using the Golub-Kahan bidiagonalization process, two types of the LSQR method were constructed in [10] to compute an approximation solution of the linear systems $Ax = b$ and unconstrained least-squares problem $\min_x \|Ax - b\|$. Two types of the LSQR method can be summarized as follows [10, 11].

Algorithm 1. ***Type 1 of LSQR method***
$\tau(0) = 1$; $\xi(0) = -1$; $\omega(0) = 0$; $w(0) = 0$; $z(0) = 0$; $\beta(1) = \|b\|$; $\beta(1)u(1) = b$; $\alpha(1) = \|A^T u(1)\|$; $\alpha(1)v(1) = A^T u(1)$;
For $i = 1, 2, \ldots$, until convergence, do:
$\xi(i) = -\xi(i-1)\beta(i)/\alpha(i)$; $z(i) = z(i-1)+\xi(i)v(i)$; $w(i) = (\tau(i-1)-\beta(i)w(i-1))/\alpha(i)$; $\omega(i) = \omega(i-1)+w(i)v(i)$; $\beta(i+1) = \|Av(i)-\alpha(i)u(i)\|$; $\beta(i+1)u(i+1) = Av(i)-\alpha(i)u(i)$; $\tau(i) = -\tau(i-1)\alpha(i)/\beta(i+1)$; $\alpha(i+1) = \|A^T u(i+1)-\beta(i+1)v(i)\|$; $\alpha(i+1)v(i+1) = A^T u(i+1) - \beta(i+1)v(i)$; $\gamma(i) = \beta(i+1)\xi(i)/(\beta(i+1)w(i) - \tau(i))$; $x(i) = z(i) - \gamma(i)\omega(i)$.

Algorithm 2. ***Type 2 of LSQR method***
$\theta(1) = \|A^T b\|$; $\theta(1)v(1) = A^T b$; $\rho(1) = \|Av(1)\|$; $\rho(1)p(1) = Av(1)$; $\omega(1) = v(1)/\rho(1)$; $\xi(1) = \theta(1)/\rho(1)$; $x(1) = \xi(1)\omega(1)$;
For $i = 1, 2, \ldots$, until convergence, do:
$\theta(i+1) = \|A^T p(i) - \rho(i)v(i)\|$; $\theta(i+1)v(i+1) = A^T p(i) - \rho(i)v(i)$; $\rho(i+1) = \|Av(i+1) - \theta(i+1)p(i)\|$; $\rho(i+1)p(i+1) = Av(i+1) - \theta(i+1)p(i)$; $\omega(i+1) = (v(i+1) - \theta(i+1)\omega(i))/\rho(i+1)$; $\xi(i+1) = -\xi(i)\theta(i+1)/\rho(i+1)$; $x(i+1) = x(i) + \xi(i+1)\omega(i+1)$.

Theorem 1. *[11] LSQR algorithms return the minimum-norm solution.*

In the above algorithms, the scalars $\alpha(i) \geq 0$, $\beta(i) \geq 0$, $\rho(i) \geq 0$ and $\theta(i) \geq 0$ are chosen to make $\|u(i)\| = 1$ and $\|v(i)\| = 1$, respectively [11].
Now we propose the matrix form of the above algorithms for solving Problem 2. With the aid of Kronecker product and vectorization operator, it is easy to see that solvability of Problem 2 is equivalent to the following system:

$$\underbrace{\begin{pmatrix} (X\Lambda^2)^T \otimes I_n & (X\Lambda)^T \otimes I_n & X^T \otimes I_n \\ I_n \otimes (X\Lambda^2)^T & I_n \otimes (X\Lambda)^T & I_n \otimes X^T \\ (X\Lambda^2)^T S_n \otimes S_n & (X\Lambda)^T S_n \otimes S_n & X^T S_n \otimes S_n \\ S_n \otimes (X\Lambda^2)^T S_n & S_n \otimes (X\Lambda)^T S_n & S_n \otimes X^T S_n \\ E_s^T \otimes E_s^T & 0 & 0 \\ 0 & E_t^T \otimes E_t^T & 0 \\ 0 & 0 & E_u^T \otimes E_u^T \\ E_s^T S_n \otimes E_s^T S_n & 0 & 0 \\ 0 & E_t^T S_n \otimes E_t^T S_n & 0 \\ 0 & 0 & E_u^T S_n \otimes E_u^T S_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \mathrm{vec}(\widehat{A}) \\ \mathrm{vec}(\widehat{B}) \\ \mathrm{vec}(\widehat{C}) \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} \mathrm{vec}(\widehat{Z}) \\ \mathrm{vec}(\widehat{Z}^T) \\ \mathrm{vec}(\widehat{Z}) \\ \mathrm{vec}(\widehat{Z}^T) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b}. \tag{2}$$

If we substitute the above system into Algorithms 1 and 2 then we obtain the following matrix LSQR algorithms for solving Problem 2.

Algorithm 3. ***Type 1 of matrix LSQR method***
$\tau(0) = 1$; $\xi(0) = -1$; $\Omega_1(0) = 0$; $\Omega_2(0) = 0$; $\Omega_3(0) = 0$; $w(0) = 0$; $Z_1(0) = 0$; $Z_2(0) = 0$; $Z_3(0) = 0$;
$\beta(1) = 2\|\widehat{Z}\|$;
$\beta(1)U_1(1) = \widehat{Z}$; $U_2(1) = 0$; $U_3(1) = 0$; $U_4(1) = 0$;
$\alpha(1) = \Big(\|U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T + (U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T)^T + S_n(U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T + (U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T)^T)S_n\|^2 + \|U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T + (U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T)^T + S_n(U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T + (U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T)^T)S_n\|^2 + \|U_1(1)X^T + E_u U_4(1)E_u^T + (U_1(1)X^T + E_u U_4(1)E_u^T)^T + S_n(U_1(1)X^T + E_u U_4(1)E_u^T + (U_1(1)X^T + E_u U_4(1)E_u^T)^T)S_n\|^2\Big)^{1/2}$;
$\alpha(1)V_1(1) = U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T + (U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T)^T + S_n(U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T + (U_1(1)(X\Lambda^2)^T + E_s U_2(1)E_s^T)^T)S_n$;
$\alpha(1)V_2(1) = U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T + (U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T)^T + S_n(U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T + (U_1(1)(X\Lambda)^T + E_t U_3(1)E_t^T)^T)S_n$;

$\alpha(1)V_3(1) = U_1(1)X^T+E_uU_4(1)E_u^T+(U_1(1)X^T+E_uU_4(1)E_u^T)^T+S_n(U_1(1)X^T+E_uU_4(1)E_u^T+(U_1(1)X^T+E_uU_4(1)E_u^T)^T)S_n;$
For $i = 1, 2, ...,$ until convergence, do:
$\xi(i) = -\xi(i-1)\beta(i)/\alpha(i);$
$Z_1(i) = Z_1(i-1) + \xi(i)V_1(i); Z_2(i) = Z_2(i-1) + \xi(i)V_2(i); Z_3(i) = Z_3(i-1) + \xi(i)V_3(i);$
$w(i) = (\tau(i-1) - \beta(i)w(i-1))/\alpha(i);$
$\Omega_1(i) = \Omega_1(i-1) + w(i)V_1(i); \Omega_2(i) = \Omega_2(i-1) + w(i)V_2(i); \Omega_3(i) = \Omega_3(i-1) + w(i)V_3(i);$
$\beta(i+1) = 2\Big(\|V_1(i)X\Lambda^2 + V_2(i)X\Lambda + V_3(i)X - \alpha(i)U_1(i)\|^2+\|E_s^TV_1(i)E_s - \alpha(i)U_2(i)\|^2 + \|E_t^TV_2(i)E_t - \alpha(i)U_3(i)\|^2 + \|E_u^TV_3(i)E_u - \alpha(i)U_4(i)\|^2\Big)^{1/2};$
$\beta(i+1)U_1(i+1) = V_1(i)X\Lambda^2 + V_2(i)X\Lambda + V_3(i)X - \alpha(i)U_1(i);$
$\beta(i+1)U_2(i+1) = E_s^TV_1(i)E_s - \alpha(i)U_2(i);$
$\beta(i+1)U_3(i+1) = E_t^TV_2(i)E_t - \alpha(i)U_3(i);$
$\beta(i+1)U_4(i+1) = E_u^TV_3(i)E_u - \alpha(i)U_4(i);$
$\tau(i) = -\tau(i-1)\alpha(i)/\beta(i+1);$
$\alpha(i+1) = \Big(\|U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T + (U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T)^T + S_n(U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T + (U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T)^T)S_n - \beta(i+1)V_1(i)\|^2 + \|U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T + (U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T)^T + S_n(U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T + (U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T)^T)S_n - \beta(i+1)V_2(i)\|^2 + \|U_1(i+1)X^T + E_uU_4(i+1)E_u^T + (U_1(i+1)X^T + E_uU_4(i+1)E_u^T)^T + S_n(U_1(i+1)X^T + E_uU_4(i+1)E_u^T + (U_1(i+1)X^T + E_uU_4(i+1)E_u^T)^T)S_n - \beta(i+1)V_3(i)\|^2\Big)^{1/2};$
$\alpha(i+1)V_1(i+1) = U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T + (U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T)^T + S_n(U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T + (U_1(i+1)(X\Lambda^2)^T + E_sU_2(i+1)E_s^T)^T)S_n - \beta(i+1)V_1(i);$
$\alpha(i+1)V_2(i+1) = U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T + (U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T)^T + S_n(U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T + (U_1(i+1)(X\Lambda)^T + E_tU_3(i+1)E_t^T)^T)S_n - \beta(i+1)V_2(i);$
$\alpha(i+1)V_3(i+1) = U_1(i+1)X^T + E_uU_4(i+1)E_u^T + (U_1(i+1)X^T + E_uU_4(i+1)E_u^T)^T + S_n(U_1(i+1)X^T + E_uU_4(i+1)E_u^T + (U_1(i+1)X^T + E_uU_4(i+1)E_u^T)^T)S_n - \beta(i+1)V_3(i);$
$\gamma(i) = \beta(i+1)\xi(i)/(\beta(i+1)w(i) - \tau(i));$
$\widehat{A}(i) = Z_1(i) - \gamma(i)\Omega_1(i);$
$\widehat{B}(i) = Z_2(i) - \gamma(i)\Omega_2(i);$
$\widehat{C}(i) = Z_3(i) - \gamma(i)\Omega_3(i).$

Algorithm 4. ***Type 2 of matrix LSQR method***
$\theta(1) = \Big(\|\widehat{Z}(X\Lambda^2)^T + (\widehat{Z}(X\Lambda^2)^T)^T + S_n(\widehat{Z}(X\Lambda^2)^T + (\widehat{Z}(X\Lambda^2)^T)^T)S_n\|^2 + \|\widehat{Z}(X\Lambda)^T + (\widehat{Z}(X\Lambda)^T)^T + S_n(\widehat{Z}(X\Lambda)^T + (\widehat{Z}(X\Lambda)^T)^T)S_n\|^2+\|\widehat{Z}X^T+(\widehat{Z}X^T)^T+S_n(\widehat{Z}X^T+(\widehat{Z}X^T)^T)S_n\|^2\Big)^{1/2}; \theta(1)V_1(1) = \widehat{Z}(X\Lambda^2)^T+(\widehat{Z}(X\Lambda^2)^T)^T+S_n(\widehat{Z}(X\Lambda^2)^T+(\widehat{Z}(X\Lambda^2)^T)^T)S_n;$
$\theta(1)V_2(1) = \widehat{Z}(X\Lambda)^T+(\widehat{Z}(X\Lambda)^T)^T+S_n(\widehat{Z}(X\Lambda)^T+(\widehat{Z}(X\Lambda)^T)^T)S_n; \theta(1)V_3(1) = \widehat{Z}X^T+(\widehat{Z}X^T)^T+S_n(\widehat{Z}X^T+(\widehat{Z}X^T)^T)S_n;$
$\rho(1) = 2\Big(\|V_1(1)X\Lambda^2 + V_2(1)X\Lambda + V_3(1)X\|^2 + \|E_s^TV_1(1)E_s\|^2 + \|E_t^TV_2(1)E_t\|^2 + \|E_u^TV_3(1)E_u\|^2\Big)^{1/2};$
$\rho(1)P_1(1) = V_1(1)X\Lambda^2 + V_2(1)X\Lambda + V_3(1)X;$
$\rho(1)P_2(1) = E_s^TV_1(1)E_s; \rho(1)P_3(1) = E_t^TV_2(1)E_t; \rho(1)P_4(1) = E_u^TV_3(1)E_u;$
$\Omega_1(1) = V_1(1)/\rho(1); \Omega_2(1) = V_2(1)/\rho(1); \Omega_3(1) = V_3(1)/\rho(1);$
$\xi(1) = \theta(1)/\rho(1); \widehat{A}(1) = \xi(1)\Omega_1(1); \widehat{B}(1) = \xi(1)\Omega_2(1); \widehat{C}(1) = \xi(1)\Omega_3(1);$
For $i = 1, 2, ...,$ until convergence, do:
$\theta(i+1) = \Big(\|P_1(i)(X\Lambda^2)^T + E_sP_2(i)E_s^T + (P_1(i)(X\Lambda^2)^T + E_sP_2(i)E_s^T)^T + S_n(P_1(i)(X\Lambda^2)^T + E_sP_2(i)E_s^T + (P_1(i)(X\Lambda^2)^T + E_sP_2(i)E_s^T)^T)S_n - \rho(i)V_1(i)\|^2 + \|P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T + (P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T)^T + S_n(P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T + (P_1(i)(X\Lambda)^T+E_tP_3(i)E_t^T)^T)S_n-\rho(i)V_2(i)\|^2+\|P_1(i)X^T+E_uP_4(i)E_u^T+(P_1(i)X^T+E_uP_4(i)E_u^T)^T+S_n(P_1(i)X^T+E_uP_4(i)E_u^T + (P_1(i)X^T + E_uP_4(i)E_u^T)^T)S_n - \rho(i)V_3(i)\|^2\Big)^{1/2};$
$\theta(i+1)V_1(i+1) = P_1(i)(X\Lambda^2)^T+E_sP_2(i)E_s^T+(P_1(i)(X\Lambda^2)^T+E_sP_2(i)E_s^T)^T+S_n(P_1(i)(X\Lambda^2)^T+E_sP_2(i)E_s^T+(P_1(i)(X\Lambda^2)^T + E_sP_2(i)E_s^T)^T)S_n - \rho(i)V_1(i);$

$\theta(i+1)V_2(i+1) = P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T + (P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T)^T + S_n(P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T + (P_1(i)(X\Lambda)^T + E_tP_3(i)E_t^T)^T)S_n - \rho(i)V_2(i)$; $\theta(i+1)V_3(i+1) = P_1(i)X^T + E_uP_4(i)E_u^T + (P_1(i)X^T + E_uP_4(i)E_u^T)^T + S_n(P_1(i)X^T + E_uP_4(i)E_u^T + (P_1(i)X^T + E_uP_4(i)E_u^T)^T)S_n - \rho(i)V_3(i)$;
$\rho(i+1) = 2\Big(\|V_1(i+1)X\Lambda^2 + V_2(i+1)X\Lambda + V_3(i+1)X - \theta(i+1)P_1(i)\|^2 + \|E_s^TV_1(i+1)E_s - \theta(i+1)P_2(i)\|^2 + \|E_t^TV_2(i+1)E_t - \theta(i+1)P_3(i)\|^2 + \|E_u^TV_3(i+1)E_u - \theta(i+1)P_4(i)\|^2\Big)^{1/2}$; $\rho(i+1)P_1(i+1) = V_1(i+1)X\Lambda^2 + V_2(i+1)X\Lambda + V_3(i+1)X - \theta(i+1)P_1(i)$; $\rho(i+1)P_2(i+1) = E_s^TV_1(i+1)E_s - \theta(i+1)P_2(i)$;
$\rho(i+1)P_3(i+1) = E_t^TV_2(i+1)E_t - \theta(i+1)P_3(i)$;
$\rho(i+1)P_4(i+1) = E_u^TV_3(i+1)E_u - \theta(i+1)P_4(i)$;
$\Omega_1(i+1) = (V_1(i+1) - \theta(i+1)\Omega_1(i))/\rho(i+1)$;
$\Omega_2(i+1) = (V_2(i+1) - \theta(i+1)\Omega_2(i))/\rho(i+1)$;
$\Omega_3(i+1) = (V_3(i+1) - \theta(i+1)\Omega_3(i))/\rho(i+1)$;
$\xi(i+1) = -\xi(i)\theta(i+1)/\rho(i+1)$;
$\widehat{A}(i+1) = \xi(i+1)\Omega_1(i+1)$;
$\widehat{B}(i+1) = \xi(i+1)\Omega_2(i+1)$;
$\widehat{C}(i+1) = \xi(i+1)\Omega_3(i+1)$.

Stopping criterion. To check convergence of Algorithms 3 and 4, we use the stopping criterion

$$\sqrt{\|\widehat{A}(i)X\Lambda^2 + \widehat{B}(i)X\Lambda + \widehat{C}(i)X - \widehat{Z}\|^2 + \|E_s^T\widehat{A}(i)E_s\|^2 + \|E_t^T\widehat{B}(i)E_t\|^2 + \|E_u^T\widehat{C}(i)E_u\|^2} \leq \texttt{tol},$$

where `tol` is a chosen fixed threshold.

3. Numerical results

In this section, to illustrate the effeteness of Algorithms 3 and 4, we give an example. The following example is taken from [2]. We consider the constrained generalized inverse eigenvalue problem $CX = BX\Lambda$ with the following parameters:

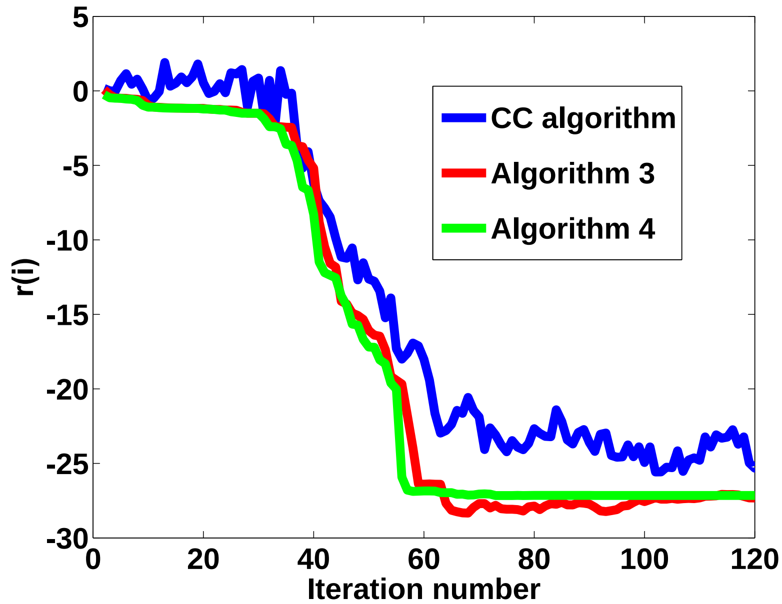
$$X = \begin{pmatrix} 0.1034 & 0.0869 & -0.4414 & 0.7860 & -0.9775 & -0.9975 \\ 0.3922 & 0.1724 & 0.6877 & -1.0000 & 1.0000 & -0.1738 \\ 0.9541 & 0.8176 & -0.8511 & 0.4781 & 0.4923 & 0.8610 \\ -1.0000 & 1.0000 & -1.0000 & -0.4625 & -0.4971 & 1.0000 \\ -0.3853 & 0.1767 & 0.7691 & 0.9927 & -0.9974 & -0.5748 \\ -0.1200 & 0.1535 & -0.6144 & -0.7733 & 0.9719 & -0.5570 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 7.5462 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6732 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1659 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0719 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0221 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0368 \end{pmatrix}.$$

Considering $s = \{3,4\}$, $t = \{2,5\}$, $C_p = \begin{pmatrix} 0.05 & 0.15 \\ 0.2 & 0.1 \end{pmatrix}$, $B_q = \begin{pmatrix} 0.25 & -0.05 \\ -0.25 & 0.25 \end{pmatrix}$, gives us

$$\widehat{C}_p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0.15 & 0 & 0 \\ 0 & 0 & 0.20 & 0.10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{B}_q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & -0.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Figure 1: Residual plots.



Now we apply the iterative methods proposed in [2] (CC algorithm) with zero initial matrices and Algorithms 3 and 4 for solving this problem. The results plot in Figure 1 where

$$r(i) = \log \sqrt{\| - \widehat{B}(i)X\Lambda + \widehat{C}(i)X - \widehat{Z}\|^2 + \|E_t^T\widehat{B}(i)E_t\|^2 + \|E_u^T\widehat{C}(i)E_u\|^2}.$$

The solutions of Problem 1 can be computed as

$$\widehat{C}^* + \widehat{C}_p = \begin{pmatrix} -0.50 & -0.30 & -0.10 & 0.10 & 0.90 & 0.70 \\ -0.30 & -0.50 & 0.15 & 0.25 & 0.70 & 0.90 \\ -0.10 & 0.15 & 0.05 & 0.15 & 0.25 & 0.10 \\ 0.10 & 0.25 & 0.20 & 0.10 & 0.15 & -0.10 \\ 0.90 & 0.70 & 0.25 & 0.15 & -0.50 & -0.30 \\ 0.70 & 0.90 & 0.10 & -0.10 & -0.30 & -0.50 \end{pmatrix},$$

$$\widehat{B}^* + \widehat{B}_q = \begin{pmatrix} -0.00 & 1.50 & -0.00 & -2.00 & 4.50 & 8.00 \\ 1.50 & 0.25 & 0.00 & 0.00 & -0.50 & 4.50 \\ -0.00 & 0.00 & 0.25 & 0.50 & 0.00 & -2.00 \\ -2.00 & 0.00 & 0.50 & 0.25 & 0.00 & -0.00 \\ 4.50 & -0.25 & 0.00 & 0.00 & 0.25 & 1.50 \\ 8.00 & 4.50 & -2.00 & -0.00 & 1.50 & -0.00 \end{pmatrix}.$$

It is found from Figure 1 that Algorithms 3 and 4 are more efficient than the CC algorithm [2].

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