Filomat 35:9 (2021), 2911–2918
https://doi.org/10.2298/FIL2109911D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Structure Space of the Ring $B_1(X)$

Atasi. Deb Ray[a], Atanu Mondal[b]

[a]*Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata - 700019, India*
[b]*Department of Commerce (E), St. Xavier's college, 30, Mother Teresa sarani, Kolkata - 700016, India*

Abstract. In this article, we continue our study of the ring of Baire one functions on a topological space (X, τ), denoted by $B_1(X)$, and extend the well known M. H. Stones's theorem from $C(X)$ to $B_1(X)$. Introducing the structure space of $B_1(X)$, an analogue of Gelfand-Kolmogoroff theorem is established. It is observed that (X, τ) may not be embedded inside the structure space of $B_1(X)$. This observation inspired us to introduce a weaker form of embedding and show that in case X is a T_4 space, X is weakly embedded as a dense subspace, in the structure space of $B_1(X)$. It is further established that the ring $B_1^*(X)$ of all bounded Baire one functions, under suitable conditions, is a C-type ring and also, the structure space of $B_1^*(X)$ is homeomorphic to the structure space of $B_1(X)$. Introducing a finer topology σ than the original T_4 topology τ on X, it is proved that $B_1(X)$ contains free maximal ideals if σ is strictly finer than τ. Moreover, in the class of all perfectly normal T_1 spaces, $\sigma = \tau$ is necessary as well as sufficient for $B_1(X) = C(X)$.

1. Introduction and Prerequisites

The collection $B_1(X)$, of all real valued Baire one functions defined on a topological space X forms a commutative lattice ordered ring with unity, if the relevant operations are defined pointwise on X. We developed a few basic properties of this ring in [1] , followed by an investigation on the duality existing between ideals / maximal ideals in $B_1(X)$ and Z_B-filters / Z_B-ultrafilters on X in [2]. A zero set in X is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$, for some $f \in B_1(X)$. A Z_B-filter on X is a family of zero sets in X enjoying the properties of a filter. A Z_B-filter on X, which is not properly contained in any Z_B-filter on X is called a maximal Z_B-filter or Z_B-ultrafilter on X. The above mentioned duality existing between ideals in $B_1(X)$ and Z_B-filters on X is manifested by the fact that if I is an ideal in $B_1(X)$, then $Z[I] = \{Z(f) : f \in B_1(X)\}$ is a Z_B-filter on X and dually for a Z_B-filter $\mathscr{F}$ on X, $Z^{-1}[\mathscr{F}] = \{f \in B_1(X) : Z(f) \in \mathscr{F}\}$ is a proper ideal in $B_1(X)$. Furthermore, the assignment $M \mapsto Z[M]$ is a bijection on the set of all maximal ideals in $B_1(X)$ and the family of all Z_B-ultrafilters on X.

A celebrated result due to M. H. Stone in the theory of rings of continuous functions $C(X)$ says that given any topological space X there exists a Tychonoff space Y such that the ring $C(X)$ is isomorphic to $C(Y)$. A natural question is, whether one can expect an analogous fact in the theory of rings of Baire one functions. Indeed it is shown in the present paper that such an isomorphism from $C(X)$ onto $C(Y)$ as mentioned above can be extended to an isomorphism from $B_1(X)$ onto $B_1(Y)$ [Theorem 2.1]. Thus in order to study the algebraic properties of the ring $B_1(X)$ one need not bother about the ambient spaces X,

2020 *Mathematics Subject Classification*. Primary 26A21; Secondary 13A15, 54C30, 54C50, 54D35
Keywords. Z_B- filter, Z_B-ultrafilter, free and fixed maximal ideals of $B_1(X)$, structure space of a ring, compactification
Received: 17 July 2020; Revised: 21 January 2021; Accepted: 27 January 2021
Communicated by Ljubiša D.R. Kočinac
Email addresses: debrayatasi@gmail.com (Atasi. Deb Ray), atanu@sxccal.edu (Atanu Mondal)

which are not Tychonoff. One of the most important facts in the theory of rings of continuous functions $C(X)$ for a Tychonoff space X is that the space of all maximal ideals in $C(X)$ equipped with the Hull Kernel topology, often called the structure space of the ring $C(X)$, is topologically equivalent to βX, the *Stone-Čech* compactification of X (7N, [4]). In the context of $B_1(X)$ the things are little restricted. In fact, we have proved that if X is a T_4-space, then it could be embedded by a weaker kind of embedding inside the structure space of the ring $B_1(X)$ [Theorem 4.4]. Nevertheless, it happens that there exists a topology σ on X, finer than the original topology τ on X for which the structure space of $B_1(X,\tau)$ is precisely $\beta(X,\sigma)$ [Theorem 4.11]. This leads to seemingly interesting conclusions. Indeed, it is realized that with a little restriction on the nature of the weak embedding map $X \to \mathcal{M}(B_1(X))$, as mentioned earlier, the ring $B_1^*(X)$ of all bounded Baire one functions on X is isomorphic to a ring of the form $C(Y)$ for some Tychonoff space Y; where $\mathcal{M}(B_1(X))$ stands for the structure space of $B_1(X)$ [Theorem 4.9]. Furthermore, it is observed that under such restriction, the structure spaces of $B_1(X)$ and $B_1^*(X)$ are homeomorphic [Corollary 4.10].

It is a standard result in the theory of $C(X)$ that X is compact if and only if each maximal ideal in $C(X)$ is fixed [Theorem 4.11 in [4]]. Circumstances are not so pleasant in the scenario of Baire one functions. Indeed, we have shown that even if (X,τ) is compact, the ring $B_1(X,\tau)$ necessarily contains free maximal ideals if $\tau \subsetneq \sigma$ [Theorem 4.15]. The paper ends with the observation that within the category of perfectly normal T_1-spaces X, $B_1(X) = C(X)$ is equivalent to the space X to be a P-space [Theorem 4.18]. Since a perfectly normal T_1 space which is also a P-space is necessarily discrete, it follows that a perfectly normal T_1 space X is discrete if and only if $B_1(X) = C(X)$.

2. Extension of M. H. Stone's Theorem

Theorem 2.1. *For each topological space X, there exists a Tychonoff space Y such that $B_1(X)$ is isomorphic to $B_1(Y)$ and $B_1^*(X)$ is isomorphic to $B_1^*(Y)$ under the same (restriction) map. Indeed, a suitable isomorphism from $C(Y)$ onto $C(X)$ extends to the desired isomorphism from $B_1(Y)$ onto $B_1(X)$.*

Proof. It follows from Theorem 3.9 in [4] that given the topological space X, there exists a Tychonoff space Y and a continuous map $\tau : X \to Y$ with $\tau(X) = Y$ such that the assignment $\psi : C(Y) \to C(X)$ given by $\psi(g) = g \circ \tau$ defines an isomorphism. We shall show that ψ has an extension to an isomorphism from $B_1(Y)$ onto $B_1(X)$. For that purpose choose $h \in B_1(Y)$. Then there exists a sequence $\{h_n\} \subseteq C(Y)$ such that $\lim\limits_{n\to\infty} h_n(y) = h(y)$ for each $y \in Y$. Clearly, $h_n \circ \tau \in C(X)$ for each $n \in \mathbb{N}$ and for each $x \in X$, $\lim\limits_{n\to\infty}(h_n \circ \tau)(x)$ exists. Define $\widehat{\psi}(h) : X \to \mathbb{R}$ by the formula : $\widehat{\psi}(h)(x) = \lim\limits_{n\to\infty}(h_n \circ \tau)(x)$, for $x \in X$. Then $\widehat{\psi}(h) \in B_1(X)$ and it is not hard to verify that the map $\widehat{\psi} : B_1(Y) \to B_1(X)$ is defined without any ambiguity. It is easy to check that $\widehat{\psi}$ is an isomorphism onto $B_1(X)$ and furthermore $\widehat{\psi}$ agrees with ψ on $C(Y)$. Since for any two topological spaces U and V, a ring homomorphism from $B_1(U)$ to $B_1(V)$ takes bounded functions to bounded functions [Theorem 3.7 in [1]], it follows that $\widehat{\psi}(B_1^*(Y)) = B_1^*(X)$. □

Since each ring homomorphism from $B_1(X)$ to $B_1(Y)$ is also a lattice homomorphism [Theorem 3.6 in [1]], following result is immediate.

Corollary 2.2. *The isomorphism $\widehat{\psi} : B_1(Y) \to B_1(X)$ in Theorem 2.1 is a lattice isomorphism.*

In view of Theorem 2.1, from now on, each topological space that will appear in this paper will be assumed to be Tychonoff.

3. The Structure Space of $B_1(X)$

We reproduce the following basic facts from [4], 7M, which we need in the present article. Let A be a commutative ring with unity and $\mathcal{M}(A)$, the set of all maximal ideals in A. For each $a \in A$, let $\mathscr{M}_a = \{M \in \mathcal{M}(A) : a \in M\}$. Then $\{\mathscr{M}_a : a \in A\}$ is a closed base for the Zariski topology or often called the hull-kernel topology τ on $\mathcal{M}(A)$. The topological space $(\mathcal{M}(A),\tau)$ or more simply $\mathcal{M}(A)$, by suppressing τ, is a compact T_1 space and is called the structure space of the ring A. It is a standard result in [5] that if A is a Gelfand ring meaning that each prime ideal in A extends to a unique maximal ideal, then $\mathcal{M}(A)$ is Hausdorff. For any subset $\mathcal{M}_0$ of $\mathcal{M}(A)$, its closure is determined by the formula : $\overline{\mathcal{M}_0} = \{M \in \mathcal{M}(A) : M \supset \bigcap \mathcal{M}_0\}$. We will use these notations in our investigation on the structure space of $B_1(X)$.

A complete description of the maximal ideals in $B_1(X)$ in the manner of Gelfand-Kolmogoroff theorem in rings of continuous functions $C(X)$ (vide Theorem 7.3 in [4]) is in our agenda. Since there is already a one-to-one match between the maximal ideals of $B_1(X)$ and the Z_B-ultrafilters on X : $M \mapsto Z[M]$, we propose to furnish an explicit description of the Z_B-ultrafilters on X. In this program we follow closely the technique adopted to prove Theorem 6.5 in [4]. Let us recall in this connection that the complete list of fixed maximal ideals in $B_1(X)$ is given by the family $\{\widehat{M}_p : p \in X\}$, where $\widehat{M}_p = \{f \in B_1(X) : f(p) = 0\}$ (Theorem 3.2, [2]). Therefore, the list of fixed Z_B-ultrafilters [i.e. those Z_B-ultrafilters $\mathscr{S}$ on X satisfying $\bigcap \mathscr{S} \neq \emptyset$] on X is $\{Z[\widehat{M}_p] \equiv \mathscr{U}_p : p \in X\}$, where $\mathscr{U}_p = \{Z \in Z(B_1(X)) : p \in Z\}$, $Z(B_1(X))$ standing for the family of all zero sets in X. Thus we see that X becomes a ready-made index set for the collection of all fixed Z_B-ultrafilters on X. We extend X to a bigger set $\widehat{X}$, to serve as an index set for the family of all Z_B-ultrafilters on X. For each $p \in \widehat{X}$, let the corresponding Z_B-ultrafilters on X be denoted by $\mathscr{U}^p$ with the understanding that in case $p \in X$, we write $\mathscr{U}^p = \mathscr{U}_p$. For each $Z \in Z(B_1(X))$ set $\overline{Z} = \{p \in \widehat{X} : Z \in \mathscr{U}^p\}$. Then $\{\overline{Z} : Z \in Z(B_1(X))\}$ forms a base for closed sets of some topology, which we wish to call the Stone topology on $\widehat{X}$. In this topology for any $Z \in Z(B_1(X))$, $\overline{Z}$ is essentially the closure of Z in $\widehat{X}$. All these results can be proved by using some routine arguments. Thus X becomes dense in $\widehat{X}$ in this topology because $\overline{X} = \{p \in \widehat{X} : X \in \mathscr{U}^p\} = \widehat{X}$ (since, X is a member of each Z_B-ultrafilter on it.)

Theorem 3.1. *The structure space $\mathcal{M}(B_1(X))$ of the ring $B_1(X)$ is homeomorphic to $\widehat{X}$ with Stone topology.*

Proof. First observe that the map $\phi : \mathcal{M}(B_1(X)) \to \widehat{X}$ defined by $\phi(\widehat{M}) = p$, where $Z[\widehat{M}] = \mathscr{U}^p$ is a bijection between the two sets under consideration; this is already mentioned in the Introductory section of this article. For any $f \in B_1(X)$ and $\widehat{M} \in \mathcal{M}(B_1(X))$, $f \in \widehat{M} \Leftrightarrow Z(f) \in Z[\widehat{M}] \Leftrightarrow Z(f) \in \mathscr{U}^p$, where $\phi(\widehat{M}) = p$ $\Leftrightarrow p \in cl_{\widehat{X}} Z(f)$. Thus ϕ exchanges the basic closed sets between $\mathcal{M}(B_1(X))$ and $\widehat{X}$ and is therefore a homeomorphism. □

The following result describes the collection of all maximal ideals in the ring $B_1(X)$.

Theorem 3.2. *Maximal ideals in the ring $B_1(X)$ are given by $\{\widehat{M}^p : p \in \widehat{X}\}$, where $\widehat{M}^p = \{f \in B_1(X) : p \in cl_{\widehat{X}} Z(f)\}$. Furthermore, if $p \neq q$ in $\widehat{X}$, then $\widehat{M}^p \neq \widehat{M}^q$.*

Proof. Since $\{\mathscr{U}^p : p \in \widehat{X}\}$ is the collection of all Z_B-ultrafilters on X, it follows that $\{Z^{-1}[\mathscr{U}^p] : p \in \widehat{X}\}$ is the family of all maximal ideals in $B_1(X)$. Let $Z^{-1}[\mathscr{U}^p] = \widehat{M}^p$. Thus $\widehat{M}^p = \{f \in B_1(X) : Z(f) \in \mathscr{U}^p\} = \{f \in B_1(X) : p \in cl_{\widehat{X}} Z(f)\}$. Again if $p \neq q$ in $\widehat{X}$, then $\mathscr{U}^p \neq \mathscr{U}^q$ which implies $Z^{-1}[\mathscr{U}^p] \neq Z^{-1}[\mathscr{U}^q]$ and so $\widehat{M}^p \neq \widehat{M}^q$. □

Theorem 3.3. *For $p \in \widehat{X}$, $\widehat{M}^p$ is a fixed maximal ideal in $B_1(X)$ if and only if $p \in X$.*

Proof. Let $p \in X$. Then $\widehat{M}^p = \{f \in B_1(X) : Z(f) \in Z[\widehat{M}^p]\} = \{f \in B_1(X) : Z(f) \in \mathscr{U}^p\} = \{f \in B_1(X) : Z(f) \in \mathscr{U}_p\}$ (because $p \in X$ implies $\mathscr{U}^p = \mathscr{U}_p) = \widehat{M}_p$.

Conversely, let $\widehat{M}^p$ be a fixed maximal ideal in $B_1(X)$. Then $\widehat{M}^p = \widehat{M}_q$, for some $q \in X$. But from the above, $\widehat{M}_q = \widehat{M}^q$. So, $\widehat{M}^p = \widehat{M}^q$. It follows from Theorem 3.2 that $p = q$. So, $p \in X$. □

4. When is $\mathcal{M}(B_1(X))$ a Compactification of X?

To introduce a weak kind of embedding of X into the space $\mathcal{M}(B_1(X))$, we reproduce the following result from [6].

Theorem 4.1. ([6]) (i) *For any topological space X and any metric space Y, $B_1(X, Y) \subseteq \mathscr{F}_\sigma(X, Y) = \{f : X \to Y : f^{-1}(G)$ is an F_σ set, for any open set $G \subseteq Y\}$, here $B_1(X, Y)$ denotes the collection of all Baire one functions from X into Y.*

(ii) *If X is a normal space, then $B_1(X, \mathbb{R}) = \mathscr{F}_\sigma(X, \mathbb{R})$.*

Definition 4.2. A function f from a topological space X into a topological space Y is called:
(i) F_σ**-continuous** if for any open set U of Y $f^{-1}(U)$ is an F_σ set in X.

(ii) **weak** F_σ **continuous** relative to an open base $\mathcal{B}$ of Y if for any $U \in \mathcal{B}$, $f^{-1}(U)$ is an F_σ set in X.
(iii) F_σ**-embedding** if f is one-to-one, F_σ continuous and $f^{-1} : f(X) \to X$ is continuous.
(iv) **weak** F_σ**-embedding** relative to an open base $\mathcal{B}$ of Y if f is one-to-one, weak F_σ continuous relative to $\mathcal{B}$ and $f^{-1} : f(X) \to X$ is continuous.

Remark 4.3. If there exists a countable open base $\mathcal{B}$ of Y, then any weak F_σ-continuous map $f : X \to Y$ relative to $\mathcal{B}$ is F_σ-continuous.

Theorem 4.4. *A T_4 space X is densely weak F_σ-embedded in $\mathcal{M}(B_1(X))$.*

Proof. Let $\psi : X \to \mathcal{M}(B_1(X))$ be defined by $\psi(x) = \widehat{M}_x$. Clearly, ψ is a one-to-one map.

We set for $f \in B_1(X)$, $\widehat{\mathscr{M}_f} = \{M \in \mathcal{M}(B_1(X)) \ : \ f \in M\}$. Then $\mathcal{B}_M \equiv \{\widehat{\mathscr{M}_f} \ : \ f \in B_1(X)\}$ is a closed base for $\mathcal{M}(B_1(X))$ and therefore $\mathcal{B}^*_M = \mathcal{M}(B_1(X)) \setminus \mathcal{B}_M$ is an open base for the same space. Since for any $f \in B_1(X)$, $\psi^{-1}(\widehat{\mathscr{M}_f}) = Z(f)$, a G_δ set in X ([1]), it follows that ψ is weak F_σ-continuous relative to $\mathcal{B}^*_M$. Furthermore, $(\psi^{-1})^{-1}(Z(g)) = \widehat{\mathscr{M}_g} \cap \psi(X)$, an easy verification for each $g \in C(X)$. This proves that $\psi^{-1} : \psi(X) \to X$ is a continuous map. Finally we observe that

$$\begin{aligned} cl_{\mathcal{M}(B_1(X))}\psi(X) &= \{\widehat{M} \in B_1(X) \ : \ \widehat{M} \supseteq \bigcap_{x \in X} \widehat{M}_x\} \\ &= \{\widehat{M} \in B_1(X) \ : \ \widehat{M} \supseteq \{0\}\} \\ &= \mathcal{M}(B_1(X)). \end{aligned}$$

Thus ψ becomes a weak F_σ embedding relative to $\mathcal{B}^*_M$ with $\psi(X)$ dense in $\mathcal{M}(B_1(X))$. □

Corollary 4.5. *If every closed set of $\mathcal{M}(B_1(X))$ is expressible as a countable intersection of basic closed sets $\{\widehat{\mathscr{M}_f} : f \in B_1(X)\}$, then the T_4-space X is densely F_σ-embedded in $\mathcal{M}(B_1(X))$.*

We now show that the above condition though a bit stringent and sufficient may not be necessary for X to be densely F_σ-embedded inside $\mathcal{M}(B_1(X))$. We construct the desired counterexample from 6S, [4] .

Remark 4.6. Take $X = \mathbb{N}$, then $B_1(X) = C(X) = C(\mathbb{N})$ and therefore $\mathcal{M}(B_1(X)) = \beta\mathbb{N}$. We recall from the discussions preceding Theorem 3.1 that $\{cl_{\beta\mathbb{N}}Z : Z \subset \mathbb{N}\}$ is a base for closed sets of the topology on $\beta\mathbb{N}$. We assert that if $p \in \beta\mathbb{N} \setminus \mathbb{N}$, then the one pointic closed set $\{p\}$ in $\beta\mathbb{N}$ can not be expressed as a countable intersection of these basic closed sets. We argue by contradiction and assume that $\{p\} = \bigcap_{n=1}^{\infty} cl_{\beta\mathbb{N}}Z_n$ for a countable family $\{Z_n \ : \ n \in \mathbb{N}\}$ of subsets of $\mathbb{N}$. Since each Z_n is clopen in $\mathbb{N}$, it follows that $cl_{\beta\mathbb{N}}Z_n$ is clopen in $\beta\mathbb{N}$ [see 6.9(c) in [4]]. Consequently, $\{p\}$ is a G_δ subset of $\beta\mathbb{N} \setminus \mathbb{N}$. Each nonempty G_δ set G in the space $\beta\mathbb{N} \setminus \mathbb{N}$ has nonempty interior and hence $|G| \geq 2^c$ [see 6S.8 in [4]], there is a contradiction. Finally we observe that the embedding $\psi : \mathbb{N} \to \mathcal{M}(B_1(\mathbb{N}))$ constructed in the proof of Theorem 4.4 is a dense F_σ-embedding.

We shall soon realise that the embedding $(\psi, \mathcal{M}(B_1(X)))$ of X in $\mathcal{M}(B_1(X))$ in Theorem 4.4 enjoys a kind of extension property analogous to that possessed by the *Stone-Čech* compactification βX of X. We need a lemma for that purpose:

Lemma 4.7. *Let X be a normal space and $\psi_0 : X \to Y$ a F_σ-continuous function. For each $h \in C(Y)$, $h \circ \psi_0 \in B_1(X)$.*

Proof. This follows from Theorem 4.1. □

Theorem 4.8. *If $f : X \to Y$ is an F_σ-continuous function from a T_4 space X to a compact Hausdorff space Y, then there exists a unique continuous function $\widehat{f} : \mathcal{M}(B_1(X)) \to Y$ such that $\widehat{f} \circ \psi = f$.*

Proof. Let $\widehat{M} \in \mathcal{M}(B_1(X))$. Set $M_1 = \{g \in C(Y) : g \circ f \in \widehat{M}\}$. There is no ambiguity in the definition of M_1 because of Lemma 4.7. It is easy to check that M_1 is a prime ideal in $C(Y)$ and therefore can be extended to a unique maximal ideal M in $C(Y)$. As Y is compact, $M = \{g \in C(Y) : g(y) = 0\}$ for a unique point $y \in Y$ (see Theorem 4.11 in [4]). Thus, $\bigcap_{g \in M_1} Z(g) = \{y\}$. Define $\widehat{f} : \mathcal{M}(B_1(X)) \to Y$ by $\widehat{f}(\widehat{M}) = y$. It is clear that $\widehat{f} \circ \psi = f$.

To show that $\widehat{f}$ is continuous at an arbitrary point $\widehat{M} \in \mathcal{M}(B_1(X))$. Let W be a neighbourhood of $\widehat{f}\big(\widehat{M}\big)$ in Y. Since Y is a compact Hausdorff space, it is Tychonoff and so, each neighbourhood of y in Y contains a zero set neighbourhood of y and also a cozero set neighbourhood of y. Thus we can write, $\widehat{f}\big(\widehat{M}\big) \in Y \setminus Z(g_1) \subseteq Z(g_2) \subseteq W$, for some $g_1, g_2 \in C(Y)$.

Now, $\widehat{f}\big(\widehat{M}\big) \in Y \setminus Z(g_1) \implies g_1 \notin M_1 \implies g_1 \circ f \notin \widehat{M}$. So, $\widehat{M} \in \mathcal{M}(B_1(X)) \setminus \widehat{\mathscr{M}_{g_1 \circ f}}$. So, $\mathcal{M}(B_1(X)) \setminus \widehat{\mathscr{M}_{g_1 \circ f}}$ is a basic open set in $\mathcal{M}(B_1(X))$ containing $\widehat{M}$.

Our claim is $\widehat{f}\Big(\mathcal{M}(B_1(X)) \setminus \widehat{\mathscr{M}_{g_1 \circ f}}\Big) \subseteq W$.

Let $\widehat{N} \in \mathcal{M}(B_1(X)) \setminus \widehat{\mathscr{M}_{g_1 \circ f}} \implies \widehat{N} \notin \widehat{\mathscr{M}_{g_1 \circ f}} \implies g_1 \circ f \notin \widehat{N} \implies g_1 \notin N_1$. Also from $g_1.g_2 = 0$ and N_1 is a prime ideal in $C(Y)$, it follows that $g_2 \in N_1$ and hence, $\widehat{f}\big(\widehat{N}\big) \subseteq Z(g_2) \subseteq W$.

The uniqueness of $\widehat{f}$ follows from its continuity and the denseness of $\psi(X)$ in $\mathcal{M}(B_1(X))$. □

Theorem 4.9. *Suppose X is a T_4 space and $\big(\psi, \mathcal{M}(B_1(X))\big)$ is an F_σ-embedding of X in $\mathcal{M}(B_1(X))$, here $\psi(x) = \widehat{M}_x$, $x \in X$. Then the ring $B_1^*(X) \equiv \{f \in B_1(X) : f \text{ is bounded over } X\}$ is isomorphic to the ring $C(\mathcal{M}(B_1(X)))$ and hence, $B_1^*(X)$ is a C-type ring.*

Proof. Let $f \in B_1^*(X)$. Since X is T_4 and f is Baire one, it follows from Theorem 4.1(ii) that f is F_σ-continuous. Using Theorem 4.8, there exists a continuous function $\widehat{f} : \mathcal{M}(B_1(X)) \to [r, s]$ such that $\widehat{f} \circ \psi = f$; here $[r, s]$ is a closed interval in $\mathbb{R}$ containing $f(X)$.

Define $\eta : B_1^*(X) \to C(\mathcal{M}(B_1(X)))$ by $\eta(f) = \widehat{f}$.

Let $f, g \in B_1^*(X)$ and $x \in X$. Then

$(\widehat{f+g})(\widehat{M}_x) = \widehat{f+g}(\psi(x)) = (f+g)(x) = f(x) + g(x) = (\widehat{f} + \widehat{g})(\psi(x)) = (\widehat{f} + \widehat{g})(\widehat{M}_x))$.

Thus the two functions agree on all fixed maximal ideals in $B_1(X)$. Since the set of all fixed maximal ideals in $B_1(X)$ is dense in $\mathcal{M}(B_1(X))$, it follows that $\widehat{f} + \widehat{g} = \widehat{f+g}$ on the whole of $\mathcal{M}(B_1(X))$. In otherwords, $\eta(f+g) = \eta(f) + \eta(g)$. Analogously, $\eta(fg) = \eta(f)\eta(g)$. Thus η is a ring homomorphism. That η is one-to-one is clear from its definition.

We finally show that $\eta(B_1^*(X)) = C(\mathcal{M}(B_1(X)))$.

For that purpose, we choose $h \in C(\mathcal{M}(B_1(X)))$. Then h is a bounded function as $\mathcal{M}(B_1(X))$ is compact. Furthermore, the hypothesis that ψ is an F_σ embedding implies in view of Lemma 4.7 that $h \circ \psi \in B_1(X)$. Hence, $h \circ \psi \in B_1^*(X)$. Since $\widehat{h \circ \psi} \circ \psi = h$, i.e., $\widehat{h \circ \psi}$ and h agree on $\psi(X)$ and $\psi(X)$ is dense in $\mathcal{M}(B_1(X))$, it follows that $\widehat{h \circ \psi} = h$, i.e., $\eta(h \circ \psi) = h$. □

As the structure spaces of two isomorphic rings are homeomorphic and also a Tychonoff space X is compact if and only if it is homeomorphic to the structure space of $C(\mathcal{M}(C(X)))$, the following result comes out as a consequence of Theorem 4.9.

Corollary 4.10. *If X is a T_4 space and ψ is an F_σ-embedding, then $\mathcal{M}(B_1(X))$ is homeomorphic to $\mathcal{M}(B_1^*(X))$.*

It follows from Theorem 4.8 that for a T_4 space X, a bounded Baire one function f on X has a unique continuous extension over the compact Hausdorff space $\mathcal{M}(B_1(X))$ through a weak embedding $\psi : X \to \mathcal{M}(B_1(X))$. It is quite natural to ask, when does $\mathcal{M}(B_1(X))$ become βX, the *Stone-Čech* compactification of X. An answer to this question is given in the last theorem (Theorem 4.18) of this article. At present, however, let us be satisfied with a somewhat weaker form of answer to this query. Indeed, the family $\mathcal{B} = \{Z(f) : f \in B_1(X)\}$ is clearly a base for closed sets for some topology σ on X. If τ is the original topology on X, then $\{Z(f) : f \in C(X,\tau)\}$ is base for its closed sets. As $C(X,\tau) \subseteq B_1(X)$ it follows that $\tau \subseteq \sigma$. Since τ is Hausdorff, it is clear that σ is also Hausdorff. Also, since the topology σ on X is determined by a set of real valued functions on X, it follows that σ is completely regular and therefore, Tychonoff [vide Theorem 3.7 in [4]].

In the following, to avoid any confusion we simply write $\mathcal{M}(B_1(X,\tau))$ instead of $\mathcal{M}(B_1(X))$. Also we use the notation βX_σ for the *Stone-Čech* compactification of the space (X,σ).

Theorem 4.11. *If X is a T_4 space, then $\mathcal{M}(B_1(X,\tau))$ is βX_σ.*

Proof. We first observe that the function $\psi^* : (X,\sigma) \to \mathcal{M}(B_1(X,\tau))$ given by $\psi^*(x) = \widehat{M}_x$ defines a topological embedding onto a dense subspace of $\mathcal{M}(B_1(X,\tau))$ and therefore the pair $(\psi^*, \mathcal{M}(B_1(X,\tau))$ is a Hausdorff compactification of X ([3]).

That the map ψ^* is one-to-one and $\psi^*(X)$ is dense in $\mathcal{M}(B_1(X,\tau))$ are already checked in Theorem 4.4. We now observe that ψ^* exchanges the basic closed sets of the spaces (X,σ) and $\psi^*(X,\sigma)$. Indeed for any $f \in B_1(X,\tau)$, $\psi^*(Z(f)) = \widehat{\mathscr{M}_f} \cap \psi^*(X)$, a fact easily verifiable. To complete this theorem, it therefore remains to prove that the above embedding ψ^* possesses the universal extension property. To that end, let $f : (X,\sigma) \to Y$ be a continuous function where Y is a compact Hausdorff space. All that we need is to define a continuous function $\widehat{f} : \mathcal{M}(B_1(X,\tau)) \to Y$ with the property $\widehat{f} \circ \psi^* = f$. So, let $\widehat{M} \in \mathcal{M}(B_1(X,\tau))$. As in the proof of Theorem 4.8, we set $M_1 = \{g \in C(Y) : g \circ f \in \widehat{M}\}$ and observe that M_1 is a prime ideal in $C(Y)$ with $\bigcap_{g \in M_1} Z(g) = \{y\}$ for a uniquely determined point $y \in Y$. Now define $\widehat{f}(\widehat{M}) = y$. Then proceeding as in the proof of Theorem 4.8 we can establish that $\widehat{f}$ is a continuous function with $\widehat{f} \circ \psi^* = f$. □

Corollary 4.12. *If X is a T_4 space, then the three spaces $\mathcal{M}(B_1(X,\tau))$, $\mathcal{M}(C(X,\sigma))$ and βX_σ are pairwise homeomorphic.*

The following gives a complete description of the maximal ideals of $B_1^*(X)$, under certain conditions.

Theorem 4.13. *Assume that X is T_4 and $\psi : X \to \mathcal{M}(B_1(X))$ given by $x \mapsto \widehat{M}_x$ is an F_σ-embedding. Then the complete list of maximal ideals in $B_1^*(X)$ is given by $\{\widehat{M}^{*p} : p \in \beta X_\sigma\}$ where $\widehat{M}^{*p} = \{f \in {B_1}^*(X) : \widehat{f}(p) = 0\}$. Also $p \neq q$ implies $\widehat{M}^{*p} \neq \widehat{M}^{*q}$. Moreover, $\widehat{M}^{*p}$ is a fixed maximal ideal if and only if $p \in X$.*

Proof. By Theorem 4.9, the map $\eta : f \to \widehat{f}$ is an isomorphism from $B_1^*(X)$ onto $C(\mathcal{M}(B_1(X)))$. So, there is a one-one correspondence between the maximal ideals of $B_1^*(X)$ and those of $C(\mathcal{M}(B_1(X)))$. $\mathcal{M}(B_1(X))$ being compact, every maximal ideal of the ring $C(\mathcal{M}(B_1(X)))$ is of the form $\{h \in C(\mathcal{M}(B_1(X))) : h(p) = 0\}$, where $p \in \mathcal{M}(B_1(X)) \cong \beta X_\sigma$. So, the maximal ideals of $B_1^*(X)$ are given by

$\eta^{-1}(\{h \in C(\mathcal{M}(B_1(X))) : h(p) = 0\}) = \{f \in {B_1}^*(X) : \widehat{f}(p) = 0\} = \widehat{M}^{*p}$ (say), for each $p \in \beta X_\sigma$; here $\eta : B_1^*(X) \to C(\mathcal{M}(B_1(X,\tau))$ is the isomorphism considered in the proof of Theorem 4.9.
$p \neq q \Rightarrow \{h \in C(\mathcal{M}(B_1(X))) : h(p) = 0\} \neq \{h \in C(\mathcal{M}(B_1(X))) : h(q) = 0\}$ and so, $\eta^{-1}(\{h \in C(\mathcal{M}(B_1(X))) : h(p) = 0\}) \neq \eta^{-1}(\{h \in C(\mathcal{M}(B_1(X))) : h(q) = 0\})$. i.e., $\widehat{M}^{*p} \neq \widehat{M}^{*q}$.

If $p \in X$, then clearly, $\widehat{M}^{*p} = \{f \in {B_1}^*(X) : f(p) = 0\} = \widehat{M}_p^*$, the fixed maximal ideal of ${B_1}^*(X)$.

If $q \in \beta X_\sigma \setminus X$, then we claim that $\widehat{M}^{*q}$ is not fixed. If possible, it is a fixed maximal ideal of ${B_1}^*(X)$. Then $\widehat{M}^{*q} = \widehat{M}_p^*$ for some $p \in X$. But in that case, $\widehat{M}^{*q} = \widehat{M}^{*p}$ and consequently, $p = q$. □

To set further insight into the interconnection between (X, σ) and $B_1(X, \tau)$, we recall from the monograph of Chandler [Chapter 1, [3]] that a compactification of a Tychonoff space X stands for a pair (ψ, Y) where Y is a compact Hausdorff space and $\psi : X \to Y$ a topological embedding with $\psi(X)$ dense in Y. Two such compactifications (ψ_1, Y_1) and (ψ_2, Y_2) of X are called topologically equivalent if there exists a homeomorphism $t : Y_1 \to Y_2$ such that $t \circ \psi_1 = \psi_2$. A close look into the proof of Theorem 4.11 reveals that the pair $(\psi^*, \mathcal{M}(B_1(X, \tau)))$ is topologically equivalent to βX_σ. On the other hand it is a standard fact that the pair $(\psi, \mathcal{M}(C(X, \sigma)))$, where $\psi(x) = M_x = \{g \in C(X, \sigma) : g(x) = 0\}$ is also topologically equivalent to βX_σ. It follows that these two pairs themselves are topologically equivalent compactifications of (X, σ). This means that there exists a homeomorphism $\zeta : \mathcal{M}(B_1(X, \tau)) \to \mathcal{M}(C(X, \sigma))$ such that $\zeta \circ \psi^* = \psi$. i.e., $\zeta(\widehat{M_x}) = M_x$, for each $x \in X$. Thus fixed maximal ideals are sent to fixed maximal ideals under the homeomorphism ζ. It is well known that a Tychonoff space Y is compact if and only if each maximal ideal of $C(Y)$ is fixed [Theorem 4.11 in [4]]. Thus we have already established the following result.

Theorem 4.14. *If (X, τ) is a T_4 space, then each maximal ideal of $B_1(X, \tau)$ is fixed if and only if (X, σ) is compact.*

Theorem 4.15. *If (X, τ) is a compact Hausdorff space and $\sigma \supsetneq \tau$, then there exists at least one free maximal ideal in the ring $B_1(X, \tau)$.*

Proof. The hypothesis imply that (X, σ) is never compact, because, if a Hausdorff space (Y, δ) is compact, then no strictly finer compact topology δ_1 can be defined on Y. It follows from Theorem 4.11 in ([4]) that there exists a free maximal ideal N in $C(X, \sigma)$. Thus $N \in \mathcal{M}(C(X, \sigma))$. Consequently, $\zeta^{-1}(N)$ is a free maximal ideal in $B_1(X, \tau)$. □

The following result decides when the two topologies τ and σ on X become identical.

Theorem 4.16. *Suppose X is a T_4 space. Then $\sigma = \tau$ if and only if $B_1(X, \tau) = C(X, \tau)$.*

Proof. If $B_1(X, \tau) = C(X, \tau)$, then since (X, τ) is Tychonoff it follows from Theorem 3.2 in [4] that $\sigma = \tau$. For the converse, let $\sigma = \tau$. Consider any $f \in B_1(X, \tau)$ and any basic open set (a, b) of $\mathbb{R}$. Then $f^{-1}(a, b) = \{x \in X : a < f(x) < b\} = X \setminus (\{x \in X : f(x) \leq a\} \cup \{x \in X : f(x) \geq b\}) = (X \setminus Z(g_1)) \cap (X \setminus Z(g_2)) = U$ (say), where $g_1, g_2 \in B_1(X, \tau)$. Since $Z(g_1)$ and $Z(g_2)$ are closed in (X, σ), U is open in (X, σ). Hence, $f \in C(X, \sigma)$. By hypothesis, $\sigma = \tau$ and therefore, $C(X, \tau) \subseteq B_1(X, \tau) \subseteq C(X, \sigma) = C(X, \tau)$. i.e., $B_1(X, \tau) = C(X, \tau)$. □

Consequently, from Theorem 4.14 and Theorem 4.16, it follows that for a T_4 space (X, τ), if a non-continuous Baire one function exists, then $B_1(X)$ has a free (maximal) ideal.

In general, $B_1(X) = C(X)$ does not always imply that X is discrete. For example, if X is a P-space, then $B_1(X) = C(X)$ ([7]). We shall now show that for a particular class of topological spaces, e.g., for perfectly normal T_1 spaces, $B_1(X) = C(X)$ is equivalent to the discreteness of the space.

Theorem 4.17. *If (X, τ) is a perfectly normal T_1-space, then σ is the discrete topology on X.*

Proof. Let $\{y\}$ be any singleton set in (X, σ). Since T_1-ness is an expansive property and (X, τ) is T_1, it follows that (X, σ) is also T_1. So, $\{y\}$ is closed in (X, σ). Since (X, τ) is perfectly normal, $\{y\} = Z(g)$, for some $g \in C(X, \tau)$. Define $\chi_y : X \to \mathcal{M}$ as follows :

$$\chi_y(x) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise.} \end{cases}$$

$\chi_y \in B_1(X, \tau)$ has been established in [2] (see Theorem 3.7). Also $Z(\chi_y) = X \setminus \{y\}$ which is an open set in (X, τ) and hence open in (X, σ). By definition of (X, σ), $Z(\chi_y)$ is a closed set in (X, σ). Hence, $\{y\}$ is both open and closed in (X, σ) which shows by arbitrariness of $\{y\}$, σ is the discrete topology on X. □

Theorem 4.18. *For a perfectly normal T_1 space (X, τ) the following statements are equivalent:*

(1) (X, τ) is discrete.

(2) $B_1(X) = C(X)$.
(3) $\mathcal{M}(B_1(X))$ *is the Stone-Čech compactification of* (X, τ).

Proof. It follows from Theorem 1 in [7] that a Tychonoff space X is a P-space if and only if $B_1(X) = C(X)$. On the other hand, it is easy to prove that if a P-space X is perfectly normal, then it becomes a discrete space. Equivalence of the statements (1) and (2) therefore follow from these observations.

Since the structure space of $C(X)$ is βX [7N in [4]], (2) $\Rightarrow$ (3) is evident.

(3) $\Rightarrow$ (1): $\mathcal{M}(B_1(X))$ is a compact Hausdorff space containing (X, σ) as a dense subspace. Then the identity map $I : \mathcal{M}(B_1(X, \tau)) \to \mathcal{M}(B_1(X, \tau))$ when restricted on (X, τ) becomes a homeomorphism between (X, τ) and (X, σ). Hence, $\sigma = \tau$. Using Theorem 4.17, (X, τ) is a discrete space. □

Acknowledgement

The authors express sincere gratitude to the learned referee for his/her valuable suggestions towards the improvement of the paper.

References

[1] A. Deb Ray, A. Mondal, On rings Of Baire one functions, Appl. Gen. Topol. 20 (2019) 237–249.
[2] A. Deb Ray, A. Mondal, Ideals in $B_1(X)$ and residue class rings of $B_1(X)$ modulo an ideal, Appl. Gen. Topol. 20 (2019) 379–393.
[3] R.E. Chandler, Hausdorff Compactifications, Lecture Notes in Pure and Applied Mathematics, Vol. 23, Marcel Dekker Inc., 1976.
[4] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand Rein-hold Co., New York, 1960.
[5] G.D. Marco, A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, Proc. Amer. Math. Soc. 30 (1971) 459–466.
[6] L. Vesely, Characterization of Baire-one functions between topological spaces, Acta Univ. Carolinae Math. Physica 33 (1992) 143–156.
[7] M. Wojtowicz, W. Sieg, *P*-spaces and an unconditional closed graph theorem, Rev. Real Acad. Cien. Exactas, Fisc. Nat. 104 (2010) 13–18.