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New Type of G-Mond-Weir Type Primal-Dual Model and Their Duality Results With Generalized Assumptions

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Abstract. In this paper, a generalization of convexity, namely G_f-invexity is considered. We formulate a Mond-Weir type symmetric dual for a class of nondifferentiable multiobjective fractional programming problem over cones. Next, we prove appropriate duality results using G_f-invexity assumptions.

1. Introduction

Convexity and generalized convexity have been playing a central role in developing optimality and duality results for multiobjective programming problems which are mathematical models for most of the real world problems occurring in the fields of engineering, economics, finance, game theory etc. Several classes of (generalized) convex functions have been defined and studied for the purpose of weakening the limitations of convexity in mathematical programming.The study of higher-order duality is significant due to the computational advantage over the first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Mukherjee [1] considered a multiobjective fractional programming problem and discussed the Mond-Weir type duality results under generalized convexity. Kaul et al. [2] derived duality results for a Mond-Weir type dual problem related to multiobjective fractional programming problem involving pseudo linear and η- pseudo linear functions.

Hanson [3] introduced the concept of invexity which is an extension of differentiable convex function and proved the sufficiency of Kuhn-Tucker conditions. Later, Hanson and Mond [4] generalized the concept of invex function by introducing type-I and type-II functions which generalized pseudo- type-I and quasi-type-I functions given by Reuda et al. [5]. Antczak [6] introduced the concept of G-invex functions and derived some optimality conditions for constrained optimization problems under G-invexity. In [7], Antczak extended the above notion by defining a vector valued G_f-invex function and proved necessary and sufficient optimality conditions for a multiobjective nonlinear programming problem. Recently, Kang

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et al. [8] defined G-invexity for a locally Lipchitz function and obtained optimality conditions for multiobjective programming using these functions.

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. Bector and Chandra [9] formulated second-order Mond-Weir type dual for a nondifferentiable fractional program and established duality results using the concept of second-order pseudo convexity and quasiconvexity. Jeykumar [10] and Yang [11] also discussed second-order dual formulation under r-convexity and its generalizations. Later on, Suneja et al. [12] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, α, ρ, d)-type I- assumptions.

Many authors have developed the necessary and sufficient conditions for pareto optimal solutions in multiobjective programming problems.Yuan et al. [13] introduced new types of generalized convex functions and sets, which are called locally (H_p, r, α)-pre-invex and locally H_p-invex sets. They also obtained optimality conditions and duality theorems for a scalar nonlinear programming problem. Further, Liu et al. [14] proposed the concept of (H_p, r)-invex function and focus his study to discuss sufficient optimality conditions to multiobjective fractional programming problem.

Recently, Mandal and Nahak [15] have introduced the concept of $(p, r) - \rho - (\eta, \theta)$-invex function and developed symmetric duality results under these assumptions. Using the same assumptions, Jayswal et al. [16] derived sufficient optimality conditions and duality theorems for multiobjective fractional programming problems. Later on, a class of nondifferentiable multiobjective fractional programming with higher-order has been discussed and usual duality results have been proved in Gulati and Saini [17]. Further, Jayswal et al. [18] formulated higher-order duality for multiobjective programming problems and established duality theorems using higher-order (F, α, ρ, d)-V-type I assumptions.

Motivated by various concepts of generalized convexity. Ferrara and Stefaneseu [19] used the (ϕ, ρ)-invexity to discuss the optimality conditions and duality results for multiobjective programming problem. Further, Stefaneseu and Ferrara [20] introduced a new class of $(\phi, \rho)^{\omega}$- invexity for a multiobjective program and established optimality conditions and duality theorems under these assumptions. Dubey and Mishra [21] introduced the symmetric duality in a nondifferentiable multiobjective programming problem and derived duality theorems under generalized assumptions. For more data on fractional programming, readers are advised to see [22–26].

In this paper, we construct a nontrivial numerical examples illustrates the existence of such functions and also formulate a pair of nondifferentiable multiobjective Mond-Weir type symmetric fractional primal-dual problems over cones. Further, under the G_f-invexity assumptions, we prove the weak, strong and strict converse duality theorems. We also formulate an example which justifies the Weak duality theorem presented in the paper.

2. Preliminaries and definitions

Let $f = (f_1, ..., f_k) : X \to R^k$ be a differentiable function defined on open set $\phi \neq X \subseteq R^n$ and $I_{f_i}(X)$ be the range of f_i, where $i = 1, 2, 3, ..., k$.

Definition 2.1. Let C be a compact convex set in R^n. The support function of C is defined by

$$S(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$S(y|C) \geq S(x|C) + z^T(y - x), \quad \forall y \in C.$$

The subdifferential of $S(x|C)$ is given by

$$\partial S(x|C) = \{z \in C : z^T x = S(x|C)\}.$$

For any set $S \subset R^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leq 0, \ \ \forall z \in S\}.$$

Obviously, for a compact convex set C, y is in $N_C(x)$ if and only if $S(y|C) = x^T y$, or equivalently, x is in $\partial S(y|C)$.

Definition 2.2 The positive polar cone $S^*$of a cone $S \subseteq R^s$ is defined by

$$S^* = \Big\{y \in R^s : x^T y \geq 0\Big\}.$$

Example 2.1 Let $C = \{(x, y) \in R^2 : \ \ x \geq 0, \ x + y \geq 0\}$ be a cone in R^2. Then, its positive polar cone $C^* = \{(x, y) \in R^2 : y \geq 0, \ x - y \geq 0\}$.

Definition 2.3[27]. The function f is said to be invex at $u \in X$ if there exists a function $\eta : X \times X \to R^n$ such that $\forall x \in X$,

$$f_i(x) - f_i(u) \geq \eta^T(x, u)\nabla_x f_i(u), \ \forall \ i = 1, 2, 3, ..., k.$$

If the above inequality sign changes to $\leq$, then f is called incave at $u \in X$ with respect to η.

Definition 2.4[7] The function f is said to be G_f-invex at $u \in X$ if there exist a differentiable function $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i} and a function $\eta : X \times X \to R^n$ such that $\forall \ x \in X$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq \eta^T(x, u)G'_{f_i}(f_i(u))\nabla_x f_i(u), \ \forall \ i = 1, 2, 3, ..., k.$$

If the above inequality sign changes to $\leq$, then f is called G_f-incave at $u \in X$ with respect to η.

If $k = 1$ in the Definition 2.4 , then the function f is called G-invex at $u \in X$ with respect to η.

Example 2.2 Let $f : [0, 1] \to R^3$ be defined as

$$f(x) = \Big\{f_1(x), f_2(x), f_3(x)\Big\}$$

where $f_1(x) = \text{arc}\ (\sin x), \ f_2(x) = x^4, \ f_3(x) = \text{arc}\ (\tan x)$ and $G_f = \Big\{G_{f_1}, G_{f_2}, G_{f_3}\Big\} : R \to R^3$ be defined as:

$$G_{f_1}(t) = \sin t, \ G_{f_2}(t) = t^9 \text{ and } G_{f_3}(t) = \tan t.$$

Let $\eta : [0, 1] \times [0, 1] \to R$ be given as:

$$\eta(x, u) = -\frac{1}{9}x^{18} + x - 8x^3u^9 - 3u.$$

Now, we will show that f is G_f-invex at $u = 0$. For this, we have to show that

$$\pi_i = G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - \eta^T(x, u)G'_{f_i}(f_i(u))\nabla_x f_i(u) \geq 0, \ \text{for } i = 1, 2, 3.$$

Substituting the values of $f_1, \ f_2, \ f_3, \ G_{f_1}, \ G_{f_2}$ and G_{f_3} in the above expressions, we obtain

$$\pi_1 = x - u - \Big(-\frac{1}{9}x^{18} + x - 8x^3u^9 - 3u\Big),$$

$$\pi_2 = x^{36} - u^{36} - \Big(-\frac{1}{9}x^{18} + x - 8x^3u^9 - 3u\Big)36u^{35}$$

and

$$\pi_3 = x - u - \Big(-\frac{1}{9}x^{18} + x - 8x^3u^9 - 3u\Big)\frac{1}{(1+u^2)^{\frac{1}{2}}}$$

which at $u = 0$ yield

$$\pi_1 = \frac{1}{9}x^{18},\ \pi_2 = x^{36} \text{ and } \pi_3 = \frac{1}{9}x^{18}.$$

Obviously, $\pi_1 \geq 0,\ \pi_2 \geq 0$ and $\pi_3 \geq 0,\ \forall\, x \in [0, 1]$.

Hence, $f = (f_1,\ f_2,\ f_3)$ is G_f-invex at $u = 0$ with respect to η.

Now, suppose

$$\psi = f_1(x) - f_1(u) - \eta^T(x, u)\nabla_x f_1(u).$$

or

$$\psi = \text{arc}\ (\tan x) - \text{arc}\ (\tan u) - \Big(-\frac{1}{9}x^{18} + x - 8x^3u^9 - 3u\Big)\Big(\frac{1}{1+u^2}\Big)$$

which at $u = 0$ yields

$$\psi = \text{arc}\ (\tan x) + \frac{1}{9}x^{18} - x.$$

This expression may not be non-negative for all $x \in [0, 1]$. For instance at $x = 1$,

$$\psi = \frac{\pi}{4} + \frac{1}{9} - 1 < 0.$$

Therefore, f_3 is not η-invex at $u = 0$. Hence, $f = (f_1,\ f_2,\ f_3)$ is not η-invex at $u = 0$.

3. G-Mond-Weir type problem

Consider the following pair of multiobjective nondifferentiable fractional symmetric programs:

(MFP) Minimize

$$L(x, y, z, r) = \Big(\frac{G_{f_1}(f_1(x, y)) + S(x|Q_1) - y^T z_1}{G_{g_1}(g_1(x, y)) - S(x|E_1) + y^T r_1}, \ldots, \frac{G_{f_k}(f_k(x, y)) + S(x|Q_k) - y^T z_k}{G_{g_k}(g_k(x, y)) - S(x|E_k) + y^T r_k} \Big)$$

subject to

$$-\sum_{i=1}^{k} \lambda_i \Big[(G'_{f_i}(f_i(x, y))\nabla_y f_i(x, y) - z_i) - \frac{G_{f_i}(f_i(x, y)) + S(x|Q_i) - y^T z_i}{G_{g_i}(g_i(x, y)) - S(x|E_i) + y^T r_i}(G'_{g_i}(g_i(x, y))\nabla_y g_i(x, y) + r_i) \Big] \in C_2^*,$$

$$y^T \sum_{i=1}^{k} \lambda_i \Big[(G'_{f_i}(f_i(x, y))\nabla_y f_i(x, y) - z_i) - \frac{G_{f_i}(f_i(x, y)) + S(x|Q_i) - y^T z_i}{G_{g_i}(g_i(x, y)) - S(x|E_i) + y^T r_i}(G'_{g_i}(g_i(x, y))\nabla_y g_i(x, y) + r_i) \Big] \geq 0,$$

$$\lambda > 0,\ x \in C_1,\ z_i \in D_i, r_i \in F_i,\ i = 1, 2, 3, \ldots, k.$$

(MFD) Maximize

$$M(u, v, w, t) = \Big(\frac{G_{f_1}(f_1(u, v)) - S(v|D_1) + u^T w_1}{G_{g_1}(g_1(u, v)) + S(v|F_1) - u^T t_1}, \ldots, \frac{G_{f_k}(f_k(u, v)) - S(v|D_k) + u^T w_k}{G_{g_k}(g_k(u, v)) + S(v|F_k) - u^T w_k} \Big)$$

subject to

$$\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+w_i)-\frac{G_{f_i}(f_i(u,v))-S(v|D_i)+u^Tz_i}{G_{g_i}(g_i(u,v))+S(v|F_i)-u^Tr_i}(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)-t_i)\Big]\in C_1^*,$$

$$u^T\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+w_i)-\frac{G_{f_i}(f_i(u,v))-S(v|D_i)+u^Tz_i}{G_{g_i}(g_i(u,v))+S(v|F_i)-u^Tr_i}(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)-t_i)\Big]\leq 0,$$

$$\lambda>0, v\in C_2, w_i\in Q_i, t_i\in E_i,\ i=1,2,3,...,k,$$

where $S_1\subseteq R^n$ and $S_2\subseteq R^m$, C_1 and C_2 are arbitrary cones in R^n and R^m, respectively such that $C_1\times C_2\subseteq S_1\times S_2$, $f_i: S_1\times S_2\to R$, $g_i: S_1\times S_2\to R$ are differentiable functions, $G_{f_i}: I_{f_i}\to R$ and $G_{g_i}: I_{g_i}\to R$ are differentiable strictly increasing functions on their domains, Q_i, E_i are compact convex sets in R^n and D_i, F_i are compact convex sets in R^m, $i=1,2,3,...,k$. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive.

The following example shows the feasibility of the primal problem (MFP) and dual problem (MFD) discussed above:

Example 3.1. Let $k=2$, $n=m=1$ and $S_1=R$, $S_2=R$. Let $f_i: S_1\times S_2\to R$, $g_i: S_1\times S_2\to R$ be defined as

$$f_1(x,y)=x^3+y^2+1,\ f_2(x,y)=2x^4+xy^2+2y^2+4,\ g_1(x,y)=2x^2y^2+4,\ g_2(x,y)=xy^4+x^2+1.$$

Suppose $G_{f_i}(t)=G_{g_i}(t)=t,\ i=1,2.$

$$Q_1=[-1,1],\ \ Q_2=[0,1],\ \ E_1=\{0\}=E_2,\ \ D_1=[-1,1],\ \ D_2=[-2,2],\ \ F_1=\{0\}=F_2.$$

Assume that $C_1=C_2=R_+$ then $C_1^*=C_2^*=R_+$. Clearly, $C_1\times C_2\subseteq S_1\times S_2$.

(EMFP) Minimize $L(x,y,z,r)=\left(\dfrac{x^3+y^2+1+|x|-yz_1}{2x^2y^2+4},\dfrac{2x^4+xy^2+2y^2+4+\frac{x+|x|}{2}-yz_2}{xy^4+x^2+1}\right)$

subject to

$$\lambda_1\Big((2y-z_1)-\frac{x^3+y^2+1+|x|-yz_1}{2x^2y^2+4}(4x^2y)\Big)$$
$$+\lambda_2\Big((2xy+4y-z_2)-\frac{2x^4+xy^2+2y^2+4+\frac{x+|x|}{2}-yz_2}{xy^4+x^2+1}(4xy^3)\Big)\leq 0, \quad (1)$$

$$y\lambda_1\Big((2y-z_1)-\frac{x^3+y^2+1+|x|-yz_1}{2x^2y^2+4}(4x^2y)\Big)$$
$$+y\lambda_2\Big((2xy+4y-z_2)-\frac{2x^4+xy^2+2y^2+4+\frac{x+|x|}{2}-yz_2}{xy^4+x^2+1}(4xy^3)\Big)\geq 0, \quad (2)$$

$$\lambda_1,\lambda_2>0, x\geq 0, -1\leq z_1\leq 1, -2\leq z_2\leq 2.$$

(EMFD) Maximize $M(u,v,w,t)=\left(\dfrac{u^3+v^2+1-|u|+uw_1}{2u^2v^2+4},\dfrac{2u^4+uv^2+2v^2+4-2|u|+uw_2}{uv^4+u^2+1}\right)$

subject to

$$\lambda_1\Big((3u^2+w_1)-\frac{u^3+v^2+1-|u|+uw_1}{2u^2v^2+4}(4uv^2)\Big)$$
$$+\lambda_2\Big((8u^3+v^2+w_2)-\frac{2u^4+uv^2+2v^2+4-2|u|+uw_2}{uv^4+u^2+1}(v^4+2u)\Big)\geq 0, \tag{3}$$

$$u\lambda_1\Big((3u^2+w_1)-\frac{u^3+v^2+1-|u|+uw_1}{2u^2v^2+4}(4uv^2)\Big)$$
$$+u\lambda_2\Big((8u^3+v^2+w_2)-\frac{2u^4+uv^2+2v^2+4-2|u|+uw_2}{uv^4+u^2+1}(v^4+2u)\Big)\leq 0, \tag{4}$$

$$\lambda_1,\lambda_2>0, v\geq 0, -1\leq w_1\leq 1, 0\leq w_2\leq 1.$$

One can easily verify that $x=3, y=0, z_1=1/2, z_2=1, \lambda_1=1, \lambda_2=2$ is (EMFP) and $u=0, v=1/2, w_1=3/4, w_2=1, \lambda_1=2, \lambda_2=3$ is (EMFD) feasible.

Now, Let $U=(U_1,U_2,...,U_k)$ and $V=(V_1,V_2,...,V_k)$. Then, we can express the programs (MFP) and (MFD) equivalently as:

$(MFP)_U$ Minimize U
subject to

$$\Big(G_{f_i}(f_i(x,y)\Big)+S(x|Q_i)-y^Tz_i\Big)-U_i\Big(G_{g_i}(g_i(x,y))-S(x|E_i)+y^Tr_i\Big)=0,\ i=1,2,3,...,k, \tag{5}$$

$$-\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)-z_i)-U_i\Big(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y)+r_i\Big)\Big]\in C_2^*, \tag{6}$$

$$y^T\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)-z_i)-U_i\Big(G'_{g_i}(g_i(x,y))\nabla_y g_i(x,y)+r_i\Big)\Big]\geq 0, \tag{7}$$

$$\lambda>0, x\in C_1, z_i\in D_i, r_i\in F_i,\ \ i=1,2,3,...,k. \tag{8}$$

$(MFD)_V$ Minimize V
subject to

$$\Big(G_{f_i}(f_i(u,v))-S(v|D_i)+u^Tw_i\Big)-V_i\Big(G_{g_i}(g_i(u,v))+S(v|F_i)-u^Tt_i\Big)=0,\ i=1,2,3,...,k, \tag{9}$$

$$\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+w_i)-V_i\Big(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)-t_i\Big)\Big]\in C_1^*, \tag{10}$$

$$u^T\sum_{i=1}^{k}\lambda_i\Big[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+w_i)-V_i\Big(G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v)-t_i\Big)\Big]\leq 0, \tag{11}$$

$$\lambda>0,\ v\in C_2,\ w_i\in Q_i,\ t_i\in E_i,\ i=1,2,3,...,k. \tag{12}$$

Next, we prove duality theorems for $(MFP)_U$ and $(MFP)_V$, which one equally apply to (MFP) and (MFD), respectively. Let $z=(z_1,z_2,...,z_k)$, $r=(r_1,r_2,...,r_k)$, $w=(w_1,w_2,...,w_k)$, $t=(t_1,t_2,...,t_k)$ and $\lambda=(\lambda_1,\lambda_2,...,\lambda_k)$.

Theorem 3.1 (Weak duality). Let (x,y,U,z,r,λ) and (u,v,V,w,t,λ) be feasible solution for $(MFP)_U$ and $(MFP)_V$, respectively. Let for $i=1,2,3,...,k$,

(*i*) $f_i(.,v)$ be G_{f_i}-invex and $(.)^T w_i$ be invex at u with respect to η_1,
(*ii*) $g_i(.,v)$ be G_{g_i}-incave and $(.)^T t_i$ be invex at u with respect to η_1,
(*iii*) $f_i(x,.)$ be G_{f_i}-incave and $(.)^T z_i$ be invex at y with respect to η_2,
(*iv*) $g_i(x,.)$ be G_{g_i}-invex and $(.)^T r_i$ be invex at y with respect to η_2,
(*v*) $\eta_1(x,u)+u \in C_1$ and $\eta_2(v,y)+y \in C_2$,
(*vi*) $G_{g_i}(g_i(x,v)) + v^T r_i - x^T t_i > 0$.

Then, the following cannot hold:

$$U_i \leq V_i, \ \forall\ i = 1,2,3,...,k \tag{13}$$

and

$$U_j < V_j, \ \text{for at least one } j = 1,2,3,...,k. \tag{14}$$

Proof. Suppose (13) and (14) hold, then

$$U_i \leq V_i, \ \forall\ i = 1,2,3,...,k \tag{15}$$

and

$$U_j < V_j, \ \text{for at least one } j = 1,2,3,...,k. \tag{16}$$

Using hypothesis (*v*) and (10), we get

$$(\eta_1(x,u)+u)^T \sum_{i=1}^{k} \lambda_i[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i) - V_i(G' g_i(g_i(u,v))\nabla_x g_i(u,v) - t_i)] \geq 0. \tag{17}$$

Also, from (11) and (17), we have

$$\eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i[(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + w_i) - V_i(G' g_i(g_i(u,v))\nabla_x g_i(u,v) - t_i)] \geq 0. \tag{18}$$

By hypothesis (*i*), we have

$$G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \geq \eta_1^T(x,u) G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v)$$

and

$$x^T w_i - u^T w_i \geq \eta_1^T(x,u) w_i, \ \ i = 1,2,3,...,k.$$

Further, it follows from $\lambda > 0$ that

$$\sum_{i=1}^{k} \lambda_i \left[G_{f_i}(f_i(x,v)) + x^T w_i - G_{f_i}(f_i(u,v)) - u^T w_i\right] \geq \eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + w_i]. \tag{19}$$

Similarly, from hypothesis (*ii*), we have

$$-G_{g_i}(g_i(x,v)) + G_{g_i}(g_i(u,v)) \geq -\eta_1^T(x,u) G'_{g_i}(g_i(u,v)) \nabla_x g_i(u,v)$$

and

$$x^T t_i - u^T t_i \geq \eta_1^T(x,u) t_i, \ i = 1,2,3,...,k.$$

Multiplying by $\lambda_i V_i$ in the above inequalities and taking summation over i, we get

$$\sum_{i=1}^{k} \lambda_i V_i \left[-G_{g_i}(g_i(x,v)) + x^T t_i + G_{g_i}(g_i(u,v)) - u^T t_i\right] \geq -\eta_1^T(x,u) \sum_{i=1}^{k} \lambda_i V_i [G'_{g_i}(g_i(u,v))\nabla_x g_i(u,v) - t_i]. \tag{20}$$

Further, using (18) in the addition of (19)-(20), we get

$$\sum_{i=1}^{k} \lambda_i[(G_{f_i}(f_i(x,v)) + x^T w_i - G_{f_i}(f_i(u,v)) - u^T w_i) - V_i(G_{g_i}(g_i(x,v)) - G_{g_i}(g_i(u,v)) - x^T t_i + u^T t_i)] \geq 0.$$

It follows from (9) and the fact that $v^T r_i \leq S(v|F_i),\ i = 1,2,3,...,k$, we get

$$\sum_{i=1}^{k} \lambda_i[(G_{f_i}(f_i(x,v)) + x^T w_i - S(v|D_i)) + V_i(x^T t_i - v^T r_i - G_{g_i}(g_i(x,v)))] \geq 0. \tag{21}$$

Similarly, using hypothesis $(iii) - (v)$ and primal constraints (5)-(8), we obtain

$$\sum_{i=1}^{k} \lambda_i[(-G_{f_i}(f_i(x,v)) + v^T z_i - S(x|Q_i)) + U_i(-x^T t_i + v^T r_i + G_{g_i}(g_i(x,v)))] \geq 0 \tag{22}$$

Adding (21) and (22), we have

$$\sum_{i=1}^{k} \lambda_i \left[v^T z_i - S(v|D_i) + x^T w_i - S(x|Q_i)\right] + \sum_{i=1}^{k} \lambda_i \left[(U_i - V_i)\{G_{g_i}(g_i(x,v)) + v^T r_i - x^T r_i\}\right] \geq 0. \tag{23}$$

Since $\lambda > 0,\ v^T z_i \leq S(v|D_i)$ and $x^T w_i \leq S(x|Q_i)$, the inequality (23) gives

$$\sum_{i=1}^{k} \lambda_i \left[(U_i - V_i)\{G_{g_i}(g_i(x,v)) + v^T r_i - x^T r_i\}\right] \geq 0.$$

Hence, the result follows from (15)-(16) and hypothesis (vi). □

Example 3.2. Let $n = m = 1,\ k = 2$ and $S_1 = S_2 = R$. Let $f_i : S_1 \times S_2 \to R,\ g_i : S_1 \times S_2 \to R$ be defined as

$$f_1(x,y) = x - y^2,\ \ f_2(x,y) = x^2 - y,\ \ g_1(x,y) = x + y^4 + 1,\ \ g_2(x,y) = x + y^2 + 1.$$

Suppose $G_{f_i}(t) = G_{g_i}(t) = t,\ i = 1,2$ and $E_1 = E_2 = Q_1 = Q_2 = D_1 = D_2 = F_1 = F_2 = \{0\}$.

Further, let $\eta_1 : S_1 \times S_1 \to R$ and $\eta_2 : S_2 \times S_2 \to R$ be defined as

$$\eta_1(x,u) = x - u, \quad \eta_2(v,y) = v - y.$$

Assume that $C_1 = C_2 = R_+$, then $C_1^* = C_2^* = R_+$. Clearly, $C_1 \times C_2 \subseteq S_1 \times S_2$.

Substituting these expressions in $(MFP)_U$ and $(MFD)_V$, we obtain

$(EMFP)_U$ Minimize $L(x,y,z,r)$=(U_1,U_2)
subject to

$$x - y^2 - U_1(x + y^4 + 1) \quad = 0, \tag{24}$$

$$x^2 - y - U_2(x + y^2 + 1) \quad = 0, \tag{25}$$

$$\lambda_1[-2y - 4y^3 U_1] + \lambda_2[-1 - 2yU_2] \quad \leq 0, \tag{26}$$

$$y\lambda_1[-2y - 4y^3 U_1] + y\lambda_2[-1 - 2yU_2] \quad \geq 0, \tag{27}$$

$$\lambda_1,\ \lambda_2 > 0,\ x \geq 0. \tag{28}$$

$(EMFD)_V$ Maximize $M(u,v,w,t)$=(V_1,V_2)
subject to

$$u - v^2 - V_1(u + v^4 + 1) \quad = 0, \tag{29}$$

$$u^2 - v - V_2(u + v^2 + 1) \quad = 0, \tag{30}$$

$$\lambda_1[1 - V_1] + \lambda_2[2u - V_2] \quad \geq 0, \tag{31}$$

$$u\lambda_1[1 - V_1] + u\lambda_2[2u - V_2] \quad \leq 0, \tag{32}$$

$$\lambda_1, \lambda_2 > 0, v \geq 0. \tag{33}$$

First, we will show that the functions defined above satisfy the hypotheses of the Theorem 2.7.

(A_1) $f_1(.,v)$ is G_{f_1}-invex at u with respect to η_1, since

$$\begin{aligned}&G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta_1(x,u)G'_{f_1}(f_1(u,v))\nabla_x f_1(u,v)\\&\quad = (x - v^2) - (u - v^2) - (x - u)\\&\quad = 0 \text{ for all } x, u \in S_1.\end{aligned}$$

Obviously, $(.)^T w_1$=0 is invex at u with respect to η_1.

Now, $f_2(.,v)$ is G_{f_2}-invex at u with respect to η_1, since

$$\begin{aligned}&G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(u,v)) - \eta_1(x,u)G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v)\\&\quad = (x^2 - v) - (u^2 - v) - (x - u) \times 2u\\&\quad = (x - u)^2\\&\quad \geq 0 \text{ for all } x, u \in S_1.\end{aligned}$$

Again, $(.)^T w_2$=0 is obviously invex at u with respect to η_1.

(A_2) $g_1(.,v)$ is G_{g_1}-incave at u with respect to η_1, since

$$\begin{aligned}&G_{g_1}(g_1(x,v)) - G_{g_1}(g_1(u,v)) - \eta_1(x,u)G'_{g_1}(g_1(u,v))\nabla_x g_1(u,v)\\&\quad = (x + v^4 + 1) - (u + v^4 + 1) - (x - u)\\&\quad = 0 \text{ for all } x, u \in S_1.\end{aligned}$$

Obviously, $(.)^T t_1$=0 is trivially invex at u with respect to η_1.

$$\begin{aligned}&G_{g_2}(g_2(x,v)) - G_{g_2}(g_2(u,v)) - \eta_1(x,u)G'_{g_2}(g_2(u,v))\nabla_x g_2(u,v)\\&\quad = (x + v^2 + 1) - (u + v^2 + 1) - (x - u)\\&\quad = 0 \text{ for all } x, u \in S_1.\end{aligned}$$

Hence, g_2 is G_{g_2}-incave at u with respect to η_1.
Naturally, $(.)^T t_2 = 0$ is invex at u with respect to η_1.

(A_3) $f_1(x,.)$ is G_{f_1}-incave at y with respect to η_2, since

$$\begin{aligned}&G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(x,y)) - \eta_2(v,y)G'_{f_1}(f_1(x,y))\nabla_y f_1(x,y)\\&\quad = (x - v^2) - (x - y^2) - (v - y) \times (-2y)\\&\quad = -(v - y)^2 \leq 0 \text{ for all } v, y \in S_2.\end{aligned}$$

Obviously, $(.)^T z_1$=0 is invex at y with respect to η_1.

$f_2(x,.)$ is G_{f_2}-incave at y with respect to η_2, since

$$\begin{aligned} & G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(x,y)) - \eta_2(v,y)G'_{f_2}(f_2(x,y))\nabla_y f_2(x,y) \\ & = (x^2 - v) - (x^2 - y) - (v - y) \times (-1) \\ & = 0 \ \text{ for all } v, y \in S_2. \end{aligned}$$

Obviously, $(.)^T z_2$=0 is invex at y with respect to η_2.

(A_4) $g_1(x,.)$ is G_{g_1}-invex at y with respect to η_2

$$\begin{aligned} & G_{g_1}(g_1(x,v)) - G_{g_1}(g_1(x,y)) - \eta_2(v,y)G'_{g_1}(g_1(x,y))\nabla_y g_1(x,y) \\ & = (x + v^4 + 1) - (x + y^4 + 1) - (v - y) \times (4y^3) \\ & = (v - y)^2[(v + y)^2 + 2y^2] \\ & \geq 0 \text{ for all } v, y \in S_2. \end{aligned}$$

$(.)^T r_1$=0 is invex at y with respect to η_1.

Again, $g_2(x,.)$ is G_{g_2}-invex at y with respect to η_2, since

$$\begin{aligned} & G_{g_2}(g_2(x,v)) - G_{g_2}(g_2(x,y)) - \eta_2(v,y)G'_{g_2}(g_2(x,y))\nabla_y g_2(x,y) \\ & = (x + v^2 + 1) - (x + y^2 + 1) - (v - y) \times (2y) \\ & = (v - y)^2 \\ & \geq 0 \ \text{ for all } v, y \in S_2. \end{aligned}$$

Obviously, $(.)^T r_2$=0 is invex at y with respect to η_2.
(A_5) $x \geq 0$ and $v \geq 0$, (from 28) and 33)),

(A_6) $G_{g_1}(g_1(x,v)) + v^T r_1 - x^T t_1 = x + v^4 + 1 > 0$,
$G_{g_2}(g_2(x,v)) + v^T r_2 - x^T t_2 = x + v + 1 > 0 \Big($from (28) and (33)$\Big)$.

Validation: To validate our result, it is enough to prove that

$$\sum_{i=1}^{2} \lambda_i(U_i - V_i)(G_{g_i}(g_i(x,v)) + v^T r_i - x^T t_i) \quad \geq 0$$

or

$$\lambda_1(U_1 - V_1)[x + v^4 + 1] + \lambda_2(U_2 - V_2)[x + v^2 + 1] \quad \geq 0.$$

Now,

$$\begin{aligned} & \lambda_1(U_1 - V_1)[x + v^4 + 1] + \lambda_2(U_2 - V_2)[x + v^2 + 1] \\ = & \ \lambda_1[(x - v^2) + V_1(-x - v^4 - 1)] + \lambda_2[(x^2 - v) + V_2(-x - v^2 - 1)] \\ & + \lambda_1[(-x + v^2 + U_1(x + v^4 + 1)] + \lambda_2[(-x^2 + v) + U_2(x + v^2 + 1)] \\ = & \ (x - u)[\lambda_1 - \lambda_1 V_1 + \lambda_2(x + u) - \lambda_2 V_2] \\ & + (v - y)[(v + y)\lambda_1 + U_1(v + y)(v^2 + y^2)\lambda_1 + \lambda_2 + \lambda_2 U_2(v + y)] \end{aligned}$$

$$\Big(\text{from feasibility conditions (24)-(25) and (29)-(30)}\Big)$$

$$\geq (x-u)[\lambda_1+\lambda_2(x+u)]-(x-u)[\lambda_1+2\lambda_2 u]+(v-y)[(v+y)\lambda_1$$
$$+U_1(v+y)(v^2+y^2)\lambda_1+\lambda_2 U_2(v+y)-2y\lambda_1-4y^3U_1\lambda_1+2U_2y\lambda_2]$$
$$\Big(\text{Using (26)-(28) and (31)-(33)}\Big)$$
$$= (x-u)^2\lambda_2+(v-y)^2[\lambda_1U_1\{(v+y)^2+2y^2\}+\lambda_1+\lambda_2U_2]$$
$$\geq 0. \text{ Hence, verified.}$$

□

Theorem 3.2 (Strong duality). Let $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{z},\bar{r})$ be an efficient solutions of $(MFP)_U$ and fix $\lambda=\bar{\lambda}$ in $(MFD)_V$. If the following conditions hold:

(*i*) the matrix $\sum_{i=1}^{k}\bar{\lambda}_i[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\left(\nabla_y f_i(\bar{x},\bar{y})\right)^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{U}_i(G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})$ $\left(\nabla_y g_i(\bar{x},\bar{y})\right)^T+G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy}g_i(\bar{x},\bar{y}))]$ is positive definite or negative definite,

(*ii*) the vectors $\left((G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i)-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i)\right)_{i=1}^{k}$ are linearly independent,

(*iii*) $\bar{U}_i>0,\ i=1,2,3,...,k.$

Then, there exist $\bar{w}_i\in Q_i$ and $\bar{t}_i\in E_i$, $i=1,2,3,...,k$ such that $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{w},\bar{t}$ feasible solution for $(MFD)_V$. Furthermore, if the hypotheses of Theorem 3.1 hold, then $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{w},\bar{t},)$ is an efficient solution of $(MFD)_V$ and the objective functions have same values.

Proof. Since $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{z},\bar{r})$ is an efficient solution of $(MFD)_U$, therefore by the Fritz John necessary optimality conditions [28], there exist $\alpha\in R^k,\beta\in R^k,\gamma\in C_2,\delta\in R,\xi\in R^k,\bar{w}_i\in R^n$ and $\bar{t}_i\in R^n$, $i=1,2,3,...,k$ such that

$$(x-\bar{x})^T\sum_{i=1}^{k}\beta_i((G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})+\bar{w}_i)-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y})-\bar{t}_i))+(y-\delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i[G''_{f_i}(f_i(\bar{x},\bar{y}))$$
$$\nabla_x f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{xy}f_i(\bar{x},\bar{y})]$$
$$-\bar{U}_i[G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T+G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{xy}g_i(\bar{x},\bar{y})]\geq 0,\ \forall\ x\in C_1, \tag{34}$$

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i))$$
$$+(\gamma-\delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})]$$
$$-\bar{U}_i[G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T+G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy}g_i(\bar{x},\bar{y})]=0, \tag{35}$$

$$\alpha_i-\beta_i(G_{g_i}(g_i(\bar{x},\bar{y}))-S(\bar{x}|E_i)+\bar{y}^T\bar{r}_i)-(\gamma-\delta\bar{y})\bar{\lambda}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i)=0,\ , \tag{36}$$

$$(\gamma-\delta\bar{y})^T[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i)-\bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i)]-\xi_i=0,\ i=1,2,3,...,k, \tag{37}$$

$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\bar{\lambda}_i\in N_{D_i}(\bar{z}_i), i=1,2,3,...,k, \tag{38}$$

$$\beta_i\bar{U}_i\bar{y}+(\gamma-\delta\bar{y})\bar{U}_i\bar{\lambda}_i\in N_{F_i}(\bar{r}_i),\ i=1,2,3,...,k, \tag{39}$$

$$\gamma^T \sum_{i=1}^{k} \bar{\lambda}_i [G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i)] = 0, \tag{40}$$

$$\delta\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i [G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i)] = 0, \tag{41}$$

$$\bar{\lambda}^T \xi = 0, \tag{42}$$

$$\bar{w}_i \in Q_i, \bar{t}_i \in E_i, \bar{x}^T \bar{t}_i = S(\bar{x}|E_i), \bar{x}^T \bar{w}_i = S(\bar{x}|Q_i),\ i = 1,2,3,...,k, \tag{43}$$

$$(\alpha,\delta,\xi) \geq 0,\ (\alpha,\beta,\gamma,\delta,\xi) \neq 0. \tag{44}$$

Since $\bar{\lambda} > 0$ and $\bar{\xi} \geq 0$, (42) implies that $\bar{\xi} = 0$.

Post-multiplication $(\gamma - \delta\bar{y})$ in (35) and using (37) and $\xi = 0$, we get

$$(\gamma - \delta\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i (G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})$$
$$-\bar{U}_i(G''_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y})(\nabla_y g_i(\bar{x},\bar{y}))^T + G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_{yy} g_i(\bar{x},\bar{y}))(\gamma - \delta\bar{y}) = 0, \tag{45}$$

which from hypothesis (i) yields

$$\gamma = \delta\bar{y}. \tag{46}$$

Using (46) in (35), we have

$$\sum_{i=1}^{k} (\beta_i - \delta\bar{\lambda}_i)[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i)] = 0.$$

It follows from hypothesis (ii) that

$$\beta_i = \delta\bar{\lambda}_i,\ i = 1,2,3,...,k. \tag{47}$$

Now, we claim that $\beta_i \neq 0$, $\forall\ i$. Otherwise, if $\beta_{t_0} = 0$, for some $i = t_0$, then from (47), since $\bar{\lambda} > 0$, we have $\delta = 0$. Again from (47), $\beta_i = 0$, $\forall\ i$. Thus from (36), we get $\alpha_i = 0$, $\forall\ i$. Also from (46), $\gamma = 0$. This contradicts (44). Hence, $\beta_i \neq 0$, for all i. Further, if $\beta_i < 0$, for any i, then from (47), $\delta < 0$, which again contradicts (44). Hence, $\beta_i > 0$, $\forall\ i$.
Further, using (44) and (47) in (34), we get

$$(x - \bar{x})^T \sum_{i=1}^{k} \bar{\lambda}_i [G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)] \geq 0,\ \forall x \in C_1. \tag{48}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ as C_1 is a closed convex cone. On substituting $x + \bar{x}$ in place of x in (48), we get

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i [(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)] \geq 0.$$

Hence,

$$\sum_{i=1}^{k} \bar{\lambda}_i [(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)] \in C_1^*. \tag{49}$$

Also, by letting $x = 0$ and $x = 2\bar{x}$ simultaneously in (48), we have

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i[(G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(G'_{g_i}(g_i(\bar{x},\bar{y}))\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)] = 0. \tag{50}$$

Since $\gamma = \delta\bar{y}$ and $\delta > 0$, we have

$$\bar{y} = \frac{\gamma}{\delta} \in C_2. \tag{51}$$

From (38), (46) and using $\beta > 0$, we get $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i = 1,2,3,...,k$. This implies

$$\bar{y}^T\bar{z}_i = S(\bar{y}|D_i),\ i = 1,2,3,...,k. \tag{52}$$

By (39) and hypothesis (*iii*), we obtain

$$\bar{y} \in N_{F_i}(\bar{r}_i),\ i = 1,2,3,...,k. \tag{53}$$

Hence,

$$\bar{y}^T\bar{r}_i = S(\bar{y}|F_i),\ i = 1,2,3,...,k. \tag{54}$$

Combining (43), (52), (54) and equation (5), it follows that

$$(G_{f_i}(f_i(\bar{x},\bar{y})) - S(\bar{y}|D_i) + \bar{x}^T\bar{w}_i) - \bar{U}_i(G_{g_i}(g_i(\bar{x},\bar{y})) + S(\bar{y}|F_i) - \bar{x}^T\bar{t}_i) = 0,\ i = 1,2,3,...,k. \tag{55}$$

This together with (49)-(50) and (55) shows that $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{w},\bar{t})$ is feasible solution for $(MFP)_V$. Now, let $(\bar{x},\bar{y},\bar{U},\bar{\lambda},\bar{w},\bar{t})$ be not an efficient solution of $(MFD)_V$. Then, there exists other $(u,v,V,\lambda,w,t) \in (MFD)_V$ such that $\bar{U}_i \leq V_i$, $\forall\ i \in K$ and $\bar{U}_j < V_j$, for some $j \in K$. This contradicts the result of the Theorem 3.1. Hence proved. □

Theorem 3.3 (Converse duality). Let $(\bar{u},\bar{v},\bar{V},\bar{\lambda},\bar{w},\bar{t})$ be an efficient solutions of $(MFD)_V$ and fix $\lambda = \bar{\lambda}$ in $(MFP)_U$. If the following conditions hold:

(*i*) the matrix $\sum_{i=1}^{k} \bar{\lambda}_i[G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})\,(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v}) - \bar{V}_i(G''_{g_i}(g_i(\bar{u},\bar{v}))\nabla_x g_i(\bar{u},\bar{v})$ $(\nabla_x g_i(\bar{u},\bar{v}))^T + G'_{g_i}(g_i(\bar{u},\bar{v}))\nabla_{xx} g_i(\bar{u},\bar{v}))]$ is positive definite or negative definite,

(*ii*) the vectors $\left(G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v}) + \bar{w}_i - \bar{V}_i(G'_{g_i}(g_i(\bar{u},\bar{v}))\nabla_x g_i(\bar{u},\bar{v}) - \bar{t}_i)\right)_{i=1}^{k}$ are linearly independent,

(*iii*) $\bar{V}_i > 0,\ i = 1,2,3,...,k$.

Then, $\exists\ \bar{z}_i \in D_i$ and $\bar{r}_i \in F_i$, $i = 1,2,3,...,k$ such that $(\bar{u},\bar{v},\bar{V},\bar{\lambda},\bar{z},\bar{r})$ is feasible solution for $(MFP)_U$. Furthermore, if the assumptions of Theorem 3.1 hold, then $(\bar{u},\bar{v},\bar{V},\bar{\lambda},\bar{z},\bar{r})$ is an efficient solution of $(MFP)_U$ and objective functions have equal values.

Proof. The results can be obtained on the lines of Theorem 3.2. □

4. Conclusions

In this paper, we have used the concept of G_f- invex functions to establish duality results for a Mond-Weir type dual model related to multiobjective nondifferentiable symmetric fractional programming problem over arbitrary cones. Numerical examples have also been illustrated to justify the weak duality theorem. The present work can further be extended to nondifferentiable second-order and higher-order symmetric fractional programming over cones. This will orient the future task for the researcher working in this area.

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