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First Eigenvalue of Weighted p-Laplacian Under Cotton Flow

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Abstract. In this paper we find out the evolution formula for the first nonzero eigenvalue of the weighted p-Laplacian operator acting on the space of functions under the Cotton flow on a closed Riemannian 3-manifold M^3.

1. Introduction

As an evolution equation for metric of a Riemannian 3-manifold (M^3, g), Kisisel et al. [10] introduced the notion of Cotton flow, whose evolution equation is given by

$$\frac{\partial}{\partial t} g_{ij} = K C_{ij} \,, \tag{1}$$

where K is a positive constant and C_{ij} is the Cotton-York tensor. The choice of K is arbitrary except being positive, the constant K can be set to 1 by scaling the evolution parameter t.

The Weyl tensor is a conformally invariant $(0, 4)$ tensor which measures conformal flatness of the manifold. As the Weyl tensor vanishes identically in dimension three, the Cotton tensor is the conformal tensor in dimension three. The Cotton tensor is given by [9]

$$C^{ij} = \frac{\eta^{ikl}}{\sqrt{g}} \nabla_k \left(R^j_l - \frac{1}{4} \delta^j_l R \right),$$

where $\epsilon^{klm} = \frac{\eta^{klm}}{\sqrt{g}}$ and η^{klm} is the complete antisymmetric tensor density of weight +1 with $\eta^{123} = 1$. C^{ij} is symmetric and covariantly conserved $\nabla_i C^{ij} = 0$ and traceless $g_{ij} C^{ij} = 0$.
In local coordinates

$$\begin{aligned} C_{ij} &= g_{ik} \epsilon^{klm} (\nabla_l R_{mj} - \frac{1}{4} \nabla_l R g_{mj}) \\ &= \frac{1}{2} g_{ik} \epsilon^{klm} C_{lmj} \,. \end{aligned}$$

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Now scaling the time parameter t to set $K = 1$, we have the following flow equation

$$\frac{\partial}{\partial t} g_{ij} = C_{ij} \,. \tag{2}$$

It is worthwhile to mention that this is not a quasilinear parabolic equation like the Ricci flow. In contrast (2) is a third order curvature flow and hence the short-time existence of the solution is yet to be proved. In this paper we assume the short-time existence for the flow.

The study of evolution and monotonicity of eigenvalue of geometric operators is a known problem. In recent years many mathematicians have studied the evolution and monotonicity of the eigenvalue of geometric operators like p-Laplacian, Witten Laplacian, weighted p-Laplacian along various geometric flows like Ricci flow, Yamabe flow, Ricci-Bourguignon flow, Ricci-harmonic flow etc., see [1–4, 6–8, 12]. The main study of evolution of eigenvalue of geometric operator along geometric flow when in [11], Perelman showed that the first eigenvalue of the geometric operator $-4\Delta + R$ is non-decreasing along the Ricci flow.

Recently in [5], Azami studied the evolution and monotonicity of the first eigenvalue of the p-Laplacain and weighted p-Laplacain along the Ricci-Bouguignon flow on a closed Riemannian manifold.

Motivated by the above mentioned studies, in this paper we study the evolution of the first nonzero eigenvalue of the weighted p-Laplacian along the Cotton flow, defined in (2) on a closed 3-manifold $(M^3, g(t))$.

2. Preliminaries

The Laplace-Beltrami operator, p-Laplacian operator, the Witten Laplacian and the weighted p-Laplacian operator are denoted by Δ, Δ_p, Δ_ϕ and $\Delta_{p,\phi}$ respectively. In local coordinates

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right),$$

where Γ^k_{ij} is the Christoffel symbols and $g^{ij} = (g_{ij})^{-1}$. We denote dv as the Riemannian volume measure on (M, g) and $d\mu = e^{-\phi}dv$, the weighted volume measure, where $\phi \in C^\infty(M)$. Throughout this study, we assume that M is a closed Riemannian manifold.

Let $f : M \to \mathbb{R}$, $f \in W^{1,p}(M)$, where $W^{1,p}(M)$ is the Sobolev space. For $p \in (1, \infty)$, the p-Laplacian of f is defined as

$$\begin{aligned}\Delta_p f &= \operatorname{div}(|\nabla f|^{p-2}\nabla f)\\ &= |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}\operatorname{Hess} f(\nabla f, \nabla f)\,.\end{aligned}$$

When $p = 2$, Δ_p is the usual Laplace-Beltrami operator.
The Witten Laplacian is defined by $\Delta_\phi = \Delta - \nabla\phi.\nabla$, which is a self adjoint operator.
For $p \in (1, \infty)$, the weighted p-Laplacian of f is defined as

$$\begin{aligned}\Delta_{p,\phi} f &= e^{\phi}\operatorname{div}(e^{-\phi}|\nabla f|^{p-2}\nabla f)\\ &= \Delta_p f - |\nabla f|^{p-2}\left\langle \nabla\phi, \nabla f\right\rangle.\end{aligned}$$

When ϕ is constant then it is just the p-Laplacian.
Choose λ such that

$$\Delta_{p,\phi} f = -\lambda |f|^{p-2} f, \tag{3}$$

for some $f \in W^{1,p}(M)$. In this case we say that λ is the eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ and f is the corresponding eigenfunction. The equation (3) implies that

$$\int_M f\Delta_{p,\phi} f\, d\mu = -\lambda \int_M |f|^p d\mu. \tag{4}$$

Using the integration by parts, (4) yields

$$\int_M |\nabla f|^p d\mu = \lambda \int_M |f|^p d\mu. \tag{5}$$

The mini-max principle states that the first nonzero eigenvalue $\lambda_1(t)$ is characterized as follows

$$\lambda_1(t) = inf\left\{\int_M |\nabla f|^p d\mu : \int_M |f|^p d\mu = 1, 0 \neq f \in W^{1,p}(M)\right\}, \tag{6}$$

where $W^{1,p}(M)$ is the completion of $C^\infty(M)$ with respect to the Sobolev norm

$$||f||_{W^{1,p}(M)} = \left(\int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu\right)^{\frac{1}{p}}$$

and f satisfies the condition $\int_M |f|^{p-2} f d\mu = 0$.
The first nonzero eigenvalue of weighted p-Laplacian and its corresponding eigenfunction are not known to be C^1-differentiable with respect to t under the Cotton flow. We assume that $\lambda(t)$ exists and is C^1-differentiable under the Cotton flow in the given interval $t \in [0, T)$.

3. First eigenvalue of the weighted p-Laplacian under Cotton flow

In this section we find a useful evolution equation for the first nonzero eigenvalue of the weighted p-Laplacian under the flow (2). Here we have assumed that the function ϕ is a time dependent function unless otherwise stated. In similar of [6], we can assume that $\lambda_1(t) = \lambda_1(f, t)$, where $\lambda_1(f, t)$ is defined by

$$\lambda_1(f, t) = -\int_M f \Delta_{p,\phi} f d\mu$$

and f is a smooth function satisfying the normalization condition

$$\int_M |f|^p d\mu = 1 \text{ and } \int_M |f|^{p-2} f d\mu = 0 \,.$$

We first recall some necessary evolution equations for the geometric quantities associated to $(M^3, g(t))$ along the Cotton flow (2).

Lemma 3.1. *[7] Let $(M^3, g(t))$ be a closed 3-manifold evolving under the Cotton flow (2). Then we have the following evolutions:*

$$(i)\ \frac{\partial g^{ij}}{\partial t} = -C^{ij} \text{ where } C^{ij} = g^{ik} g^{jl} C_{kl}, \tag{7}$$

$$(ii)\ \frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\nabla_i C_{jl} + \nabla_j C_{il} - \nabla_l C_{ij}), \tag{8}$$

$$(iii)\ \frac{\partial}{\partial t} R_{ij} = 3R_{li} C^l_j - R^{lm} C_{lm} g_{ij} - \frac{1}{2} R C_{ij} - \frac{1}{2}\nabla^2 C_{ij}, \tag{9}$$

$$(iv)\ \frac{\partial R}{\partial t} = -C^{ij} R_{ij}, \tag{10}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature.

Lemma 3.2. *Let $(M^3, g(t))$ be a closed 3-manifold evolving under the Cotton flow (2). Then the associated geometric quantities evolve by*

$$(i)\quad \frac{\partial}{\partial t}|\nabla f|^p = \frac{p}{2}|\nabla f|^{p-2}\left\{-C^{ij}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\right\}, \tag{11}$$

$$(ii)\quad \frac{\partial}{\partial t}|\nabla f|^{p-2} = \frac{p-2}{2}|\nabla f|^{p-4}\left\{-C^{ij}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\right\}, \tag{12}$$

$$(iii)\quad \frac{\partial}{\partial t}(\Delta f) = -C^{ij}\nabla_i\nabla_j f + \Delta f_t, \tag{13}$$

$$(iv)\quad \frac{\partial}{\partial t}(\Delta_p f) = -C^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t), \tag{14}$$

$$\begin{aligned}(v)\quad \frac{\partial}{\partial t}(\Delta_{p,\phi} f) = &-C^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - Z_t\left\langle\nabla\phi, \nabla f\right\rangle \\ &+ZC^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f - Zg^{ij}\nabla_i\phi\nabla_j f_t,\end{aligned} \tag{15}$$

where $Z := |\nabla f|^{p-2}$, $Z_t = \frac{\partial}{\partial t}Z$ and $f_t = \frac{\partial}{\partial t}f$.

Proof. By direct computations we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^p &= \frac{\partial}{\partial t}(g^{ij}\nabla_i f\nabla_j f)^{\frac{p}{2}} \\ &= \frac{p}{2}|\nabla f|^{p-2}\left\{\frac{\partial}{\partial t}(g^{ij})\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\right\} \\ &= \frac{p}{2}|\nabla f|^{p-2}\left\{-C^{ij}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\right\}.\end{aligned}$$

Using (11) we get

$$\frac{\partial}{\partial t}|\nabla f|^{p-2} = \frac{p-2}{2}|\nabla f|^{p-4}\left\{-C^{ij}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t\right\}.$$

Using (8), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}\left[g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k)f\right] \\ &= \frac{\partial}{\partial t}(g^{ij})\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)f + g^{ij}\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)f_t - g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij}\partial_k f \\ &= -C^{ij}\nabla_i\nabla_j f + \Delta f_t - \frac{1}{2}g^{ij}g^{kl}\left(\nabla_i C_{jl} + \nabla_j C_{il} - \nabla_l C_{ij}\right)\nabla_k f \\ &= -C^{ij}\nabla_i\nabla_j f + \Delta f_t - \frac{1}{2}g^{kl}\left(2g^{ij}\nabla_i C_{jl} - g^{ij}\nabla_l C_{ij}\right)\nabla_k f \\ &= -C^{ij}\nabla_i\nabla_j f + \Delta f_t,\end{aligned}$$

as Cotton tensor is trace free and divergence free.
Let us take $Z = |\nabla f|^{p-2}$. Then we get

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_p f) &= \frac{\partial}{\partial t}(\mathrm{div}(|\nabla f|^{p-2}\nabla f)) \\ &= \frac{\partial}{\partial t}(g^{ij}\nabla_i(Z\nabla_j f)) \\ &= \frac{\partial}{\partial t}(g^{ij})\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f + Z(\Delta f)_t \\ &= -C^{ij}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f + Z(-C^{ij}\nabla_i\nabla_j f + \Delta f_t) \\ &= -C^{ij}\nabla_i Z\nabla_j f - ZC^{ij}\nabla_i\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f + Z\Delta f_t \\ &= -C^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t).\end{aligned}$$

We have

$$\Delta_{p,\phi} f = \Delta_p f - |\nabla f|^{p-2} \left\langle \nabla\phi, \nabla f \right\rangle .$$

Taking derivative with respect to the time t of both sides of the above equation and using (14) we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta_{p,\phi} f) &= \frac{\partial}{\partial t}(\Delta_p f) - \frac{\partial}{\partial t}(Z \left\langle \nabla\phi, \nabla f \right\rangle) \\
&= \frac{\partial}{\partial t}(\Delta_p f) - Z_t \left\langle \nabla\phi, \nabla f \right\rangle - Z\frac{\partial}{\partial t}(g^{ij}\nabla_i\phi\nabla_j f) \\
&= \frac{\partial}{\partial t}(\Delta_p f) - Z_t \left\langle \nabla\phi, \nabla f \right\rangle + ZC^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f - Zg^{ij}\nabla_i\phi\nabla_j f_t \\
&= -C^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - Z_t \left\langle \nabla\phi, \nabla f \right\rangle \\
&\quad +ZC^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f - Zg^{ij}\nabla_i\phi\nabla_j f_t.
\end{aligned}$$

□

Now we derive an evolution formula for the first nonzero eigenvalue of the weigthed p-Laplacian $\Delta_{p,\phi}$ of the eigenvalue problem (3) along the Cotton flow (2).

Theorem 3.3. *Let $(M^3(t), g(t))$, $t \in [0,T)$ be a solution of the Cotton flow (2) on a closed Riemannian manifold M^3. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian and $f(x,t)$ its corresponding eigenfunction. Then $\lambda_1(t)$ satisfies the following evolution equation*

$$\frac{d}{dt}\lambda_1(t) = -\frac{p}{2}\int_M |\nabla f|^{p-2}C^{ij}\nabla_i f\nabla_j f d\mu + \lambda_1(t)\int_M \phi_t|f|^p d\mu - \int_M \phi_t|\nabla f|^p d\mu \tag{16}$$

for all time $t \in [0,T), T < \infty$.

Proof. We now have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f d\mu &= \int_M \frac{\partial}{\partial t}(\Delta_{p,\phi} f) f d\mu + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f d\mu) \\
&= -\int_M C^{ij}\nabla_i(Z\nabla_j f) f d\mu + \int_M g^{ij}\nabla_i(Z_t\nabla_j f) f d\mu \\
&\quad + \int_M g^{ij}\nabla_i(Z\nabla_j f_t) f d\mu - \int_M Z_t \left\langle \nabla\phi, \nabla f \right\rangle f d\mu \\
&\quad + \int_M ZC^{ij}\nabla_i\phi\nabla_j f f d\mu - \int_M Z \left\langle \nabla\phi_t, \nabla f \right\rangle f d\mu \\
&\quad - \int_M Z \left\langle \nabla\phi, \nabla f_t \right\rangle f d\mu + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f d\mu).
\end{aligned} \tag{17}$$

Using the integration by parts the first integral of the right hand side of the above equation, we get

$$\begin{aligned}
-\int_M C^{ij}\nabla_i(Z\nabla_j f) f d\mu &= \int_M Z\nabla_i f\nabla_j(C^{ij} f e^{-\phi}) dv \\
&= \int_M ZC^{ij}\nabla_i f\nabla_j f d\mu + \int_M Z\nabla_i C^{ij}\nabla_j f f d\mu - \int_M ZC^{ij}\nabla_i f\nabla_j\phi f d\mu \\
&= \int_M ZC^{ij}\nabla_i f\nabla_j f d\mu - \int_M ZC^{ij}\nabla_i f\nabla_j\phi f d\mu \ ,
\end{aligned} \tag{18}$$

as the Cotton tensor is divergence free, i.e. $\nabla_i C^{ij} = 0$.
Using integration by parts in the second and third integral in the right hand side of (17) we obtain

$$\begin{aligned}\int_M g^{ij}\nabla_i(Z_t\nabla_j f)fd\mu &= -\int_M Z_t\nabla_j f\nabla^i(fe^{-\phi})d\nu \\ &= -\int_M Z_t|\nabla f|^2 d\mu + \int_M Z_t\left\langle \nabla f, \nabla\phi\right\rangle fd\mu\,, \end{aligned} \tag{19}$$

and

$$\begin{aligned}\int_M g^{ij}\nabla_i(Z\nabla_j f_t)fd\mu &= -\int_M Z\nabla_j f_t\nabla^i(fe^{-\phi})d\nu \\ &= -\int_M Z\left\langle \nabla f_t, \nabla f\right\rangle d\mu + \int_M Z\left\langle \nabla f_t, \nabla\phi\right\rangle fd\mu. \end{aligned} \tag{20}$$

In view of (18), (19) and (20), it follows form (17) that

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}fd\mu &= \int_M ZC^{ij}\nabla_i f\nabla_j fd\mu - \int_M Z_t|\nabla f|^2 d\mu - \int_M Z\left\langle \nabla f_t, \nabla f\right\rangle d\mu \\ &\quad -\int_M Z\left\langle \nabla\phi_t, \nabla f\right\rangle fd\mu + \int_M \Delta_{p,\phi}f\frac{\partial}{\partial t}(fd\mu). \end{aligned} \tag{21}$$

From (12) we get

$$|\nabla f|^2\frac{\partial}{\partial t}Z = -\frac{p-2}{2}ZC^{ij}\nabla_i\nabla_j f + (p-2)Z\left\langle \nabla f, \nabla f_t\right\rangle. \tag{22}$$

In veiw of (22), (21) yields

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}fd\mu &= \int_M ZC^{ij}\nabla_i f\nabla_j fd\mu + \frac{p-2}{2}\int_M ZC^{ij}\nabla_i f\nabla_j fd\mu \\ &\quad -(p-2)\int_M Z\left\langle \nabla f, \nabla f_t\right\rangle d\mu - \int_M Z\left\langle \nabla f_t, \nabla f\right\rangle d\mu \\ &\quad -\int_M Z\left\langle \nabla\phi_t, \nabla f\right\rangle fd\mu + \int_M \Delta_{p,\phi}f\frac{\partial}{\partial t}(fd\mu) \\ &= \frac{p}{2}\int_M ZC^{ij}\nabla_i f\nabla_j fd\mu - (p-1)\int_M Z\left\langle \nabla f, \nabla f_t\right\rangle d\mu \\ &\quad -\int_M Z\left\langle \nabla\phi_t, \nabla f\right\rangle fd\mu + \int_M \Delta_{p,\phi}f\frac{\partial}{\partial t}(fd\mu). \end{aligned} \tag{23}$$

Using

$$-(p-1)\int_M Z\left\langle \nabla f, \nabla f_t\right\rangle d\mu = (p-1)\int_M \Delta_{p,\phi}ff_t d\mu, \tag{24}$$

and

$$-\int_M Z\left\langle \nabla\phi_t, \nabla f\right\rangle fd\mu = \int_M \phi_t\Delta_{p,\phi}ffd\mu + \int_M \phi_t|\nabla f|^p d\mu, \tag{25}$$

in (23), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f d\mu &= \frac{p}{2}\int_M ZC^{ij}\nabla_i f\nabla_j f d\mu + (p-1)\int_M \Delta_{p,\phi} f f_t d\mu \\
&+ \int_M \phi_t \Delta_{p,\phi} f f d\mu + \int_M \phi_t |\nabla f|^p d\mu + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f d\mu) \\
&= \frac{p}{2}\int_M ZC^{ij}\nabla_i f\nabla_j f d\mu + \int_M \phi_t \Delta_{p,\phi} f f d\mu + \int_M \phi_t |\nabla f|^p d\mu \\
&+ \int_M \Delta_{p,\phi} f((p-1) f_t d\mu + \frac{\partial}{\partial t}(f d\mu)).
\end{aligned}$$

Differentiating with respect to t of the integrability condition $\int_M |f|^p d\mu = 1$ we get

$$\frac{\partial}{\partial t}\int_M |f|^p d\mu = 0,$$

i.e.

$$\int_M |f|^{p-2} f((p-1) f_t d\mu + \frac{\partial}{\partial t}(f d\mu)) = 0.$$

Using the facts that

$$\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f d\mu = -\frac{\partial}{\partial t}\lambda_1(t) \text{ and } \Delta_{p,\phi} f = -\lambda_1(t)|f|^{p-2} f\ ,$$

we comes to

$$\frac{d}{dt}\lambda_1(t) = -\frac{p}{2}\int_M |\nabla f|^{p-2} C^{ij}\nabla_i f\nabla_j f d\mu + \lambda_1(t)\int_M \phi_t |f|^p d\mu - \int_M \phi_t |\nabla f|^p d\mu\ .$$

□

Corollary 3.4. *Let $(M^3(t), g(t))$, $t \in [0, T)$ be a solution of the Cotton flow (2) on a closed Riemannian manifold M^3. If the weight function ϕ is independent of t and $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian and $f(x, t)$ its corresponding eigenfunction, then*

$$\frac{d}{dt}\lambda_1(t) = -\frac{p}{2}\int_M |\nabla f|^{p-2} C^{ij}\nabla_i f\nabla_j f d\mu \tag{26}$$

holds for all $t \in [0, T), T < \infty$.

Corollary 3.5. *Let (M^3, g_0) be an Einstein manifold. If the weight function $\phi = \psi(x)$ is time-independent, i.e. $d\mu = e^{-\psi(x)}dv$, then the evolution equation (26) reduces to*

$$\frac{d}{dt}\lambda_1(t) = 0,$$

i.e. the first nonzero eigenvalue of the weighted p-Laplacian under the Cotton flow is constant.

Theorem 3.6. *Let $(M^3(t), g(t))$, $t \in [0, T)$ be a solution of the Cotton flow (2) on a closed Riemannian manifold M^3. Let $C^{ij} \geq a\phi_t g_{ij}$ along the flow (2) in $M \times [0, T)$ for some positive constant a. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian. Then the quantity $\lambda_1(\tau)e^{-(\frac{p}{2}a+2)\int_0^t \max_M |\phi_t|(\tau)d\tau}$ is non-increasing under the Cotton flow.*

Proof. According to (26) of Theorem 3.3, we have

$$\begin{aligned}\frac{d}{dt}\lambda_1(t) &\leq -(\frac{p}{2}a+1)\int_M \phi_t|\nabla f|^p d\mu + \lambda_1(t)\int_M \phi_t|f|^p d\mu \\ &\leq (\frac{p}{2}a+2)\lambda_1(t)\max_M|\phi_t|(t).\end{aligned}$$

Therefore, $\frac{d}{dt}\left(\lambda_1(\tau)e^{-(\frac{p}{2}a+2)\int_0^t \max_M|\phi_t|(\tau)d\tau}\right) \leq 0$ and it implies the quantity $\lambda_1(\tau)e^{-(\frac{p}{2}a+2)\int_0^t \max_M|\phi_t|(\tau)d\tau}$ is non-increasing under the Cotton flow. □

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